Introduction

We present nine lectures that are introductory and foundational in nature. The basic inspiration comes from the Riemann zeta function, which is the starting point. Along the way there are sprinkled some connections of the material to physics. The asymptotics of Fourier coefficients of zero weight modular forms, for example, are considered in regards to black hole entropy. Thus we have some interests also connected with Einstein's general relativity. References are listed that cover much more material, of course, than what is attempted here.

Although his papers were few in number during his brief life, which was cut short by tuberculosis, Georg Friedrich Bernhard Riemann (1826–1866) ranks prominently among the most outstanding mathematicians of the nineteenth century. In particular, Riemann published only one paper on number theory [32]: "Über die Anzahl der Primzahlen unter einer gegebenen Grösse", that is, "On the number of primes less than a given magnitude". In this short paper prepared for Riemann's election to the Berlin Academy of Sciences, he presented a study of the distribution of primes based on complex variables methods. There the now famous *Riemann zeta function*

$$\zeta(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{0.1}$$

defined for $\operatorname{Re} s > 1$, appears along with its analytic continuation to the full complex plane $\mathbb{C}$, and a proof of a functional equation (FE) that relates the values $\zeta(s)$ and $\zeta(1-s)$. The FE in fact was conjectured by Leonhard Euler, who also obtained in 1737 (over 120 years before Riemann) an *Euler product*

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representation

$$\zeta(s) = \prod_{p>0} \frac{1}{1-p^{-s}} \qquad (\mathrm{Re}\, s > 1) \tag{0.2}$$

of $\zeta(s)$ where the product is taken over the primes p. Moreover, Riemann introduced in that seminal paper a query, now called the *Riemann Hypothesis* (RH), which to date has defied resolution by the best mathematical minds. Namely, as we shall see, $\zeta(s)$ vanishes at the values $s = -2n$, where $n = 1, 2, 3, \ldots$; these are called the *trivial* zeros of $\zeta(s)$. The RH is the (yet unproved) statement that if s is a zero of ζ that is *not* trivial, the real part of s must have the value $\frac{1}{2}$!

Regarding Riemann's analytic approach to the study of the distribution of primes, we mention that his main goal was to set up a framework to facilitate a proof of the *prime number theorem* (which was also conjectured by Gauss) which states that if $\pi(x)$ is the number of primes $\leq x$, for $x \in \mathbb{R}$ a real number, then $\pi(x)$ behaves asymptotically (as $x \to \infty$) as $x/\log x$. That is, one has (precisely) that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \tag{0.3}$$

which was independently proved by Jacques Hadamard and Charles de la Vallée-Poussin in 1896. A key role in the proof of the monumental result (0.3) is the fact that at least all nontrivial zeros of $\zeta(s)$ reside in the interior of the *critical strip* $0 \leq \mathrm{Re}\, s \leq 1$.

Riemann's deep contributions extend to the realm of physics as well - Riemannian geometry, for example, being the perfect vehicle for the formulation of Einstein's gravitational field equations of general relativity. Inspired by the definition (0.1), or by the Euler product in (0.2), one can construct various other zeta functions (as is done in this volume) with a range of applications to physics. A particular zeta function that we shall consider later will bear a particular relation to a particular solution of the Einstein field equations — namely a *black hole* solution; see my Speaker's Lecture.

There are quite many ways nowadays to find the analytic continuation and FE of $\zeta(s)$. We shall basically follow Riemann's method. For the reader's benefit, we collect some standard background material in various appendices. Thus, to a large extent, we shall attempt to provide details and completeness of the material, although at some points (later for example, in the lecture on modular forms) the goal will be to present a general picture of results, with some (but not all) proofs.

Special thanks are extended to Jennie D'Ambroise for her competent and thoughtful preparation of all my lectures presented in this volume.

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Lecture 1. Analytic continuation and functional equation of the Riemann zeta function

Since $|1/n^s| = 1/n^{\operatorname{Re} s}$, the series in (0.1) converges absolutely for $\operatorname{Re} s > 1$. Moreover, by the Weierstrass M-test, for any $\delta > 0$ one has uniform convergence of that series on the strip

$$S_\delta \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \operatorname{Re} s > 1 + \delta\},$$

since $|1/n^s| = 1/n^{\operatorname{Re} s} < 1/n^{1+\delta}$ on S_δ, with

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty.$$

Since any compact subset of the domain $S_0 \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \operatorname{Re} s > 1\}$ is contained in some S_δ, the series, in particular, converges absolutely and uniformly on compact subsets of S_0. By Weierstrass's general theorem we can conclude that the Riemann zeta function $\zeta(s)$ in (0.1) is holomorphic on S_0 (since the terms $1/n^s$ are holomorphic in s) and that termwise differentiation is permitted: for $\operatorname{Re} s > 1$

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}. \tag{1.1}$$

We wish to analytically continue $\zeta(s)$ to the full complex plane. For that purpose, we begin by considering the world's simplest *theta function* $\theta(t)$, defined

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for $t > 0$:

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t} \tag{1.2}$$

where $\mathbb{Z}$ denotes the ring of integers. It enjoys the remarkable property that its values at t and t inverse (i.e. $1/t$) are related:

$$\theta(t) = \frac{\theta(1/t)}{\sqrt{t}}. \tag{1.3}$$

The very simple formula (1.3), which however requires some work to prove, is called the *Jacobi inversion formula*. We set up a proof of it in Appendix C, based on the *Poisson Summation Formula* proved in Appendix C. One can of course define more complicated theta functions, even in the context of higher-dimensional spaces, and prove analogous Jacobi inversion formulas.

For $s \in \mathbb{C}$ define

$$J(s) \stackrel{\text{def}}{=} \int_1^{\infty} \frac{\theta(t) - 1}{2} t^{s} \, dt. \tag{1.4}$$

By Appendix A, $J(s)$ is an entire function of s, whose derivative can be obtained, in fact, by differentiation under the integral sign. One can obtain both the analytic continuation and the functional equation of $\zeta(s)$ by introducing the sum

$$I(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \int_0^{\infty} (\pi n^2)^{-s} e^{-t} t^{s-1} \, dt, \tag{1.5}$$

which we will see is well-defined for $\operatorname{Re} s > \frac{1}{2}$, and by computing it in different ways, based on the inversion formula (1.3). Recalling that the gamma function $\Gamma(s)$ is given for $\operatorname{Re} s > 0$ by

$$\Gamma(s) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-t} t^{s-1} \, dt \tag{1.6}$$

we clearly have

$$I(s) \stackrel{\text{def}}{=} \pi^{-s} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \right) \Gamma(s) = \pi^{-s} \zeta(2s) \Gamma(s), \tag{1.7}$$

so that $I(s)$ is well-defined for $\operatorname{Re} 2s > 1$: $\operatorname{Re} s > \frac{1}{2}$. On the other hand, by the change of variables $u = t/\pi n^2$ we transform the integral in (1.5) to obtain

$$I(s) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s-1} \, dt.$$

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We can interchange the summation and integration here by noting that

$$\sum_{n=1}^{\infty}\int_0^{\infty}\left|e^{-\pi n^2 t}t^{s-1}\right|dt=\sum_{n=1}^{\infty}\int_0^{\infty}e^{-\pi n^2 t}t^{\operatorname{Re}s-1}\,dt=I(\operatorname{Re}s)<\infty$$

for $\operatorname{Re}s>\frac{1}{2}$; thus

$$\begin{aligned}I(s)&=\int_0^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 t}\,t^{s-1}\,dt=\int_0^{\infty}\frac{\theta(t)-1}{2}t^{s-1}\,dt\\&=\int_0^{1}\frac{\theta(t)-1}{2}\,t^{s-1}\,dt+\int_1^{\infty}\frac{\theta(t)-1}{2}\,t^{s-1}\,dt,\end{aligned}\tag{1.8}$$

by (1.2). Here

$$\int_0^1 t^{s-1}\,dt=\lim_{\varepsilon\to0+}\int_{\varepsilon}^1 t^{s-1}\,dt=\frac{1}{s}\tag{1.9}$$

for $\operatorname{Re}s>0$. In particular (1.9) holds for $\operatorname{Re}s>\frac{1}{2}$, and we have

$$\int_0^1\frac{\theta(t)-1}{2}\,t^{s-1}\,dt=\frac{1}{2}\int_0^1\theta(t)t^{s-1}\,dt-\frac{1}{2s}.\tag{1.10}$$

By the change of variables $u=1/t$, coupled with the Jacobi inversion formula (1.3), we get

$$\begin{aligned}\int_0^1\theta(t)t^{s-1}\,dt&=\int_1^{\infty}\theta\left(\frac{1}{t}\right)t^{-1-s}\,dt=\int_1^{\infty}\theta(t)t^{\frac{1}{2}}t^{-1-s}\,dt\\&=\int_1^{\infty}(\theta(t)-1)\,t^{-\frac{1}{2}-s}\,dt+\int_1^{\infty}t^{-\frac{1}{2}-s}\,dt\\&=\int_1^{\infty}(\theta(t)-1)\,t^{-\frac{1}{2}-s}\,dt+\int_0^{1}u^{-\frac{3}{2}+s=(s-\frac{1}{2})-1}\,du\\&=\int_1^{\infty}(\theta(t)-1)\,t^{-\frac{1}{2}-s}\,dt+\frac{1}{s-\frac{1}{2}},\end{aligned}$$

where we have used (1.9) again for $\operatorname{Re}s>\frac{1}{2}$. Together with equations (1.8) and (1.10), this gives

$$\begin{aligned}I(s)&=\frac{1}{2}\int_1^{\infty}(\theta(t)-1)\,t^{-\frac{1}{2}-s}\,dt+\frac{1}{2(s-\frac{1}{2})}-\frac{1}{2s}+\int_1^{\infty}\frac{\theta(t)-1}{2}\,t^{s-1}\,dt\\&=\int_1^{\infty}\frac{\theta(t)-1}{2}\left(t^{s-1}+t^{-\frac{1}{2}-s}\right)dt+\frac{1}{2s-1}-\frac{1}{2s},\end{aligned}$$

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which with equation (1.7) gives

$$\pi^{-s}\zeta(2s)\Gamma(s) = \int_1^\infty \frac{\theta(t)-1}{2}\left(t^{s-1}+t^{-\frac{1}{2}-s}\right)dt + \frac{1}{2s-1} - \frac{1}{2s}, \tag{1.11}$$

for $\operatorname{Re} s > \frac{1}{2}$. Finally, in (1.11) replace s by $s/2$, to obtain

$$\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \int_1^\infty \frac{\theta(t)-1}{2}\left(t^{\frac{s}{2}-1}+t^{-\frac{s}{2}-\frac{1}{2}}\right)dt + \frac{1}{s-1} - \frac{1}{s} \tag{1.12}$$

for $\operatorname{Re} s > 1$. Since $z\Gamma(z) = \Gamma(z+1)$, we have $\Gamma\left(\frac{s}{2}\right)s = 2\Gamma\left(\frac{s}{2}+1\right)$, which proves:

THEOREM 1.13. *For* $\operatorname{Re} s > 1$ *we can write*

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\int_1^\infty \frac{\theta(t)-1}{2}\left(t^{\frac{s}{2}-1}+t^{-\frac{s}{2}-1}\right)dt + \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)(s-1)} - \frac{\pi^{\frac{s}{2}}}{2\Gamma\left(\frac{s}{2}+1\right)}.$$

The integral $\int_1^\infty$ in this equality is an entire function of s, since, by (1.4), it equals $J\left(\frac{s}{2}-1\right)+J\left(-\frac{s}{2}-1\right)$. Also, since $1/\Gamma(s)$ is an entire function of s, it follows that the right-hand side of the equality in Theorem 1.13 provides for the analytic continuation of $\zeta(s)$ to the full complex plane, where it is observed that $\zeta(s)$ has only one singularity: $s = 1$ is a *simple pole*.

The fact that $\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$ allows one to compute the corresponding residue:

$$\lim_{s\to 1}(s-1)\zeta(s) = \lim_{s\to 1}\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = 1.$$

An equation that relates the values $\zeta(s)$ and $\zeta(1-s)$, called a *functional equation*, easily follows from the preceding discussion. In fact define

$$X_{\boldsymbol{R}}(s) \stackrel{\text{def}}{=} \pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) \tag{1.14}$$

for $\operatorname{Re} s > 1$ and note that the right-hand side of equation (1.12) (which provides for the analytic continuation of $X_{\boldsymbol{R}}(s)$ as a meromorphic function whose simple poles are at $s = 0$ and $s = 1$) is *unchanged* if s there is replaced by $1-s$:

THEOREM 1.15 (THE FUNCTIONAL EQUATION FOR $\zeta(s)$). *Let* $X_{\boldsymbol{R}}(s)$ *be given by* (1.14) *and analytically continued by the right-hand side of the* (1.12). *Then* $X_{\boldsymbol{R}}(s) = X_{\boldsymbol{R}}(1-s)$ *for* $s \neq 0, 1$.

One can write the functional equation as

$$\zeta(1-s) = \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)} = \frac{\pi^{-s+\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\Gamma\left(\frac{1-s}{2}\right)} \tag{1.16}$$

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for $s \neq 0, 1$, multiply the right-hand side here by $1 = -\left(\frac{s-1}{2}\right)/\left(\frac{1-s}{2}\right)$, use the identity $\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \Gamma\left(\frac{3-s}{2}\right)$, and thus also write

$$\zeta(1-s) = -\frac{\pi^{-s+\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)(s-1)\zeta(s)}{2\Gamma\left(\frac{3-s}{2}\right)}, \tag{1.17}$$

an equation that will be useful later when we compute $\zeta'(0)$.

For the computation of $\zeta'(0)$ we make use of the following result, which is of independent interest. $[x]$ denotes the largest integer that does not exceed $x \in \mathbb{R}$.

THEOREM 1.18. *For* $\operatorname{Re} s > 1$,

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \frac{1}{2} + s\int_1^\infty \frac{\left([x]-x+\frac{1}{2}\right)dx}{x^{s+1}} \\ &= \frac{1}{s-1} + 1 + s\int_1^\infty \frac{[x]-x}{x^{s+1}}\,dx. \end{aligned} \tag{1.19}$$

That these two expressions for $\zeta(s)$ are equal follows from the equality $\int_1^\infty \frac{dx}{x^{s+1}} = \frac{1}{s}$ for $\operatorname{Re} s > 0$; this with the inequalities $0 \leq x - [x] < 1$ allows one to deduce that the improper integrals there converge absolutely for $\operatorname{Re} s > 0$. We base the proof of Theorem 1.18 on a general observation:

LEMMA 1.20. *Let $\phi(x)$ be continuously differentiable on a closed interval $[a, b]$. Then, for $c \in \mathbb{R}$,*

$$\int_a^b \left(x - c - \tfrac{1}{2}\right)\phi'(x)\,dx = \left(b - c - \tfrac{1}{2}\right)\phi(b) - \left(a - c - \tfrac{1}{2}\right)\phi(a) - \int_a^b \phi(x)\,dx.$$

In particular for $[a, b] = [n, n+1]$, with $n \in \mathbb{Z}$ one gets

$$\int_n^{n+1} \left(x - [x] - \tfrac{1}{2}\right)\phi'(x)\,dx = \frac{\phi(n+1)+\phi(n)}{2} - \int_n^{n+1} \phi(x)\,dx.$$

PROOF. The first assertion is a direct consequence of integration by parts. Using it, one obtains for the choice $c = n$ the second assertion: $\int_n^{n+1}[x]\phi'(x)\,dx = \int_n^{n+1} n\phi'(x)\,dx$ (since $[x] = n$ for $n \leq x < n+1$); hence

$$\begin{aligned} &\int_n^{n+1} \left(x - [x] - \tfrac{1}{2}\right)\phi'(x)\,dx \\ &\qquad = \int_n^{n+1} \left(x - n - \tfrac{1}{2}\right)\phi'(x)\,dx \\ &\qquad \overset{\therefore}{=} \left(n + 1 - n - \tfrac{1}{2}\right)\phi(n+1) - \left(n - n - \tfrac{1}{2}\right)\phi(n) - \int_n^{n+1} \phi(x)\,dx \\ &\qquad = \tfrac{1}{2}\phi(n+1) + \tfrac{1}{2}\phi(n) - \int_n^{n+1} \phi(x)\,dx, \end{aligned}$$

$\square$

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As a first application of the lemma, note that for integers m_2, m_1 with $m_2 > m_1$,

$$\begin{aligned}
\sum_{n=m_1}^{m_2} [\phi(n+1) + \phi(n)] \\
&= \sum_{n=m_1}^{m_2} \phi(n+1) + \sum_{n=m_1}^{m_2} \phi(n) \\
&= \phi(m_1+1)+\phi(m_1+2)+\cdots+\phi(m_2+1)+\phi(m_1)+\phi(m_1+1)+\cdots+\phi(m_2) \\
&= \phi(m_2+1) + \phi(m_1)+2 \sum_{n=m_1+1}^{m_2} \phi(n).
\end{aligned}$$

Also $\sum_{n=m_1}^{m_2} \int_n^{n+1} = \int_{m_1}^{m_2+1}$. Therefore

$$\begin{aligned}
\frac{\phi(m_2+1)+\phi(m_1)}{2} + \sum_{n=m_1+1}^{m_2} \phi(n) &= \frac{1}{2} \sum_{n=m_1}^{m_2} [\phi(n+1)+\phi(n)] \\
&= \sum_{n=m_1}^{m_2} \int_n^{n+1} (x-[x]-\tfrac{1}{2})\phi'(x)\,dx + \sum_{n=m_1}^{m_2} \int_n^{n+1} \phi(x)\,dx
\end{aligned}$$

(by Lemma 1.20), which equals $\int_{m_1}^{m_2+1}(x-[x]-\frac{1}{2})\phi'(x)\,dx + \int_{m_1}^{m_2+1}\phi(x)\,dx$. Thus

$$\begin{aligned}
\sum_{n=m_1+1}^{m_2} \phi(n) &= \frac{-\phi(m_2+1)-\phi(m_1)}{2} \\
&\quad + \int_{m_1}^{m_2+1} \phi(x)\,dx + \int_{m_1}^{m_2+1} (x-[x]-\frac{1}{2})\phi'(x)\,dx \qquad (1.21)
\end{aligned}$$

for $\phi(x)$ continuously differentiable on $[m_1, m_2+1]$. Now choose $m_1 = 1$ and

$$\phi(x) \stackrel{\text{def}}{=} x^{-s}$$

for $x > 0$, $\operatorname{Re} s > 1$. Then $\int_1^\infty \frac{dx}{x^s} = \frac{1}{s-1}$. Also $\phi(m_2+1) = (m_2+1)^{-s} \to 0$ as $m_2 \to \infty$, since $\operatorname{Re} s > 0$. Thus in (1.21) let $m_2 \to \infty$:

$$\sum_{n=2}^{\infty} \frac{1}{n^s} = -\tfrac{1}{2} + \frac{1}{s-1} + \int_1^\infty (x-[x]-\tfrac{1}{2})(-sx^{-s-1})\,dx.$$

That is, for $\operatorname{Re} s > 1$ we have

$$\zeta(s) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^s} = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{([x]-x+\frac{1}{2})}{x^{s+1}}dx,$$

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which proves Theorem 1.18.

We turn to the second integral in equation (1.19), which we denote by

$$f(s) \stackrel{\text{def}}{=} \int_1^\infty \frac{([x]-x)}{x^{s+1}} dx$$

for $\operatorname{Re} s > 0$. We can write $f(s) = \lim_{n\to\infty} \int_1^n \frac{([x]-x)}{x^{s+1}} dx$, where

$$\int_1^n \frac{([x]-x)}{x^{s+1}} dx = \sum_{j=1}^{n-1} \int_j^{j+1} \frac{([x]-x)}{x^{s+1}} dx = \sum_{j=1}^{n-1} \int_j^{j+1} \frac{j-x}{x^{s+1}} dx, \qquad (1.22)$$

since $[x] = j$ for $j \le x < j+1$. That is, $f(s) = \sum_{j=1}^\infty a_j(s)$ where

$$a_j(s) \stackrel{\text{def}}{=} \int_j^{j+1} \frac{j-x}{x^{s+1}} dx = \frac{j}{s}\left(\frac{1}{j^s} - \frac{1}{(j+1)^s}\right) - \frac{1}{s-1}\left(\frac{1}{j^{s-1}} - \frac{1}{(j+1)^{s-1}}\right)$$

for $s \neq 0, 1$, and where for the second term here $s = 1$ is a removable singularity:

$$\lim_{s\to 1} (s-1)\frac{1}{s-1}\left(\frac{1}{j^{s-1}} - \frac{1}{(j+1)^{s-1}}\right) = 0.$$

Similarly, for the first term $s = 0$ is a removable singularity. That is, the $a_j(s)$ are entire functions. In particular each $a_j(s)$ is holomorphic on the domain $D^+ \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. At the same time, for $\sigma := \operatorname{Re} s > 0$ we have

$$|a_j(s)| \le \int_j^{j+1} \frac{dx}{x^{\sigma+1}} = \frac{1}{\sigma}\left(\frac{1}{j^\sigma} - \frac{1}{(j+1)^\sigma}\right)$$

(where the inequality comes from $|j-x| = x-j \le 1$ for $j \le x \le j+1$); moreover

$$\sum_{j=1}^{n}\left(\frac{1}{j^\sigma} - \frac{1}{(j+1)^\sigma}\right) = 1 - \frac{1}{(n+1)^\sigma} \Rightarrow \sum_{j=1}^{\infty}\left(\frac{1}{j^\sigma} - \frac{1}{(j+1)^\sigma}\right) = 1$$

(i.e. $1/(n+1)^\sigma \to 0$ as $n \to \infty$ for $\sigma > 0$). Hence, by the M-test, $\sum_{j=1}^\infty a_j(s)$ converges absolutely and uniformly on D^+ (and in particular on compact subsets of D^+). $f(s)$ is therefore holomorphic on D^+, by the Weierstrass theorem. Of course, in equation (1.19),

$$s\int_1^\infty \frac{([x]-x+\frac{1}{2})}{x^{s+1}} dx = sf(s) + \tfrac{1}{2}$$

is also a holomorphic function of s on D^+.

We have deduced:

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COROLLARY 1.23. *Let*

$$f(s) \overset{\text{def}}{=} \int_1^\infty \frac{([x]-\frac{1}{2})}{x^{s+1}}dx.$$

Then $f(s)$ *is well-defined for* $\operatorname{Re} s > 0$ *and is a holomorphic function on the domain* $D^+ \overset{\text{def}}{=} \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. *For* $\operatorname{Re} s > 1$ *one has* (*by Theorem* 1.18)

$$\zeta(s) = \frac{1}{s-1} + 1 + sf(s). \tag{1.24}$$

From this we see that $\zeta(s)$ *admits an analytic continuation to* D^+. *Its only singularity there is a simple pole at* $s = 1$ *with residue* $\lim_{s\to 1}(s-1)\zeta(s) = 1$, *as before*.

This result is obviously weaker than Theorem 1.13. However, as a further application we show that

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma \tag{1.25}$$

where

$$\gamma \overset{\text{def}}{=} \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) \tag{1.26}$$

is the *Euler–Mascheroni constant*; $\gamma \simeq 0.577215665$. By the continuity (in particular) of $f(s)$ at $s = 1$, $f(1) = \lim_{s\to 1} f(s)$. That is, by (1.24), we have

$$\begin{aligned}
\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) &= \lim_{s\to 1}(1 + sf(s)) \\
&= 1 + f(1) \overset{(1.22)}{=} 1 + \lim_{n\to\infty}\sum_{j=1}^{n-1}\int_j^{j+1}\frac{j-x}{x^2}dx \\
&= 1 + \lim_{n\to\infty}\sum_{j=1}^{n-1}\left(\frac{1}{j+1} - \big(\log(j+1) - \log j\big)\right) \\
&= 1 + \lim_{n\to\infty}\left(\sum_{j=1}^{n-1}\frac{1}{j+1} - \sum_{j=1}^{n-1}\big(\log(j+1) - \log j\big)\right) \\
&= 1 + \lim_{n\to\infty}\left(\sum_{j=1}^{n-1}\frac{1}{j+1} - \log n\right) \\
&= 1 + \lim_{n\to\infty}\left(-1 + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \log n\right) \\
&= \gamma,
\end{aligned}$$

as desired.

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Since $s = 1$ is a simple pole with residue 1, $\zeta(s)$ has a Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \sum_{k=1}^{\infty} \gamma_k (s-1)^k \tag{1.27}$$

on a deleted neighborhood of 1. By equation (1.25), $\gamma_0 = \gamma$. One can show that, in fact, for $k = 0, 1, 2, 3, \ldots$

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{n\to\infty} \left(\sum_{l=1}^{n} \frac{(\log l)^k}{l} - \frac{(\log n)^{k+1}}{k+1} \right), \tag{1.28}$$

a result we will not need (except for the case $k = 0$ already proved) and thus which we will not bother to prove.

The inversion formula (1.3), which was instrumental in the approach above to the analytic continuation and FE of $\zeta(s)$, provides for a function $F(t)$, $t > 0$, that is invariant under the transformation $t \to 1/t$. Namely, let $F(t) \stackrel{\text{def}}{=} t^{1/4}\theta(t)$. Then (1.3) is equivalent to statement that $F(1/t) = F(t)$, for $t > 0$.

Lecture 2. Special values of zeta

In 1736, L. Euler discovered the celebrated special values result

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!} \tag{2.1}$$

for $n = 1, 2, 3, \ldots$, where B_j is the j-th *Bernoulli number*, defined by

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^j,$$

for $|z| < 2\pi$, which is the Taylor expansion about $z = 0$ of the holomorphic function $h(z) \stackrel{\text{def}}{=} z/(e^z - 1)$, which is defined to be 1 at $z = 0$. Since $e^z - 1$ vanishes if and only if $z = 2\pi i n$, for $n \in \mathbb{Z}$, the restriction $|z| < 2\pi$ means that the denominator $e^z - 1$ vanishes only for $z = 0$. The B_j were computed by Euler up to $j = 30$. Here are the first few values:

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0

(2.2)

In general, $B_{\text{odd}>1} = 0$. To see this let $H(z) \stackrel{\text{def}}{=} h(z) + z/2$ for $|z| < 2\pi$, which we claim is an *even* function. Namely, for $z \neq 0$ the sum $z/(e^z - 1) + z/(e^{-z} - 1)$ equals $= -z$ by simplification:

$$H(-z) = \frac{-z}{e^{-z} - 1} - \frac{z}{2} = \frac{z}{e^z - 1} + z - \frac{z}{2} = H(z).$$

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Then

$$\begin{aligned}
&\frac{z}{2} + B_0 + B_1 z + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} z^{2j} + \sum_{j=1}^{\infty} \frac{B_{2j+1}}{(2j+1)!} z^{2j+1} \\
&\qquad = \frac{z}{2} + \sum_{j=0}^{\infty} \frac{B_j}{j!} z^j = H(z) = H(-z) \\
&\qquad = -\frac{z}{2} + B_0 + B_1(-z) + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} (-z)^{2j} + \sum_{j=1}^{\infty} \frac{B_{2j+1}}{(2j+1)!} (-z)^{2j+1},
\end{aligned}$$

which implies

$$0 = (1 + 2B_1) z + 2 \sum_{j=1}^{\infty} \frac{B_{2j+1}}{(2j+1)!} z^{2j+1},$$

and consequently $B_1 = -\frac{1}{2}$ and $B_{2j+1} = 0$ for $j \geq 1$, as claimed. By formula (2.1) (in particular)

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}, \tag{2.3}$$

the first formula, $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, being well-known apart from knowledge of the zeta function $\zeta(s)$. We provide a proof of (2.1) based on the summation formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth \pi a - \frac{1}{2a^2} \tag{2.4}$$

for $a > 0$; see Appendix E on page 92. Before doing so, however, we note some other special values of zeta.

As we have noted, $1/\Gamma(s)$ is an entire function of s. It has zeros at the points $s = 0, -1, -2, -3, -4, \ldots$. By Theorem 1.13 and the remarks that follow its statement we therefore see that for $n = 1, 2, 3, 4, \ldots,$

$$\zeta(-2n) = \frac{-\pi^{-n}}{2\Gamma(-n+1)} = 0, \quad \zeta(0) = \frac{-1}{2\Gamma(1)} = -\frac{1}{2}. \tag{2.5}$$

Thus, as mentioned in the Introduction, $\zeta(s)$ vanishes at the real points $s = -2, -4, -6, -8, \ldots$, called the *trivial* zeros of $\zeta(s)$. The value $\zeta(0)$ is nonzero — it equals $-\frac{1}{2}$ by (2.5). Later we shall check that

$$\zeta'(0) = \zeta(0) \log 2\pi = -\tfrac{1}{2} \log 2\pi. \tag{2.6}$$

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Turning to the proof of (2.1), we take $0 < t < 2\pi$ and choose $a = \dfrac{t}{2\pi}$ in (2.4), obtaining successively

$$\frac{\pi^2}{t}\coth\frac{t}{2} - \frac{2\pi^2}{t^2} = 4\pi^2 \sum_{n=1}^{\infty} \frac{1}{t^2+4\pi^2 n^2},$$

$$\frac{1}{2}\coth\frac{t}{2} - \frac{1}{t} = 2t \sum_{n=1}^{\infty} \frac{1}{t^2+4\pi^2 n^2},$$

$$\frac{1}{e^t-1} + \frac{1}{2} = \frac{2+e^t-1}{2(e^t-1)}\left(\frac{e^{-t/2}}{e^{-t/2}}\right) = \frac{e^{-t/2}+e^{t/2}}{2(e^{t/2}-e^{-t/2})}$$

$$= \frac{1}{2}\frac{\cosh(t/2)}{\sinh(t/2)} = \frac{1}{2}\coth\frac{t}{2} = \frac{1}{t} + 2t\sum_{n=1}^{\infty}\frac{1}{t^2+4\pi^2 n^2},$$

$$\frac{t}{e^t-1} + \frac{t}{2} = 1 + 2t^2 \sum_{n=1}^{\infty} \frac{1}{t^2+4\pi^2 n^2}. \tag{2.7}$$

Since $B_0 = 1$ and $B_1 = -\frac{1}{2}$ (see (2.2)), and since $B_{2k+1} = 0$ for $k \geq 1$, we can write

$$\frac{t}{e^t-1} \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k},$$

and (2.7) becomes

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k} = 2t^2 \sum_{n=1}^{\infty} \frac{1}{t^2+4\pi^2 n^2}. \tag{2.8}$$

For $0 < t < 2\pi$, we can use the convergent geometric series

$$\sum_{k=0}^{\infty}\left(\frac{-t^2}{4\pi^2 n^2}\right)^k = \frac{1}{1+\frac{t^2}{4\pi^2 n^2}} = \frac{4\pi^2 n^2}{t^2+4\pi^2 n^2}, \tag{2.9}$$

to rewrite (2.8) as

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k} = 2t^2 \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{1}{4\pi^2 n^2}\left(\frac{-t^2}{4\pi^2 n^2}\right)^k \tag{2.10}$$

$$= 2t^2 \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{4\pi^2 n^2}\left(\frac{-t^2}{4\pi^2 n^2}\right)^{k-1}.$$

The point is to commute the summations on n and k in this equation. Now

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{1}{4\pi^2 n^2}\left(-\frac{t^2}{4\pi^2 n^2}\right)^{k-1}\right| = \sum_{k=1}^{\infty}\frac{t^{2(k-1)}}{(4\pi^2)^k}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} \leq \sum_{n=1}^{\infty}\frac{t^{2(k-1)}}{(4\pi^2)^k}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

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which is finite since $\sum_{n=1}^{\infty} 1/n^2 = \zeta(2) < \infty$ and $\sum_{n=1}^{\infty} t^{2(k-1)}/(4\pi^2)^k < \infty$, by the ratio test (again for $0 < t < 2\pi$). Commutation of the summation is therefore justified:

$$\sum_{k=1}^{\infty} \frac{B_{2k} t^{2k}}{(2k)!} = 2t^2 \sum_{k=1}^{\infty} \frac{(-t^2)^{k-1}}{4\pi^2 (4\pi^2)^{k-1}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{(4\pi^2)^k} \zeta(2k) t^{2k}$$

on $(0, 2\pi)$. By equating coefficients, we obtain

$$\frac{B_{2k}}{(2k)!} = \frac{2(-1)^{k-1}\zeta(2k)}{(4\pi^2)^k} \quad \text{for } k \geq 1,$$

which proves Euler's formula (2.1).

Next we turn to a proof of equation (2.6). We start with an easy consequence of the quotient and product rules for differentiation.

LEMMA 2.11 (LOGARITHMIC DIFFERENTIATION WITHOUT LOGS). *If*

$$F(s) = \frac{\phi_1(s)\phi_2(s)\phi_3(s)}{\phi_4(s)},$$

on some neighborhood of $s_0 \in \mathbb{C}$, *where the* $\phi_j(s)$ *are nonvanishing holomorphic functions there*, *then*

$$\frac{F'(s_0)}{F(s_0)} = \frac{\phi_1'(s_0)}{\phi_1(s_0)} + \frac{\phi_2'(s_0)}{\phi_2(s_0)} + \frac{\phi_3'(s_0)}{\phi_3(s_0)} - \frac{\phi_4'(s_0)}{\phi_4(s_0)}.$$

Now choose $\phi_1(s) \stackrel{\text{def}}{=} \pi^{\frac{1}{2}-s}$, $\phi_2(s) \stackrel{\text{def}}{=} \Gamma\left(\frac{s}{2}\right)$, $\phi_4(s) = 2\Gamma\left(\frac{3-s}{2}\right)$, say on a small neighborhood of $s = 1$. For the choice of $\phi_3(s)$, we write $\zeta(s) = g(s)/(s-1)$ on a neighborhood N of $s = 1$, for $s \neq 1$, where $g(s)$ is holomorphic on N and $g(1) = 1$. This can be done since $s = 1$ is a simple pole of $\zeta(s)$ with residue $= 1$; for example, see equation (1.27). Assume $0 \notin N$ and take $\phi_3(s) \stackrel{\text{def}}{=} g(s)$ on N. By equation (1.17), $-\zeta(1-s) = \phi_1(s)\phi_2(s)\phi_3(s)/\phi_4(s)$ near $s = 1$, so that by Lemma 2.11 and introducing the function $\psi(s) \stackrel{\text{def}}{=} \Gamma'(s)/\Gamma(s)$, we obtain

$$\left.\frac{\zeta'(1-s)}{-\zeta(1-s)}\right|_{s=1}$$
$$= \left.\frac{\pi^{\frac{1}{2}-s}(-\log\pi)}{\pi^{\frac{1}{2}-s}}\right|_{s=1} + \left.\psi\left(\frac{s}{2}\right)\frac{1}{2}\right|_{s=1} + \left.\frac{g'(s)}{g(s)}\right|_{s=1} - \left.\psi\left(\frac{3-s}{2}\right)\left(-\frac{1}{2}\right)\right|_{s=1}. \quad (2.12)$$

If γ is the Euler–Mascheroni constant of (1.26), the facts $-\psi(1) = \gamma$ and $\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2$ are known to prevail, which reduces equation (2.12) to

$$\zeta'(0) = \zeta(0)\left(\log\pi + \frac{\gamma}{2} + \log 2 - g'(1) + \frac{\gamma}{2}\right)$$
$$= -\tfrac{1}{2}\left(\log\pi + \gamma + \log 2 - g'(1)\right),$$

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since $g(1) = 1$ and $\zeta(0) = -\frac{1}{2}$; see (2.5). But $g'(1) = \gamma$, as we will see in a minute; hence we have reached the conclusion that $\zeta'(0) = -\frac{1}{2}\log 2\pi$, which is (2.6). There remains to check that $g'(1) = \gamma$. We have

$$g'(1) \stackrel{\text{def}}{=} \lim_{s\to 1} \frac{g(s) - 1}{s - 1},$$

again since $g(1) = 1$; this in turn equals $\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma$, by equation (1.25).

To obtain further special values of zeta we appeal to the special values formula

$$\Gamma\left(\tfrac{1}{2} - n\right) = \frac{(-1)^n \sqrt{\pi}\, 2^{2n} n!}{(2n)!} \tag{2.13}$$

for the gamma function, where $n = 1, 2, 3, 4, \ldots$. This we couple with (2.1) and the functional equation (1.16) to show that

$$\zeta(-1) = -\frac{1}{12} \quad \text{and} \quad \zeta(1 - 2n) = -\frac{B_{2n}}{2n} \quad \text{for } n = 1, 2, 3, 4, \ldots. \tag{2.14}$$

Namely, $\zeta(1-2n) = \pi^{-2n+\frac{1}{2}}\Gamma(n)\zeta(2n)/\Gamma\left(\frac{1}{2}-n\right)$, by (1.16); this in turn equals

$$\frac{\pi^{-2n+\frac{1}{2}}(n-1)!\,\zeta(2n)(2n)!}{(-1)^n\sqrt{\pi}\,2^{2n}n!},$$

by (2.13); whence (2.1) gives

$$\zeta(1 - 2n) = \frac{(-1)^n(2\pi)^{-2n}\zeta(2n)(2n)!}{n} = -\frac{B_{2n}}{2n}.$$

Taking $n = 1$ gives $\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}$, by (2.2), which confirms (2.14).

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Lecture 3. An Euler product expansion

For a function $f(n)$ defined on the set $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ of positive integers one has a corresponding zeta function or *Dirichlet series*

$$\phi_f(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

defined generically for $\operatorname{Re} s$ sufficiently large. If $f(n) = 1$ for all $n \in \mathbb{Z}^+$, for example, then for $\operatorname{Re} s > 1$, $\phi_f(s)$ is of course just the Riemann zeta function $\zeta(s)$, which according to equation (0.2) of the Introduction has an Euler product expansion $\zeta(s) = \prod_{p\in P} \frac{1}{1-p^{-s}}$ over the primes P in $\mathbb{Z}^+$. It is natural to inquire whether, more generally, there are conditions that permit an analogous Euler product expansion of a given Dirichlet series $\phi_f(s)$. Very pleasantly, there is an affirmative result when, for example, the $f(n)$ are *Fourier coefficients* (see Theorem 4.32, where the n-th Fourier coefficient there is denoted by a_n) of certain types of *modular forms*, due to a beautiful theory of E. Hecke. Also see equations (3.20), (3.21) below. Rather than delving directly into that theory at this point we shall instead set up an *abstract* condition for a product expansion. The goal is to show that under suitable conditions on $f(n)$ of course the desired expansion assumes the form

$$\phi_f(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{f(1)}{\prod_{p\in P}\left(1+\alpha(p)p^{-2s}-f(p)p^{-s}\right)} \tag{3.1}$$

for some function $\alpha(p)$ on P; see Theorem 3.17 below. Here we would want to have, in particular, that $f(1) \neq 0$. Before proceeding toward a precise statement and proof of equation (3.1), we note that (again) if $f(n) = 1$ for all $n \in \mathbb{Z}^+$, for example, then for the choice $\alpha(p) = 0$ for all $p \in P$, equation (3.1) reduces to the classical Euler product expansion of equation (0.2).

Given $f : \mathbb{Z}^+ \to \mathbb{R}$ or $\mathbb{C}$, and $\alpha : P \to \mathbb{R}$ or $\mathbb{C}$, we assume the following abstract *multiplicative* condition:

$$f(n)f(p) = \begin{cases} f(np) & \text{if } p \nmid n, \\ f(np) + \alpha(p) f\left(\frac{n}{p}\right) & \text{if } p \mid n, \end{cases} \tag{3.2}$$

for $(n, p) \in \mathbb{Z}^+ \times P$; here $p \mid n$ means that p divides n and $p \nmid n$ means the opposite. Given condition (3.2) we observe first that if $f(1) = 0$ then f vanishes identically, the proof being as follows. For a prime $p \in P$, (3.2) requires that $f(1)f(p) = f(p)$, since $p \nmid 1$; that is, $f(p) = 0$. If $n \in \mathbb{Z}^+$ with $n \geq 2$, there exists $p \in P$ such that $p \mid n$, say $ap = n$, $a \in \mathbb{Z}^+$. Proceed inductively. If $p \nmid a$, $f(a)f(p) = f(ap) = f(n)$, by (3.2), so $f(n) = 0$, as $f(p) = 0$. If $p \mid a$, we have $0 = f(a)f(p)$ (again as $f(p) = 0$), and this equals $f(ap) + \alpha(p)f(a/p) = f(n) + \alpha(p)f(a/p)$, where $1 < p \leq a$ (so $1 \leq a/p < a = n/p < n$). Thus $f(a/p) = 0$, by induction, so $f(n) = 0$, which completes the induction. Thus we see that if $f \not\equiv 0$ then $f(1) \neq 0$.

As in Appendix D (page 88) we set, $m, n \in \mathbb{Z}^+$,

$$d(m,n) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } m \mid n, \\ 0 & \text{if } m \nmid n. \end{cases}$$

Fix a finite set of distinct primes $S = \{p_1, p_2, \ldots, p_l\} \subset P$ and define $g(n) = g_S(n)$ on $\mathbb{Z}^+$ by

$$g(n) = f(n) \prod_{j=1}^{l} \left(1 - d(p_j, n)\right). \tag{3.3}$$

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Fix $p \in P - \{p_1, p_2, \ldots, p_l\}$. Then the next observation is that, for $n \in \mathbb{Z}^+$,

$$g(n)f(p) = \begin{cases} g(np) & \text{if } p \nmid n, \\ g(np) + \alpha(p)g\left(\frac{n}{p}\right) & \text{if } p \mid n \end{cases}, \tag{3.4}$$

which compares with equation (3.2).

PROOF. If $p_j \mid n$ then of course $p_j \mid pn$. If $p_j \nmid n$ then $p_j \nmid pn$; for if $p_j | pn$ then $p \mid n$ since p_j, p are relatively prime, given that $p \neq$ each p_j. Thus

$$d(p_j, n) = d(p_j, pn) \quad \text{for } n \in \mathbb{Z}^+,\ 1 \leq j \leq l. \tag{3.5}$$

Similarly suppose $p \mid n$, say $bp = n$, with $b \in \mathbb{Z}^+$. If $p_j \nmid n/p$ then $p_j \nmid n$; for otherwise $p_j \mid n = bp$ again with p_j, p relatively prime, implying that $p_j | b = n/p$. Thus we similarly have

$$d(p_j, n/p) = d(p_j, n) \quad \text{for } n \in \mathbb{Z}^+,\ 1 \leq j \leq l, \text{ such that } p \mid n. \tag{3.6}$$

Now if $n \in \mathbb{Z}^+$ is such that $p \nmid n$, then

$$g(n)f(p) \overset{(3.3)}{=} f(n)f(p)\prod_{j=1}^{l}\left(1 - d(p_j, n)\right) \underset{(3.5)}{\overset{(3.2)}{=}} f(np)\prod_{j=1}^{l}\left(1 - d(p_j, pn)\right) \overset{(3.3)}{=} g(np).$$

On the other hand, if $p \mid n$, then

$$\begin{aligned} g(n)f(p) &\overset{(3.3)}{=} f(n)f(p)\prod_{j=1}^{l}\left(1 - d(p_j, n)\right) \\ &\overset{(3.2)}{=} \left(f(np) + \alpha(p)f\left(\tfrac{n}{p}\right)\right)\prod_{j=1}^{l}\left(1 - d(p_j, n)\right) \\ &\underset{(3.6)}{\overset{(3.5)}{=}} f(np)\prod_{j=1}^{l}\left(1 - d(p_j, pn)\right) + \alpha(p)f\left(\tfrac{n}{p}\right)\prod_{j=1}^{l}\left(1 - d\left(p_j, \tfrac{n}{p}\right)\right) \\ &\overset{(3.3)}{=} g(np) + \alpha(p)g\left(\tfrac{n}{p}\right), \end{aligned}$$

which proves (3.4). □

Let $\phi_h(s) = \sum_{n=1}^{\infty} h(n)/n^s$ be a Dirichlet series that converges absolutely, say at some fixed point $s_0 \in \mathbb{C}$. Fix $p \in P$ and some complex number $\lambda(p)$ corresponding to p such that

$$h(n)\lambda(p) = \begin{cases} h(np) & \text{if } p \nmid n, \\ h(np) + \alpha(p)h\left(\frac{n}{p}\right) & \text{if } p \mid n, \end{cases} \tag{3.7}$$

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for $n \in \mathbb{Z}^+$. Then

$$\phi_h(s_0)\big(1+\alpha(p)p^{-2s_0}-\lambda(p)p^{-s_0}\big)=\sum_{n=1}^{\infty}\frac{\big(1-d(p,n)\big)h(n)}{n^{s_0}}. \tag{3.8}$$

PROOF. Define

$$a_n \overset{\text{def}}{=} \begin{cases} \dfrac{\alpha(p)h\big(\frac{n}{p}\big)}{(pn)^{s_0}} & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n, \end{cases}$$

for $n \in \mathbb{Z}^+$. Since $p \mid pn$, we have

$$a_{pn}=\frac{\alpha(p)h\big(\frac{pn}{p}\big)}{(ppn)^{s_0}}=\alpha(p)p^{-2s_0}\frac{h(n)}{n^{s_0}},$$

which shows that $\sum_{n=1}^{\infty} a_{pn}$ converges. The Scholium of Appendix D (page 91) then implies that the series $\sum_{n=1}^{\infty} d(p,n)a_n$ converges, and one has

$$\sum_{n=1}^{\infty} a_{pn}=\sum_{n=1}^{\infty} d(p,n)a_n, \tag{3.9}$$

where the left-hand side here is $\alpha(p)p^{-2s_0}\phi_h(s_0)$. Since both $d(p,n), a_n=0$ if $p \nmid n$, $d(p,n)a_n=a_n$, which is also clear if $p \mid n$. On the other hand, if $p \mid n$,

$$a_n \overset{\text{def}}{=} \alpha(p)h\big(\tfrac{n}{p}\big)(pn)^{-s_0}=\big(h(n)\lambda(p)-h(np)\big)(pn)^{-s_0} \tag{3.10}$$

by equation (3.7). Equation (3.10) also holds by (3.7) in case $p \nmid n$, for then both sides are zero. That is, (3.10) holds for all $n \geq 1$ and equation (3.9) reduces to the statement

$$\alpha(p)p^{-2s_0}\phi_h(s_0)=\sum_{n=1}^{\infty}[h(n)\lambda(p)-h(np)](pn)^{-s_0}. \tag{3.11}$$

We apply the Scholium a second time, where this time we define $a_n \overset{\text{def}}{=} h(n)/n^{s_0}$; since $|d(p,n)a_n| \leq |a_n|$, the sum $\sum_{n=1}^{\infty} d(p,n)a_n$ converges. By the Scholium, $\sum_{n=1}^{\infty} a_{pn}$ converges and $\sum_{n=1}^{\infty} a_{pn}=\sum_{n=1}^{\infty} d(p,n)a_n$; that is,

$$\sum_{n=1}^{\infty} h(pn)(pn)^{-s_0}=\sum_{n=1}^{\infty} d(p,n)h(n)n^{-s_0},$$

which one plugs into (3.11), to obtain $\alpha(p)p^{-2s_0}\phi_h(s_0)=\lambda(p)p^{-s_0}\phi_h(s_0)-\sum_{n=1}^{\infty} d(p,n)h(n)n^{-s_0}$. This proves equation (3.8).

The proof of the main result does involve various moving parts, and it is a bit lengthy as we have chosen to supply full details. We see, however, that the proof is elementary. One further basic ingredient is needed. Again let $\{p_1,\dots,p_l\}$ by a fixed, finite set of distinct primes in P. With f,α subject to the multiplicative

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condition (3.2), we assume that $\phi_f(s_0)$ converges absolutely where $s_0 \in \mathbb{C}$ is some fixed number. For $n \geq 1$, we have $0 \leq \prod_{j=1}^{l}\big(1-d(p_j,n)\big) \leq 1$; therefore the series $\sum_{n=1}^{\infty}\big(\prod_{j=1}^{l}(1-d(p_j,n))\big)f(n)/n^{s_0}$ converges absolutely. We now show by induction on l that

$$\prod_{j=1}^{l}\big(1+\alpha(p_j)p_j^{-2s_0}-f(p_j)p_j^{-s_0}\big)\phi_f(s_0) = \sum_{n=1}^{\infty}\Big(\prod_{j=1}^{l}\big(1-d(p_j,n)\big)\Big)\frac{f(n)}{n^{s_0}}. \tag{3.12}$$

For $l=1$, the claim follows by (3.8) with $p=p_1$, $h(n)=f(n)$, $\lambda(p)=f(p)$. Proceeding inductively, we consider a set $\{p_1,p_2,\ldots,p_l,p_{l+1}\}$ of $l+1$ distinct primes in P. Then

$$\begin{aligned}
&\prod_{j=1}^{l+1}\big(1+\alpha(p_j)p_j^{-2s_0}-f(p_j)p_j^{-s_0}\big)\phi_f(s_0)\\
&=\big(1+\alpha(p_{l+1})p_{l+1}^{-2s_0}-f(p_{l+1})p_{l+1}^{-s_0}\big)\prod_{j=1}^{l}\big(1+\alpha(p_j)p_j^{-2s_0}-f(p_j)p_j^{-s_0}\big)\phi_f(s_0)\\
&=\big(1+\alpha(p_{l+1})p_{l+1}^{-2s_0}-f(p_{l+1})p_{l+1}^{-s_0}\big)\sum_{n=1}^{\infty}\Big(\prod_{j=1}^{l}\big(1-d(p_j,n)\big)\Big)\frac{f(n)}{n^{s_0}}\\
&=\big(1+\alpha(p_{l+1})p_{l+1}^{-2s_0}-f(p_{l+1})p_{l+1}^{-s_0}\big)\sum_{n=1}^{\infty}\frac{g(n)}{n^{s_0}},
\end{aligned} \tag{3.13}$$

where the second equality follows by induction and the last one by definition (3.3). We noted, just above (3.12), that $\sum_{n=1}^{\infty}\big(\prod_{j=1}^{l}(1-d(p_j,n))\big)f(n)/n^{s_0}$ converges absolutely; that is, $\sum_{n=1}^{\infty}g(n)/n^{s_0}$ converges absolutely. Thus we choose $h(n)=g(n)$, $p=p_{l+1}$, $\lambda(p)=f(p)$. Condition (3.7) is then a consequence of equation (3.4), since $p_{l+1}\in P-\{p_1,p_2,\ldots,p_l\}$, and one is therefore able to apply formula (3.8) again:

$$\begin{aligned}
\big(1+\alpha(p_{l+1})p_{l+1}^{-2s_0}-f(p_{l+1})p_{l+1}^{-s_0}\big)\phi_g(s_0)
&=\sum_{n=1}^{\infty}\frac{\big(1-d(p_{l+1},n)\big)g(n)}{n^{s_0}}\\
&\overset{(3.3)}{=}\sum_{n=1}^{\infty}\frac{\big(1-d(p_{l+1},n)\big)}{n^{s_0}}\prod_{j=1}^{l}\big(1-d(p_j,n)\big)f(n)\\
&=\sum_{n=1}^{\infty}\prod_{j=1}^{l+1}\big(1-d(p_j,n)\big)\frac{f(n)}{n^{s_0}}
\end{aligned} \tag{3.14}$$

which, together with equation (3.13), allows one to complete the induction, and thus the proof of the claim (3.12). □

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Now let p_l be the l-th positive prime, so $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7, \ldots$. We can write the right-hand side of (3.12) as

$$f(1) + \sum_{n=2}^{\infty}\Big(\prod_{j=1}^{l}\big(1 - d(p_j, n)\big)\Big)\frac{f(n)}{n^{s_0}},$$

since no p_j divides 1. We show that if $2 \le n \le l$ then

$$\prod_{j=1}^{l}\big(1 - d(p_j, n)\big) = 0. \tag{3.15}$$

Namely, for $n \ge 2$ choose $q \in P$ such that $q \mid n$. If no p_j divides n, $1 \le j \le l$, then $q \ne p_1, \ldots, p_l$ (since $q \mid n$); hence $q \ge p_{l+1}$ (since p_l is the l-th prime) and so $q \ge l+1$. But this is impossible since $n \le l$ (by hypothesis) and $q \le n$ (since $q \mid n$). This contradiction proves that some p_j divides n, that is, $1 = d(p_j, n)$, which gives (3.15).

It follows that

$$\sum_{n=1}^{\infty}\Big(\prod_{j=1}^{l}\big(1 - d(p_j, n)\big)\Big)\frac{f(n)}{n^{s_0}} - f(1) = \sum_{n=l+1}^{\infty}\Big(\prod_{j=1}^{l}\big(1 - d(p_j, n)\big)\Big)\frac{f(n)}{n^{s_0}},$$

where (using again that $0 \le \prod_{j=1}^{l}\big(1 - d(p_j, n)\big) \le 1$) we have

$$\left|\sum_{n=l+1}^{\infty}\Big(\prod_{j=1}^{l}\big(1 - d(p_j, n)\big)\Big)\frac{f(n)}{n^{s_0}}\right| \le \sum_{n=l+1}^{\infty}\left|\frac{f(n)}{n^{s_0}}\right| = \sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{s_0}}\right| - \sum_{n=1}^{l}\left|\frac{f(n)}{n^{s_0}}\right|.$$

But this difference tends to 0 as $l \to \infty$. That is, by equation (3.12), the limit

$$\prod_{p \in P}\big(1 + \alpha(p)p^{-2s_0} - f(p)p^{-s_0}\big)\phi_f(s_0) \stackrel{\text{def}}{=} \lim_{l \to \infty}\prod_{j=1}^{l}\big(1 + \alpha(p_j)p_j^{-2s_0} - f(p_j)p^{-s_0}\big)\phi_f(s_0) \tag{3.16}$$

exists, where p_j = the j-th positive prime, and it equals $f(1)$.

We have therefore finally reached the main theorem.

THEOREM 3.17. (As before, $\mathbb{Z}^+ \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$ and P denotes the set of positive primes.) *Let $f : \mathbb{Z} \to \mathbb{R}$ or $\mathbb{C}$ and $\alpha : P \to \mathbb{R}$ or $\mathbb{C}$ be functions where f is not identically zero and where f is subject to the multiplicative condition* (3.2). *Then $f(1) \ne 0$. Let $D \subset \mathbb{C}$ be some subset on which the corresponding Dirichlet series $\phi_f(s) = \sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely. Then on D,*

$$\prod_{p \in P}\big(1 + \alpha(p)p^{-2s} - f(p)p^{-s}\big)\phi_f(s) = f(1)$$

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(see equation (3.16)). *In particular both* $\prod_{p\in P}\left(1+\alpha(p)p^{-2s}-f(p)p^{-s}\right)$ *and* $\phi_f(s)$ *are nonzero on* D *and*

$$\phi_f(s)=\frac{f(1)}{\prod_{p\in P}\left(1+\alpha(p)p^{-2s}-f(p)p^{-s}\right)} \tag{3.18}$$

on D (which is equation (3.1)).

We remark (again) that the proof of Theorem 3.17 is entirely elementary, if a bit long-winded; it only requires a few basic facts about primes, and a weak version of the fundamental theorem of arithmetic — namely that an integer $n \geq 2$ is divisible by a prime.

As a simple example of Theorem 3.17, suppose $f : \mathbb{Z}^+ \to \mathbb{R}$ or $\mathbb{C}$ is not identically zero, and is *completely multiplicative*: $f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{Z}^+$. Assume also that $s \in \mathbb{C}$ is such that $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely. Then

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\frac{1}{\prod\limits_{p\in P}\left(1-\frac{f(p)}{p^s}\right)}. \tag{3.19}$$

To see this, first we note *directly* that $f(1) \neq 0$. In fact since $f \not\equiv 0$, choose $n \in \mathbb{Z}^+$ such that $f(n) \neq 0$. Then $f(n) = f(n\cdot 1) = f(n)f(1)$, so $f(1) = 1$. Also f satisfies condition (3.2) for the choice $\alpha = 0$ (that is, $\alpha(p) = 0$ for all $p \in P$). Equation (3.19) therefore follows by (3.18). In Lecture 5, we apply (3.19) to *Dirichlet L-functions*.

Before concluding this lecture, we feel some obligation to explain the pivotal, abstract multiplicative condition (3.2). This will involve, however, some facts regarding modular forms that will be discussed in the next lecture, Lecture 4. Thus suppose that $f(z)$ is a holomorphic modular form of weight $k = 4, 6, 8, 10, \ldots,$ with Fourier expansion

$$f(z)=\sum_{n=0}^{\infty}a_n e^{2\pi i n z} \tag{3.20}$$

on the upper half-plane π^+. Then there is naturally attached to $f(z)$ a Dirichlet series

$$\phi_f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}, \tag{3.21}$$

called a *Hecke L-function*, which is known to converge absolutely for $\operatorname{Re} s > k$, and which is holomorphic on this domain. Actually, if $f(z)$ is a *cusp form* (i.e., $a_0 = 0$) then $\phi_f(s)$ is holomorphic on the domain $\operatorname{Re} s > 1 + \frac{k}{2}$. As in the case of the Riemann zeta function, Hecke theory provides for the meromorphic

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continuation of $\phi_f(s)$ to the full complex plane, and for an appropriate functional equation for $\phi_f(s)$.

For each positive integer $n = 1, 2, 3, \ldots$, there is an operator $T(n)$ (called a *Hecke operator*) on the space of modular forms of weight k given by

$$(T(n)f)(z) = n^{k-1} \sum_{\substack{d>0 \\ d \mid n}} \sum_{a \in \mathbb{Z}/d\mathbb{Z}} f\left(\frac{nz+da}{d^2}\right) d^{-k}. \tag{3.22}$$

Here the inner sum is over a complete set of representatives a in $\mathbb{Z}$ for the cosets $\mathbb{Z}/d\mathbb{Z}$. We shall be interested in the case when $f(z)$ is an eigenfunction of all Hecke operators: $T(n)f = \lambda(n)f$ for all $n \geq 1$, where $f \neq 0$ and $\lambda(n) \in \mathbb{C}$. We assume also that $a_1 = 1$, in which case $f(z)$ is called a *normalized simultaneous eigenform*. For such an eigenform it is known from the theory of Hecke operators that the Fourier coefficients and eigenvalues coincide for $n \geq 1 : a_n = \lambda(n)$ for $n \geq 1$. Moreover the Fourier coefficients satisfy the "multiplicative" condition

$$a_{n_1} a_{n_2} = \sum_{\substack{d>0 \\ d \mid n_1, d \mid n_2}} d^{k-1} a_{n_1 n_2/d^2} \tag{3.23}$$

for $n_1, n_2 \geq 1$. In particular for a prime $p \in P$ and an integer $n \geq 1$, condition (3.23) clearly reduces to the simpler condition

$$a_n a_p = \begin{cases} a_{np} & \text{if } p \nmid n, \\ a_{np} + p^{k-1} a_{np/p^2} & \text{if } p \mid n, \end{cases} \tag{3.24}$$

which is the origin of condition (3.2), where we see that in the present context we have $f(n) = a_n$ and $\alpha(p) = p^{k-1}$; $f(n)$ here is the function $f : \mathbb{Z} \to \mathbb{C}$ of condition (3.2), of course, and is not the eigenform $f(z)$. By Theorem 3.17, therefore, the following strong result is obtained.

THEOREM 3.25 (EULER PRODUCT FOR HECKE L-FUNCTIONS). *Let $f(z)$ be a normalized simultaneous eigenform of weight k (as desribed above), and let $\phi_f(s)$ be its corresponding Hecke L-function given by definition* (3.21) *for* $\operatorname{Re} s > k$. *Then, for* $\operatorname{Re} s > k$,

$$\phi_f(s) = \frac{1}{\prod_{p \in P}\left(1 + p^{k-1-2s} - a_p p^{-s}\right)}, \tag{3.26}$$

where a_p is the p-th Fourier coefficient $f(z)$; see equation (3.20). *If, moreover, $f(z)$ is a cusp form* (*i.e., the* 0*-th Fourier coefficient a_0 of $f(z)$ vanishes*), *then for* $\operatorname{Re} s > 1 + \frac{k}{2}$, *$\phi_f(s)$ converges* (*in fact absolutely*) *and formula* (3.26) *holds.*

The following example is important, though no proofs (which are quite involved)

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are supplied. If

$$\eta(z) \overset{\text{def}}{=} e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) \tag{3.27}$$

is the *Dedekind eta function* on π^+, then the *Ramanujan tau function* $\tau(n)$ on $\mathbb{Z}^+$ is defined by the Fourier expansion

$$\eta(z)^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}, \tag{3.28}$$

which is an example of equation (3.20), where in fact $\eta(z)^{24}$ is a normalized simultaneous eigenform of weight $k = 12$. It turns out, remarkably, that every $\tau(n)$ is real and is in fact an integer. For example, $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252$, $\tau(4) = -1472$, $\tau(5) = 4830$. Note also that since the sum in (3.27) starts at $n = 1$, $\eta(z)^{24}$ is a cusp form. By Theorem 3.25 we get:

COROLLARY 3.29. *For* $\operatorname{Re} s > 1 + \frac{k}{2} = 7$,

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \frac{1}{\prod_{p \in P} \left(1 + p^{11-2s} - \tau(p) p^{-s}\right)}. \tag{3.30}$$

The Euler product formula (3.30) was actually proved first by L. Mordell (in 1917, before E. Hecke) although it was claimed earlier to be true by S. Ramanujan.

Since we have introduced the Dedekind eta function $\eta(z)$ in (3.27), we check, as a final point, that it is indeed holomorphic on π^+. For

$$a_n(z) \overset{\text{def}}{=} -e^{2\pi i n z} \quad \text{and} \quad G(z) \overset{\text{def}}{=} \prod_{n=1}^{\infty} (1 + a_n(z)),$$

write $\eta(z) = e^{\pi i z/12} G(z)$. The product $G(z)$ converges absolutely on π^+ since $\sum_{n=1}^{\infty} |a_n(z)| = \sum_{n=1}^{\infty} e^{-2\pi n y}$ (for $z = x + iy$, $x, y \in \mathbb{R}$, $y > 0$) is a convergent geometric series as $e^{-2\pi y} < 1$. We note also that $a_n(z) \neq -1$ since (again) $|a_n(z)| = e^{-2\pi n y} < 1$ for $n \geq 1$. If $K \subset \pi^+$ is any compact subset, then the continuous function $\operatorname{Im} z$ on K has a positive lower bound B: $\operatorname{Im} z \geq B > 0$ for every $z \in K$. Hence

$$|a_n(z)| = e^{-2\pi n \operatorname{Im} z} \leq e^{-2\pi n B} \quad \text{on } K,$$

where $\sum_{n=1}^{\infty} e^{-2\pi n B}$ is a convergent geometric series as $e^{-2\pi B} < 1$ for $B > 0$. Therefore the series $\sum_{n=1}^{\infty} a_n(z)$ converges uniformly on compact subsets of π^+ (by the M-test), which means that the product $G(z)$ converges uniformly on compact subsets of π^+. That is, $G(z)$ is holomorphic on π^+ (as the $a_n(z)$ are holomorphic on π^+), and therefore $\eta(z)$ is holomorphic on π^+.

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Lecture 4. Modular forms: the movie

In the previous lecture we proved an Euler product formula for Hecke L-functions, in Theorem 3.25 which followed as a concrete application of Theorem 3.17. That involved, in part, some notions/results deferred to the present lecture for further discussion. Here the attempt is to provide a brief, kaleidoscopic tour of the modular universe, whose space is Lobatchevsky–Poincáre hyperbolic space, the upper half-plane π^+, and whose galaxies of stars are modular forms. As no universe would be complete without zeta functions, Hecke L-functions play that role. In particular we gain, in transit, an enhanced appreciation of Theorem 3.25.

There are many fine texts and expositions on modular forms. These obviously venture much further than our modest attempt here which is designed to serve more or less as a limited introduction and reader's guide. We recommend, for example, the books of Audrey Terras [35], portions of chapter three, (also note her lectures in this volume) and Tom Apostol [2], as supplements.

We begin the story by considering a holomorphic function $f(z)$ on π^+ that satisfies the periodicity condition $f(z+1) = f(z)$. By the remarks following Theorem B.7 of the Appendix (page 84), $f(z)$ admits a Fourier expansion (or q-expansion)

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q(z)^n = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z} \tag{4.1}$$

on π^+, where $q(z) \stackrel{\text{def}}{=} e^{2\pi i z}$, and where the a_n are given by formula (B.6). We say that $f(z)$ is *holomorphic at infinity* if $a_n = 0$ for every $n \le -1$:

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \tag{4.2}$$

on π^+. Let $G = \mathrm{SL}(2,\mathbb{R})$ denote the group of 2×2 real matrices $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with determinant $= 1$, and let $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \subset G$ denote the subgroup of elements $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $a, b, c, d \in \mathbb{Z}$. The standard linear fractional action of G on π^+, given by

$$g\cdot z \stackrel{\text{def}}{=} \frac{az+b}{cz+d} \in \pi^+ \quad \text{for } (g,z) \in G\times\pi^+ \tag{4.3}$$

restricts to any subgroup of G, and in particular it restricts to Γ.

A (holomorphic) *modular form of weight* $k \in \mathbb{Z}$, $k \ge 0$, *with respect to* Γ, is a holomorphic function $f(z)$ on π^+ that satisfies the following two conditions:

(M1) $f(\gamma\cdot z) = (cz+d)^k f(z)$ for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$, $z \in \pi^+$.

(M2) $f(z)$ is holomorphic at infinity.

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Here we note that for the case of $\gamma = T \stackrel{\text{def}}{=} \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma$, we have $\gamma \cdot z = z + 1$ by (4.3). Then $f(z+1) = f(z)$ by (M1), which means that condition (M2) is well-defined, and therefore $f(z)$ satisfies equation (4.2), which justifies the statement of equation (3.20) of Lecture 3. One can also consider *weak* modular forms, where the assumption that $a_n = 0$ for every $n \leq -1$ is relaxed to allow *finitely many* negative Fourier coefficients to be nonzero. One can consider, moreover, modular forms with respect to various subgroups of Γ. There are two other quick notes to make. First, if $\gamma = -1 \stackrel{\text{def}}{=} \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \Gamma$, then by (4.3) and (M1) we must have $f(z) = (-1)^k f(z)$ which means that $f(z) \equiv 0$ if k is odd. For this reason we always assume that k is *even*. Secondly, condition (M1) is equivalent to the following two conditions:

(M1)$'$ $f(z+1) = f(z)$, and

(M1)$''$ $f(-1/z) = z^k f(z)$ for $z \in \pi^+$.

For we have already noted that (M1) $\Rightarrow$ (M1)$'$, by the choice $\gamma = T$. Also choose $\gamma = S \stackrel{\text{def}}{=} \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \Gamma$. Then by (4.3) and (M1), condition (M1)$''$ follows. Conversely, the conditions (M1)$'$ and (M1)$''$ together, for $k \geq 0$ even, imply condition (M1) since the two elements $T, S \in \Gamma$ generate Γ; a proof of this is provided in Appendix F (page 96).

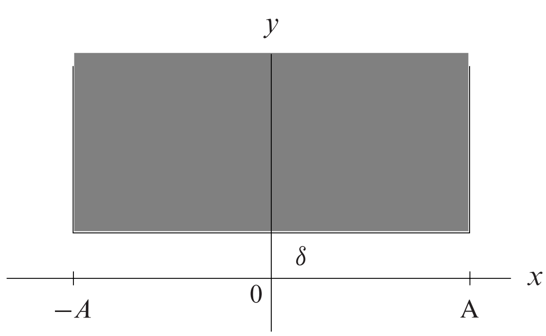
Basic examples of modular forms are provided by the *holomorphic Eisenstein series* $G_k(z)$, which serve in fact as building blocks for other modular forms:

$$G_k(z) \stackrel{\text{def}}{=} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(m+nz)^k} \tag{4.4}$$

for $z \in \pi^+$, $k = 4, 6, 8, 10, 12, \ldots$. The issue of absolute or uniform convergence of these series rests mainly on the next observation, whose proof goes back to Chris Henley [2]. Given $A, \delta > 0$ let

$$S_{A,\delta} \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 \mid |x| \leq A, y \geq \delta\} \tag{4.5}$$

be the region $\subset \pi^+$, as illustrated:



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LEMMA 4.6. *There is a constant* $K = K(A, \delta) > 0$, *depending only on* A *and* δ, *such that for any* $(x, y) \in S_{A,\delta}$ *and* $(a, b) \in \mathbb{R}^2$ *with* $b \neq 0$ *the inequality*

$$\frac{(a+bx)^2 + b^2y^2}{a^2+b^2} \geq K \tag{4.7}$$

holds. In fact one can take

$$K \overset{\text{def}}{=} \frac{\delta^2}{1+(A+\delta)^2}. \tag{4.8}$$

PROOF. Given $(x, y) \in S_{A,\delta}$ and $(a, b) \in \mathbb{R}^2$ with $b \neq 0$, let $q \overset{\text{def}}{=} a/b$. Then (4.7) amounts to

$$\frac{(q+x)^2+y^2}{1+q^2} \geq K.$$

Two cases are considered. First, if $|q| \leq A+\delta$, then $1+q^2 \leq 1+(A+\delta)^2$, so

$$\frac{1}{1+q^2} \geq \frac{1}{1+(A+\delta)^2}.$$

Also $(q+x)^2+y^2 \geq y^2 \geq \delta^2$ (since $y \geq \delta$ for $(x, y) \in S_{A,\delta}$). Therefore

$$\frac{(q+x)^2+y^2}{1+q^2} \geq \frac{\delta^2}{1+(A+\delta)^2} = K,$$

with K as in (4.8).

If instead $|q| > A+\delta$, we have $1/|q| < 1/(A+\delta)$, so $-|x|/q \geq -|x|/(A+\delta)$. Use the triangular inequality and the fact that $|x| \leq A$ for $(x, y) \in S_{A,\delta}$ to write

$$\left|1+\frac{x}{q}\right| \geq 1-\left|\frac{x}{q}\right| \geq 1-\frac{|x|}{A+\delta} \geq 1-\frac{A}{A+\delta} = \frac{\delta}{A+\delta}.$$

That is, $|q+x| \geq |q|\dfrac{\delta}{A+\delta}$, which implies $(q+x)^2 \geq \dfrac{q^2\delta^2}{(A+\delta)^2}$, or again

$$\frac{(q+x)^2+y^2}{1+q^2} \geq \frac{(q+x)^2}{1+q^2} \geq \frac{\delta^2}{(A+\delta)^2}\,\frac{q^2}{1+q^2}. \tag{4.9}$$

On the other hand, $f(x) \overset{\text{def}}{=} x^2/(1+x^2)$ is a strictly increasing function on $(0, \infty)$ since $f'(x) = 2x/((1+x^2)^2)$ is positive for $x > 0$. Thus, since $|q| > A+\delta$, we have

$$\frac{q^2}{1+q^2} = f(|q|) > f(A+\delta) = \frac{(A+\delta)^2}{1+(A+\delta)^2},$$

which leads to

$$\frac{(q+x)^2+y^2}{1+q^2} \geq \frac{\delta^2}{(A+\delta)^2}\,\frac{(A+\delta)^2}{1+(A+\delta)^2}$$

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by the second inequality in (4.9). But the right-hand side is again the constant K of (4.8). This concludes the proof. $\square$

Now suppose $b = 0$, but $a \neq 0$. Then $\dfrac{(a+bx)^2+b^2y^2}{a^2+b^2} = \dfrac{a^2}{a^2} = 1 > \frac{1}{2}$, which by Lemma 4.6 says that

$$\frac{(a+bx)^2+b^2y^2}{a^2+b^2} \geq K_1 \stackrel{\text{def}}{=} \min\left(\tfrac{1}{2}, K\right)$$

for $(x, y) \in S_{A,\delta}$, $(a, b) \in \mathbb{R}\times\mathbb{R}-\{(0,0)\}$. Hence if $z \in S_{A,\delta}$, say $z = x+iy$, and $(m, n) \in \mathbb{Z}\times\mathbb{Z}-\{(0,0)\}$, we get $|m+nz|^2 = (m+nx)^2+n^2y^2 \geq (m^2+n^2)K_1 = K_1|m+ni|^2$. This implies, for $\alpha \geq 0$, that $|m+nz|^\alpha \geq K_1^{\alpha/2}|m+ni|^\alpha$, or

$$\frac{1}{|m+nz|^\alpha} \leq \frac{1}{K_1^{\alpha/2}|m+ni|^\alpha}. \tag{4.10}$$

Moreover — setting for convenience $\mathbb{Z}_*^2 \stackrel{\text{def}}{=} \mathbb{Z}\times\mathbb{Z}-\{(0,0)\}$ — we know from results in Appendix G that $\sum_{(m,n)\in\mathbb{Z}_*^2} 1/|m+ni|^\alpha$ converges for $\alpha > 2$. This shows that $G_k(z)$ converges absolutely and uniformly on every $S_{A,\delta}$ for $k > 2$, which (since k is even) is why we take $k = 4, 6, 8, 10, 12, \ldots$ in (4.4). In particular, since any compact subset of π^+ is contained in some $S_{A,\delta}$, the holomorphicity of $G_k(z)$ on π^+ is established.

Since the map $(m, n) \mapsto (m-n, n)$ is a bijection of $\mathbb{Z}_*^2$, we have

$$G_k(z+1) = \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{1}{(m+n+nz)^k} = \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{1}{(m-n+n+nz)^k} = G_k(z).$$

Similarly,

$$G_k\left(-\frac{1}{z}\right) = \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{1}{(m-n/z)^k} = z^k \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{1}{(mz-n)^k} = z^k \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{1}{(nz+m)^k},$$

since the map $(m, n) \mapsto (n, -m)$ is a bijection of $\mathbb{Z}_*^2$. This shows that $G_k(z)$ satisfies the conditions (M1)$'$ and (M1)$''$.

To complete the argument that the $G_k(z)$, for k even ≥ 4, are modular forms of weight k, we must check condition (M2). Although this could be done more directly, we take the route whereby the Fourier coefficients of $G_k(z)$ are actually computed explicitly. For this, consider the function

$$\phi_k(z) \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}} \frac{1}{(z+m)^k} \tag{4.11}$$

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on π^+ for $k \in \mathbb{Z}$, $k \geq 2$. The inequality (4.10) gives

$$\frac{1}{|m+z|^k} \leq \frac{1}{K_1^{k/2}|m+i|^k} = \frac{1}{K_1^{k/2}(m^2+1)^{k/2}} \tag{4.12}$$

for $z \in S_{A,\delta}$. Since

$$\sum_{m\in\mathbb{Z}} \frac{1}{(m^2+1)^{k/2}} = 1 + 2\sum_{m=1}^{\infty} \frac{1}{(m^2+1)^{k/2}} \leq 1 + 2\sum_{m=1}^{\infty} \frac{1}{m^{2(k/2)} = m^k} < \infty$$

for $k > 1$, we see that $\phi_k(z)$ converges absolutely and uniformly on every $S_{A,\delta}$, and is therefore a holomorphic function on π^+ such that

$$\phi_k(z+1) = \sum_{m\in\mathbb{Z}} \frac{1}{(z+m+1)^k} = \sum_{m\in\mathbb{Z}} \frac{1}{(z+m)^k} = \phi_k(z).$$

Thus (again) there is a Fourier expansion

$$\phi_k(z) = \sum_{n\in\mathbb{Z}} a_n(k) e^{2\pi i n z} \tag{4.13}$$

on π^+, where by formula (B.6) of page 84 (with the choice $b_1 = 0$, $b_2 = \infty$)

$$a_n(k) = \int_0^1 \phi_k(t+ib) e^{-2\pi i n(t+ib)}\, dt \tag{4.14}$$

for $n \in \mathbb{Z}$, $b > 0$.

PROPOSITION 4.15. *In the Fourier expansion* (4.13), $a_n(k) = 0$ *for* $n \leq 0$ *and* $a_n(k) = (-2\pi i)^k n^{k-1}/(k-1)!$ *for* $n \geq 1$. *Therefore*

$$\phi_k(z) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}$$

is the Fourier expansion of the function $\phi_k(z)$ *of* (4.11) *on* π^+. *Here* $k \in \mathbb{Z}$, $k \geq 2$ *as in* (4.11).

PROOF. For fixed $n \in \mathbb{Z}$ and $b > 0$, define $h_m(t) \overset{\text{def}}{=} e^{-2\pi i n t}/(t+ib+m)^k$ on $[0,1]$, for $m \in \mathbb{Z}$. Since $(t,b) \in S_{1,b}$ for $t \in [0,1]$ according to (4.5), the inequality in (4.12) gives

$$|h_m(t)| \leq \frac{1}{K_1^{k/2}(m^2+1)^{k/2}}.$$

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As we have seen, $\sum_{m\in\mathbb{Z}} 1/(m^2+1)^{k/2} < \infty$ for $k > 1$, so $\sum_{m\in\mathbb{Z}} h_m(t)$ converges uniformly on [0, 1]. By (4.11) and (4.14), therefore, we see that

$$a_n(k) = \int_0^1 \sum_{m\in\mathbb{Z}} h_m(t) e^{2\pi nb}\,dt = e^{2\pi nb} \sum_{m\in\mathbb{Z}} \int_0^1 h_m(t)\,dt$$
$$= e^{2\pi nb} \sum_{m\in\mathbb{Z}} \int_0^1 \frac{e^{-2\pi int}}{(t+ib+m)^k}\,dt. \tag{4.16}$$

By the change of variables $x = t+m$, we get

$$\int_0^1 \frac{e^{-2\pi int}\,dt}{(t+ib+m)^k} = \int_m^{m+1} \frac{e^{-2\pi in(x-m)}}{(x+ib)^k} dx = \int_m^{m+1} \frac{e^{-2\pi inx}}{(x+ib)^k} dx,$$

so

$$\sum_{m\in\mathbb{Z}} \int_0^1 \frac{e^{-2\pi int}}{(t+ib+m)^k}\,dt$$
$$= \sum_{m\in\mathbb{Z}} \int_m^{m+1} \frac{e^{-2\pi inx}}{(x+ib)^k} dx$$
$$= \sum_{m=0}^{\infty} \int_m^{m+1} \frac{e^{-2\pi inx}}{(x+ib)^k} dx + \sum_{m=1}^{\infty} \int_{-m}^{-m+1} \frac{e^{-2\pi inx}}{(x+ib)^k} dx$$
$$= \int_0^{\infty} \frac{e^{-2\pi inx}}{(x+ib)^k} dx + \int_{-\infty}^{0} \frac{e^{-2\pi inx}}{(x+ib)^k} = \int_{-\infty}^{\infty} \frac{e^{-2\pi inx}}{(x+ib)^k} dx. \tag{4.17}$$

(Note that the integrals on the last line are finite for $k > 1$, since

$$\left|\frac{e^{-2\pi inx}}{(x+ib)^k}\right| = \frac{1}{(x^2+b^2)^{k/2}} \tag{4.18}$$

and the map $x \mapsto 1/(x^2+b^2)^\sigma$ lies in $L^1(\mathbb{R}, dx)$ for $2\sigma > 1$.) From (4.16) and (4.17), we have

$$a_n(k) = e^{2\pi nb} \int_{-\infty}^{\infty} \frac{e^{-2\pi inx}}{(x+ib)^k} dx. \tag{4.19}$$

In particular,

$$|a_n(k)| \le e^{2\pi nb} \int_{\mathbb{R}} \frac{dx}{b^k\left(\left(\frac{x}{b}\right)^2+1\right)^{k/2}},$$

by (4.18). Setting $t = x/b$, we can rewrite the right-hand side as

$$\frac{e^{2\pi nb}}{b^k} \int_{\mathbb{R}} \frac{b\,dt}{(t^2+1)^{k/2}} = \frac{e^{2\pi nb} c_k}{b^{k-1}}, \tag{4.20}$$

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where $c_k \overset{\text{def}}{=} \int_{\mathbb{R}} dt/(t^2+1)^{k/2} < \infty$ for $k > 1$. Since $b > 0$ is arbitrary, we let $b \to \infty$. For $n = 0$ or for $n < 0$, we see by the inequality (4.20) that $a_n(k) = 0$. Also for $n = 1, 2, 3, 4, \ldots$ and $k > 1$, it is known that

$$\int_{\mathbb{R}} \frac{e^{-2\pi i n x}}{(x+ib)^k} dx = e^{-2\pi n b} \frac{(-2\pi i)^k}{(k-1)!} n^{k-1}; \tag{4.21}$$

see the Remark below. The proof of Proposition 4.15 is therefore completed by way of equation (4.19). □

REMARK. Equation (4.21) follows from a contour integral evaluation:

$$\int_{-\infty+ib}^{\infty+ib} \frac{e^{-2\pi i \mu z}}{z^k} dz = \frac{(2\pi)^k \mu^{k-1} e^{-k\pi i/2}}{\Gamma(k)} \tag{4.22}$$

where $\mu, b > 0,\ k > 1$. The left-hand side here is

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i \mu (x+ib)}}{(x+ib)^k} dx = e^{2\pi \mu b} \int_{-\infty}^{\infty} \frac{e^{-2\pi i \mu x}}{(x+ib)^k} dx.$$

Thus we can write $\int_{-\infty}^{\infty} e^{-2\pi i \mu x} dx/(x+ib)^k = e^{-2\pi \mu b}(-2\pi i)^k \mu^{k-1}/(k-1)!$ for $\mu, b > 0$ and $k > 1$ an integer. The choice $\mu = n$ $(n = 1, 2, 3, 4, \ldots)$ gives (4.21).

For $n \in \mathbb{Z}$, define

$$\psi_n(z) \overset{\text{def}}{=} \phi_k(nz) = \sum_{m \in \mathbb{Z}} \frac{1}{(nz+m)^k}$$

(see (4.11) for the last equality) on π^+. In definition (4.4), the $n = 0$ contribution to the sum is $\sum_{m \in \mathbb{Z}-\{0\}} 1/m^k = \sum_{m=1}^{\infty} 1/m^k + \sum_{m=1}^{\infty} 1/(-m)^k = 2\sum_{m=1}^{\infty} 1/m^k$ (since k is even) $= 2\zeta(k)$. Thus we can write $G_k(z) = 2\zeta(k) + \sum_{n \in \mathbb{Z}-\{0\}} \sum_{m \in \mathbb{Z}} 1/(m+nz)^k = 2\zeta(k) + \sum_{n=1}^{\infty} \psi_n(z) + \sum_{n=1}^{\infty} \psi_{-n}(z)$. But $\psi_{-n}(z) = \psi_n(z)$, again because k is even (the easy verification is left to the reader). Therefore

$$\begin{aligned} G_k(z) &= 2\zeta(k) + 2\sum_{n=1}^{\infty} \psi_n(z) = 2\zeta(k) + 2\sum_{n=1}^{\infty} \phi_k(nz) \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m n z} \end{aligned}$$

(by Proposition 4.15, for k even), leading to

$$G_k(z) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

by formula (D.8) of Appendix D. This proves:

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THEOREM 4.23. *The holomorphic Eisenstein series* $G_k(z)$, $k = 4, 6, 8, 10, 12, \ldots$, *defined in* (4.4) *satisfy conditions* (M1)$'$, (M1)$''$, *and are holomorphic at infinity. In fact,* $G_k(z)$ *has Fourier expansion* (4.2), *where* $a_0 = 2\zeta(k)$ *and*

$$a_n = \frac{2(2\pi i)^k}{(k-1)!}\sigma_{k-1}(n) \overset{\text{def}}{=} \frac{2(2\pi i)^k}{(k-1)!} \sum_{\substack{d>0 \\ d \mid n}} d^{k-1}$$

for $n \geq 1$. *The* $G_k(z)$ *are therefore modular forms of weight* k.

Since k is even in Theorem 4.23, formula (2.1) applies to $\zeta(k)$.

As mentioned, a modular form is a *cusp form* if its initial Fourier coefficient a_0 in equation (4.2) vanishes. By Theorem 4.23 the $G_k(z)$, for example, are *not* cusp forms since $\zeta(k) \neq 0$ for k even, $k \geq 4$. In fact we know (by Theorem 3.17) that since $\zeta(s)$ is given by an Euler product, it is nonvanishing for $\operatorname{Re} s > 1$.

We return now to the discussion of Hecke L-functions, where we begin with results on estimates of Fourier coefficients of modular forms. The $G_k(z)$ already provide the example of how the general estimate looks. This involves only an estimate of the divisor function $\sigma_\nu(n)$ for $\nu > 1$, where we first note that $d > 0$ runs through the divisors of $n \in \mathbb{Z}$, $n \geq 1$, as does n/d:

$$\sigma_\nu(n) \overset{\text{def}}{=} \sum_{\substack{0<d \\ d \mid n}} d^\nu = \sum_{\substack{0<d \\ d \mid n}} \left(\frac{n}{d}\right)^\nu = n^\nu \sum_{\substack{0<d \\ d \mid n}} \frac{1}{d^\nu} \leq n^\nu \sum_{0<d\in\mathbb{Z}} \frac{1}{d^\nu} = n^\nu \zeta(\nu).$$

Therefore for a_n the n-th Fourier coefficient of $G_k(z)$, Theorem 4.23 gives, for $n \geq 1$, $|a_n| \leq 2(2\pi)^k/(k-1)!\zeta(k-1)n^{k-1} = C(k)n^{k-1}$, where we have set

$$C(k) \overset{\text{def}}{=} \frac{2(2\pi)^k}{(k-1)!}\zeta(k-1).$$

In general:

THEOREM 4.24. *For a modular form* $f(z)$ *of weight* $k = 4, 6, 8, 10, \ldots$, *with Fourier expansion given by equation* (4.2), *there is a constant* $C(f,k) > 0$, *depending only on* f *and* k, *such that* $|a_n| < C(f,k)n^{k-1}$ *for* $n \geq 1$. *If* $f(z)$ *is a cusp form,* $C(f,k) > 0$ *can be chosen so that* $|a_n| < C(f,k)n^{k/2}$ *for* $n \geq 1$.

The idea of the proof is to first establish the inequality

$$|a_n| < C(f,k)n^{k/2}, \quad n \geq 1, \tag{4.25}$$

for a cusp form $f(z)$ by estimating $f(z)$ on a *fundamental domain* $F \subset \pi^+$ for the action of Γ on π^+ (given by restriction of the action of G in equation

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(4.3) to Γ), whence an estimate of $f(z)$ on π^+ is readily obtained. Then a_n is estimated from the formula

$$a_n = \int_0^1 f(t+ib)e^{-2\pi in(t+ib)}\,dt \tag{4.26}$$

for any $b > 0$; compare formula (4.14). By these arguments one can discover, in fact, that a modular form $f(z)$ of weight k *is a cusp form* if and only if there is a constant $M(f,k) > 0$, depending only on f and k, such that

$$|f(z)| < M(f,k)(\operatorname{Im} z)^{-k/2} \tag{4.27}$$

on π^+. Once (4.25) is established for a cusp form, the weaker result $|a_n| < C(f,k)n^{k-1}$, $n \geq 1$, for an arbitrary modular form $f(z)$ of weight k follows from the fact that it holds for $G_k(z)$ (as shown above), and the fact that $f(z)$ differs from a cusp form (where one can apply the inequality (4.25)) by a constant multiple of $G_k(z)$. In fact, write $f(z) = a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi inz}$, $G_k(z) = b_0 + \sum_{n=1}^{\infty} b_n e^{2\pi inz}$ (by equation (4.2)) where $b_0 = 2\zeta(k)$, for example (by Theorem 4.23). Then $f_0(z) \stackrel{\text{def}}{=} f(z) - (a_0/b_0)\,G_k(z)$ is a modular form of weight k, with Fourier expansion

$$\begin{aligned} f_0(z) &= a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi inz} - \frac{a_0}{b_0}b_0 - \sum_{n=1}^{\infty}\frac{a_0}{b_0}b_n e^{2\pi inz} \\ &= \sum_{n=1}^{\infty}\left(a_n - \frac{a_0 b_n}{b_0}\right)e^{2\pi inz}, \end{aligned}$$

which shows that $f_0(z)$ is a cusp form such that

$$f(z) = f_0(z) + \frac{a_0}{b_0}G_k(z). \tag{4.28}$$

To complete our sketch of the proof of Theorem 4.24, details of which can be found in section 6.15 of [2], for example, we should add further remarks regarding F. By definition, a fundamental domain for the action of Γ on π^+ is an open set $F \subset \pi^+$ such that (F1) no two distinct points of F lie in the same Γ-orbit: if $z_1, z_2 \in F$ with $z_1 \neq z_2$, then there is no $\gamma \in \Gamma$ such that $z_1 = \gamma \cdot z_2$. We also require condition: (F2) given $z \in \pi^+$, there exists some $\gamma \in \Gamma$ such that $\gamma \cdot z \in \bar{F}$ (= the closure of F). The standard fundamental domain, as is well-known, is given by

$$F \stackrel{\text{def}}{=} \{z \in \pi^+ \mid |z| > 1, |\operatorname{Re} z| < \tfrac{1}{2}\}, \tag{4.29}$$

and is shown at the the top of the next page.

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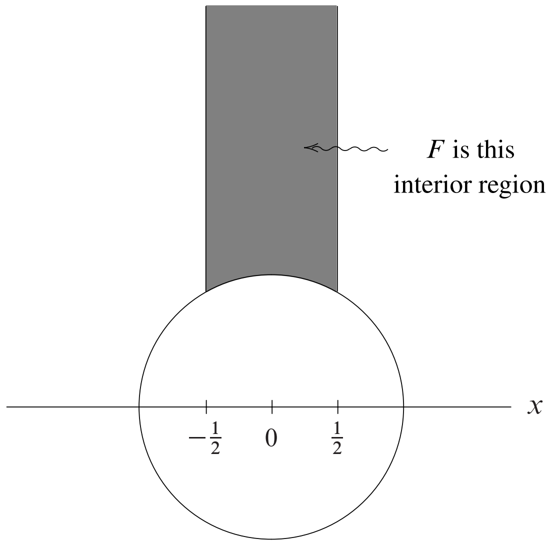

If $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the space of modular forms and cusp forms of weight $k = 4, 6, 8, 10, \ldots$, respectively, there is the $\mathbb{C}$-vector space direct sum decomposition

$$M_k(\Gamma) = S_k(\Gamma) \oplus \mathbb{C}G_k, \tag{4.30}$$

by (4.28). The sum in (4.30) is indeed direct since (as we have seen) $G_k(z)$ is not a cusp form.

If $f \in M_k(\Gamma)$ with Fourier expansion (4.2), the corresponding Hecke L-function $L(s; f)$ is given by definition (3.21):

$$L(s; f) = \phi_f(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{4.31}$$

By Theorem 4.24 this series converges absolutely for Re $s > k$, and for Re $s > 1+\frac{k}{2}$ if $f \in S_k(\Gamma)$. On these respective domains $L(s; f)$ is holomorphic in s (by an argument similar to that for the Riemann zeta function), as we have asserted in Lecture 3. Since Theorem 3.25 is based on equation (3.24), which is based on equation (3.23), our proof of it actually shows the following reformulation:

THEOREM 4.32. *Suppose the Fourier coefficients a_n of $f \in M_k(\Gamma)$ satisfy the multiplicative condition* (3.23), *with at least one a_n nonzero*:

$$a_{n_1} a_{n_2} = \sum_{\substack{d>0 \\ d \,|\, n_1,\ d \,|\, n_2}} d^{k-1} \frac{a_{n_1 n_2}}{d^2} \tag{4.33}$$

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for $n_1, n_2 \geq 1$. *Then* $L(s; f)$ *has the Euler product representation*

$$L(s; f) = \prod_{p \in P} \frac{1}{1 + p^{k-1-2s} - a_p p^{-s}} \tag{4.34}$$

for $\operatorname{Re} s > k$. *If* $f \in S_k(\Gamma)$, *equation* (4.34) *holds for* $\operatorname{Re} s > 1 + k/2$.

Here, we only need to note that $a_1 = 1$. For if some $a_n \neq 0$, then by (4.33) $a_n a_1 = a_{n \cdot 1}/1^2 = a_n$ and so $a_1 = 1$.

Theorem 4.32 raises the question of finding modular forms whose Fourier coefficients satisfy the multiplicative condition (3.23) = condition (4.33). This question was answered by Hecke (in 1937), who found, in fact, all such forms. As was observed in Lecture 3, the multiplicative condition is satisfied by normalized simultaneous eigenforms: nonzero forms $f(z)$ with $a_1 = 1$, that are simultaneous eigenfunctions of all the Hecke operators $T(n), n \geq 1$; see definition (3.22). More concretely, among the non-cusp forms the normalized simultaneous eigenforms turn out to be the forms

$$f(z) = \frac{(k-1)!}{2(2\pi i)^k} G_k(z),$$

where indeed, by Theorem 4.23, $a_1 = \sigma_{k-1}(1) = 1$.

We mention that the Hecke operators $\{T(n)\}_{n \geq 1}$ map the space $M_k(\Gamma)$ to itself, and also map the space $S_k(\Gamma)$ to itself. For $f \in M_k(\Gamma)$ with Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ on π^+, as in (4.2), $(T(n)f)(z)$ has Fourier expansion

$$(T(n)f)(z) = \sum_{m=0}^{\infty} a_m^{(n)} e^{2\pi i m z}$$

on π^+, where $a_0^{(n)} = a_0 \sigma_{k-1}(n)$ and $a_m^{(n)} = \sum_{0<d,\, d \mid n,\, d \mid m} d^{k-1} a_{\frac{n}{d} \frac{m}{d}}$ for $m \geq 1$ — a result that leads to condition (4.33); in particular $a_1^{(n)} = a_n$.

Besides his striking observations regarding the connection between modular forms $f(z)$ and their associated Dirichlet series $L(s; f)$, that we have briefly discussed so far, Enrich Hecke also obtained the analytic continuation and functional equation of $L(s; f)$, which we now describe. Hecke showed that

$$(2\pi)^{-s} L(s; f) \Gamma(s) = \int_1^{\infty} (f(it) - a_0)(t^{s-1} + i^k t^{k-s-1})\, dt + a_0 \left(\frac{i^k}{s-k} - \frac{1}{s} \right) \tag{4.35}$$

for $\operatorname{Re} s > k$. Here the integral

$$J_f(s) \stackrel{\text{def}}{=} \int_1^{\infty} (f(it) - a_0) t^{s-1}\, dt \tag{4.36}$$

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is an entire function of s. Notice that equation (4.35) is similar in form to (1.12). Again since $1/\Gamma(s)$ is an entire function, and since $(1/s)\,(1/\Gamma(s)) = 1/\Gamma(s+1)$, we can write equation (4.35) as

$$L(s;f) = \frac{(2\pi)^s}{\Gamma(s)}\big(J_f(s) + i^k J_f(k-s)\big) + \frac{(2\pi)^s}{\Gamma(s)}\frac{a_0 i^k}{(s-k)} - \frac{(2\pi)^s a_0}{\Gamma(s+1)} \quad (4.37)$$

for $\operatorname{Re} s > k$. If $f(z)$ is a cusp form, we see that $L(s;f)$ extends to an *entire* function by way of the first term in (4.37). In general, we see that $L(s;f)$ extends meromorphically to $\mathbb{C}$, with a single (simple) pole at $s = k$ with residue $(2\pi)^k a_0 i^k/\Gamma(k) = (2\pi)^k a_0 i^k/(k-1)!$. Also by equation (4.37), for $k-s \neq k$ (i.e., $s \neq 0$), we have

$$\begin{aligned} &i^k (2\pi)^{s-k}\Gamma(k-s)L(k-s;f) \\ &\quad = i^k (2\pi)^{s-k}\Gamma(k-s)\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\left(J_f(k-s) + i^k J_f(s) + \frac{a_0 i^k}{-s} - \frac{a_0\Gamma(k-s)}{\Gamma(k-s+1)}\right) \\ &\quad = i^k J_f(k-s) + J_f(s) + \frac{a_0}{-s} - \frac{i^k a_0}{k-s} \end{aligned}$$

(again since $\Gamma(w+1) = w\Gamma(w)$, and since $i^{2k} = 1$ for k even); the right-hand side in turn equals $(2\pi)^{-s}\Gamma(s)L(s;f)$. That is,

$$(2\pi)^{-s}\Gamma(s)L(s;f) = i^k(2\pi)^{s-k}\Gamma(k-s)L(k-s;f) \quad (4.38)$$

for $s \neq 0$, which is the functional equation for $L(s;f)$, which compares with the functional equation for the Riemann zeta function; see [17; 18].

The Eisenstein series $G_k(z)$ can be used as building blocks to construct other modular forms. It is known that any modular form $f(z)$ is, in fact, a finite sum of the form $f(z) = \sum_{n,m\geq 0} c_{nm} G_4(z)^n G_6(z)^m$ for suitable complex numbers c_{nm}. Of particular interest are the *discriminant form*

$$\Delta(z) \overset{\text{def}}{=} (60G_4(z))^3 - 27\,(140G_6(z))^2 \quad (4.39)$$

and the *modular invariant*

$$J(z) \overset{\text{def}}{=} (60G_4(z))^3 \,/\Delta(z) \quad (4.40)$$

which is well-defined since it is true that $\Delta(z)$ never vanishes on π^+. $\Delta(z)$ is a modular form of weight 12, since if $f_1(z), f_2(z)$ are modular forms of weight k_1, k_2, then $f_1(z)f_2(z)$ is a modular form of weight k_1+k_2. Similarly $J(z)$ is a weak modular form of weight $k = 0$: $J(\gamma, z) = J(z)$ for $\gamma \in \Gamma$, $z \in \pi^+$. The form $J(z)$ was initially constructed by R. Dedekind in 1877, and by F. Klein in 1878. Associated with it is the equally important *modular j-invariant*

$$j(z) \overset{\text{def}}{=} 1728J(z). \quad (4.41)$$

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$\Delta(z)$ is connected with the Dedekind eta function $\eta(z)$ (see definition (3.27)) by the *Jacobi identity*

$$\Delta(z) = (2\pi)^{12}\eta(z)^{24}, \tag{4.42}$$

which with equation (3.28) shows that $\Delta(z)$ has Fourier expansion

$$\Delta(z) = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)e^{2\pi inz}, \tag{4.43}$$

where $\tau(n)$ is the Ramanujan tau function, and which in particular shows that $\Delta(z)$ is a cusp form: $\Delta(z) \in S_{12}(\Gamma)$, which can also be proved directly by definition (4.39) and Theorem 4.23, for $k = 4, 6$. There are no nonzero cusp forms of weight < 12.

Note that by Theorem 4.24, there is a constant $C > 0$ such that $|\tau(n)| < Cn^6$ for $n \geq 1$. However P. Deligne proved the *Ramanujan conjecture* $|\tau(n)| \leq \sigma_0(n)n^{11/2}$ for $n \geq 1$, where $\sigma_0(n)$ is the number of positive divisors of n. As we remarked in Lecture 3, the $\tau(n)$ (remarkably) are all integers. This can be proved using definition (4.39) and Theorem 4.23, for $k = 4, 6$. It is also true that, thanks to the factor 1728 in definition (4.41), *all of the Fourier coefficients of the modular j-invariant* are integers:

$$j(z) = 1e^{-2\pi iz} + \sum_{n=0}^{\infty} a_n e^{2\pi inz} \tag{4.44}$$

with each $a_n \in \mathbb{Z}$:

$$a_0 = 744,\ a_1 = 196{,}884,\ a_2 = 21{,}493{,}760,\ a_3 = 864{,}299{,}970,$$
$$a_4 = 20{,}245{,}856{,}256,\ a_5 = 333{,}202{,}640{,}600,\ \ldots \tag{4.45}$$

An application of the modular invariant $j(z)$, and of the values in (4.45), to three-dimensional gravity with a negative cosmological constant will be given in my Speaker's Lecture; see especially equation (5-8) on page 343 and the subsequent discussion.

In the definition (4.4) of the holomorphic Eisenstein series $G_k(z)$, one cannot take $k = 2$ for convergence reasons. However, Theorem 4.23 provides a suggestion of how one might proceed to construct a series $G_2(z)$. Namely, take $k = 2$ there and thus define

$$G_2(z) \overset{\text{def}}{=} 2\zeta(2) + 2(2\pi i)^2\sum_{n=1}^{\infty}\sigma(n)e^{2\pi inz} = \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty}\sigma(n)e^{2\pi inz} \tag{4.46}$$

on π^+, where $\sigma(n) \overset{\text{def}}{=} \sigma_1(n) \overset{\text{def}}{=} \sum_{0<d,\ d\,|\,n} d$. Note that the series on the right does converge on π^+ and, in fact, the convergence is absolute: since $\sigma(n) \leq \sum_{d=1}^{n} d = \frac{1}{2}n(n+1)$ we have $|\sigma(n)e^{2\pi inz}| \leq \frac{1}{2}n(n+1)e^{-2\pi n\,\mathrm{Im}\,z}$, and convergence is assured by the ratio test. Given any compact subset $K \subset \pi^+$, a positive lower bound B for the continuous function $\mathrm{Im}\,z$ on K exists: we have

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$\operatorname{Im} z \geq B > 0$ on K, so $|\sigma(n)e^{2\pi inz}| \leq \frac{1}{2}n(n+1)e^{-2\pi nB}$ on K, and again $\sum_{n=1}^{\infty} n(n+1)e^{-2\pi nB} < \infty$ for $B > 0$. By the M-test, $\sum_{n=1}^{\infty} \sigma(n)e^{2\pi inz}$ converges uniformly on K, which (by the Weierstrass theorem) means that $G_2(z)$ is a holomorphic function on π^+.

Another expression for $G_2(z)$ is

$$G_2(z) = 2\zeta(2) + \sum_{n\in\mathbb{Z}-\{0\}} \sum_{m\in\mathbb{Z}} \frac{1}{(m+nz)^2}. \tag{4.47}$$

To check this, start by taking $k = 2$ in definition (4.11) and in Proposition 4.15:

$$\frac{1}{z^2} + \sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(z+m)^2} = \phi_2(z) = (-2\pi i)^2 \sum_{k=1}^{\infty} ke^{2\pi ikz}. \tag{4.48}$$

Replace z by nz in (4.48) and sum on n from 1 to ∞:

$$\frac{1}{z^2}\zeta(2) + \sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(nz+m)^2} = (-2\pi i)^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} ke^{2\pi iknz}. \tag{4.49}$$

By (D.6) (see page 90), we obtain

$$\sigma(n) = \sum_{k=1}^{\infty} d(k,n)k. \tag{4.50}$$

For $a_n \stackrel{\text{def}}{=} e^{2\pi inz}$, $n \geq 1$, $z \in \pi^+$, and for $k \geq 1$ fixed the series $\sum_{n=1}^{\infty} d(k,n)a_n$ clearly converges absolutely, since $\operatorname{Im} z > 0$ and $0 \leq d(k,n) \leq 1$. Then the series $\sum_{n=1}^{\infty} a_{kn}$ converges and equals $\sum_{n=1}^{\infty} d(k,n)a_n$, by the Scholium of Appendix D (page 91):

$$\sum_{n=1}^{\infty} e^{2\pi iknz} = \sum_{n=1}^{\infty} d(k,n)e^{2\pi inz} \tag{4.51}$$

which gives, by equation (4.50)

$$\begin{aligned}\sum_{n=1}^{\infty} \sigma(n)e^{2\pi inz} &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} d(k,n)e^{2\pi inz} = \sum_{k=1}^{\infty} k \sum_{n=1}^{\infty} d(k,n)e^{2\pi inz} \\ &= \sum_{k=1}^{\infty} k \sum_{n=1}^{\infty} e^{2\pi iknz} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} ke^{2\pi iknz},\end{aligned} \tag{4.52}$$

which in turn allows for the expression

$$\begin{aligned}\frac{2}{z^2}\zeta(2) + 2\sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(nz+m)^2} &= 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma(n)e^{2\pi inz} \\ &= G_2(z) - 2\zeta(2)\end{aligned} \tag{4.53}$$

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by equation (4.49), provided the commutations of the summations over k, n in (4.52) are legal. But for $y = \operatorname{Im} z$,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |d(k,n)ke^{2\pi i n z}| = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} d(k,n)ke^{-2\pi n y} = \sum_{n=1}^{\infty} \sigma(n)e^{-2\pi n y}$$

by (4.50), and this equals $\sum_{n=1}^{\infty} |\sigma(n)e^{2\pi i n z}|$, which is finite, as we have seen. This justifies the first commutation. Similarly,

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |ke^{2\pi i k n z}| = \sum_{k=1}^{\infty} k \sum_{n=1}^{\infty} (e^{-2\pi k y})^n = \sum_{k=1}^{\infty} k\left(\frac{e^{-2\pi k y}}{1-e^{-2\pi k y}}\right) = \sum_{k=1}^{\infty} \frac{k}{e^{2\pi k y}-1}$$

is finite by the integral test:

$$\int_1^{\infty} \frac{t\,dt}{e^{2\pi y t}-1} = \frac{1}{(2\pi y)^2}\int_{2\pi y}^{\infty} \frac{u\,du}{e^u-1} < \infty, \tag{4.54}$$

as we shall see later by Theorem 6.1, for example. This justifies the second commutation in equation (4.52). Since for $n \geq 1$

$$\sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(m-nz)^2} = \sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(-m-nz)^2} = \sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(m+nz)^2}, \tag{4.55}$$

the double sum on the right-hand side of (4.47) can be written as

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}-\{0\}}^{\infty}\left(\frac{1}{(nz)^2} + \sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(m+nz)^2}\right) \\
&\quad = 2\sum_{n=1}^{\infty}\frac{1}{n^2z^2} + \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(m+nz)^2} + \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(m-nz)^2} \\
&\quad = 2\frac{\zeta(2)}{z^2} + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{(m+nz)^2},
\end{aligned} \tag{4.56}$$

which is the left-hand side of (4.53). That is, $G_2(z) - 2\zeta(2)$ equals the double sum on the right-hand side of (4.47), proving (4.47).

Using equation (4.47) one can eventually show that $G_2(z)$ satisfies the rule

$$G_2\left(-\frac{1}{z}\right) = z^2 G_2(z) - 2\pi i z. \tag{4.57}$$

Because of the term $-2\pi i z$ in (4.57), $G_2(z)$ is *not* a modular form of weight 2. That is, condition (M1)$''$ above is not satisfied, although condition (M1)$'$ is: $G_2(z+1) = G_2(z)$ by definition (4.46). Equation (4.57) also follows by a transformation property of the Dedekind eta function, whose logarithmic derivative turns out to be a constant multiple of $G_2(z)$. Thus we indicate now an alternative derivation of the rule (4.57).

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On the domain $D \overset{\text{def}}{=} \{w \in \mathbb{C} \mid |w| < 1\}$, the holomorphic function $1 + w$ is nonvanishing and it therefore has a holomorphic logarithm $g(w)$ that can be chosen so as to vanish at $w = 0$: $e^{g(w)} \overset{\text{def}}{=} 1 + w$ on D. In fact $g(w) = -\sum_{n=1}^{\infty} (-w)^n/n$. Again for $q(z) \overset{\text{def}}{=} e^{2\pi i z}$, $z \in \pi^+$, consider the n-th partial sum $s_n(z) \overset{\text{def}}{=} \sum_{k=1}^{n} g(-q(z)^k)$, which is well-defined, because $q(z) \in D$ for $z \in \pi^+$ implies $-q(z)^k \in D$ for $k > 0$. We claim that the series

$$\psi(z) \overset{\text{def}}{=} -\sum_{k=1}^{\infty} g(-q(z)^k) = -\lim_{n\to\infty} s_n(z) \tag{4.58}$$

converges. We have $\psi(z) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} (q(z)^k)^n/n$, where for $y = \operatorname{Im} z$ (again)

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(q(z)^k)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} (e^{-2\pi n y})^k$$
$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{e^{-2\pi n y}}{1 - e^{-2\pi n y}}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{e^{2\pi n y} - 1}\right),$$

which is finite by the integral test; compare with (4.54), for example. This allows us to write $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (q(z)^k)^n/n = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} (q(z)^k)^n/n$, and shows the finiteness of these series. Hence $\psi(z)$ is finite. Now $\prod_{n=1}^{\infty}(1 - q(z)^n) = \lim_{n\to\infty} \prod_{k=1}^{n} (1 - q(z)^k) = \lim_{n\to\infty} \prod_{k=1}^{n} e^{g(-q(z)^k)}$, by the definition of $g(w)$, and this equals $e^{-\psi(z)}$ by (4.58). That is, for the Dedekind eta function

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - q(z)^n) \tag{4.59}$$

on π^+ defined in (3.27) we see that

$$\eta(z) = e^{\pi i z/12 - \psi(z)}. \tag{4.60}$$

Differentiation of the equation $e^{g(w)} \overset{\text{def}}{=} 1 + w$ gives $g'(w) = 1/e^{g(w)} = 1/(1+w)$ (of course), which with termwise differentiation of (4.58) (whose justification we skip) gives

$$\psi'(z) = -\sum_{k=1}^{\infty} g'(-q(z)^k)(-kq(z)^{k-1}q'(z)) = 2\pi i \sum_{k=1}^{\infty} \frac{kq(z)^k}{1 - q(z)^k}$$
$$= 2\pi i \sum_{k=1}^{\infty} k \sum_{n=1}^{\infty} (q(z)^k)^n = 2\pi i \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z},$$

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by equation (4.52). Therefore, by (4.60), we obtain

$$\eta'(z) = \eta(z)\Big(\frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \sigma(n)e^{2\pi i n z}\Big),$$

or

$$\frac{\eta'(z)}{\eta(z)} = -\frac{G_2(z)}{4\pi i} \tag{4.61}$$

by definition (4.46).

Now $\eta(z)$ satisfies the known transformation rule

$$\eta\Big(-\frac{1}{z}\Big) = e^{-i\pi/4}\sqrt{z}\,\eta(z),$$

where we take $\arg z \in (-\pi, \pi)$. Differentiation gives

$$\frac{\eta'\Big(-\frac{1}{z}\Big)\frac{1}{z^2}}{\eta\Big(-\frac{1}{z}\Big)} = \frac{e^{-i\pi/4}\Big(\sqrt{z}\,\eta'(z) + \frac{\sqrt{z}}{2z}\eta(z)\Big)}{\eta\Big(-\frac{1}{z}\Big)} = \frac{\eta'(z)}{\eta(z)} + \frac{1}{2z} \tag{4.62}$$

which by equation (4.61) says that $-G_2\big(-\frac{1}{z}\big)/4\pi i z^2 = -G_2(z)/4\pi i + \frac{1}{2z}$. This is immediately seen to imply the transformation rule (4.57).

In the lectures of Geoff Mason and Michael Tuite the particular normalization $G_k(z)/(2\pi i)^k$ of the Eisenstein series is considered, which they denote by $E_k(z)$. In particular, by definition (4.46),

$$E_2(z) = -\frac{1}{12} + 2\sum_{n=1}^{\infty} \sigma(n)e^{2\pi i n z}.$$

However, other normalized Eisenstein series appear in the literature that also might be denoted by $E_k(z)$. For example, in [20] there is the normalization (and notation) $E_k(z) \stackrel{\text{def}}{=} G_k(z)/2\zeta(k)$.

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Lecture 5. Dirichlet L-functions

Equation (3.19) has an application to *Dirichlet L-functions*, which we now consider. To construct such a function, we need first a *character* χ *modulo* m, where $m > 0$ is a fixed integer. This is defined as follows. Let U_m denote the group of *units* in the commutative ring $\mathbb{Z}_{(m)} \stackrel{\text{def}}{=} \mathbb{Z}/m\mathbb{Z}$. Thus if $\bar{n} = n + m\mathbb{Z}$ denotes the coset of $n \in \mathbb{Z}$ in $\mathbb{Z}_{(m)}$, we have $\bar{n} \in U_m \iff \exists \bar{a} \in \mathbb{Z}_{(m)}$ such that $\bar{a}\bar{n} = \bar{1}$. One knows of course that $\bar{n} \in U_m \iff (n, m) = 1$ (i.e. n and m are relatively prime). A character modulo m is then (by definition) a group homomorphism $\chi : U_m \to \mathbb{C}^* \stackrel{\text{def}}{=} \mathbb{C} - \{0\}$. For our purpose, however, there is an equivalent way of thinking about characters modulo m. In fact, given χ, define $\chi_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ by

$$\chi_{\mathbb{Z}}(n) \stackrel{\text{def}}{=} \begin{cases} \chi(\bar{n}) & \text{if } (n,m) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

for $n \in \mathbb{Z}$. For $n, n_1, n_2 \in \mathbb{Z}$, $\chi_{\mathbb{Z}}$ satisfies:

(D1) $\chi_{\mathbb{Z}}(n) = 0 \iff (n,m) \neq 1$;

(D2) $\chi_{\mathbb{Z}}(n_1) = \chi_{\mathbb{Z}}(n_2)$ when $\bar{n}_1 = \bar{n}_2$ in $\mathbb{Z}_{(m)}$;

(D3) $\chi_{\mathbb{Z}}(n_1 n_2) = \chi_{\mathbb{Z}}(n_1)\chi_{\mathbb{Z}}(n_2)$ when $(n_1, m) = 1$ and $(n_2, m) = 1$.

Conversely, suppose $\chi_0 : \mathbb{Z} \to \mathbb{C}$ is a function that satisfies the three conditions (D1), (D2), and (D3). Define $\chi : U_m \to \mathbb{C}$ by $\chi(\bar{n}) = \chi_0(n)$ for $n \in \mathbb{Z}$ such that $(n,m) = 1$. The character χ is well-defined by (D2), and $\chi : U_m \to \mathbb{C}^*$ by (D1). By (D3), $\chi(ab) = \chi(a)\chi(b)$ for $a, b \in U_m$, so we see that χ is a character modulo m. Moreover the induced map $(\chi_0)_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ given by definition (5.1) coincides with χ_0.

Note that $\chi_{\mathbb{Z}}$ is completely multiplicative:

(D4) $\chi_{\mathbb{Z}}(n_1 n_2) = \chi_{\mathbb{Z}}(n_1)\chi_{\mathbb{Z}}(n_2)$ for all $n_1, n_2 \in \mathbb{Z}$.

For if either $(n_1, m) \neq 1$ or $(n_2, m) \neq 1$, then $(n_1 n_2, m) \neq 1$, so that by (D1) both $\chi_{\mathbb{Z}}(n_1)\chi_{\mathbb{Z}}(n_2)$ and $\chi_{\mathbb{Z}}(n_1 n_2)$ are zero. If both $(n_1, m) = 1$ and $(n_2, m) = 1$, then already $\chi_{\mathbb{Z}}(n_1 n_2) = \chi_{\mathbb{Z}}(n_1)\chi_{\mathbb{Z}}(n_2)$ by (D3).

Note also that since $\chi(a) \in \mathbb{C}^*$ for every $a \in U_m$ (that is, $\chi(a) \neq 0$), we have $0 \neq \chi(\bar{1}) = \chi(\bar{1}\bar{1}) = \chi(\bar{1})\chi(\bar{1})$ by (D4), so $\chi(\bar{1}) = 1$. Moreover since $(1, m) = 1$, $\chi_{\mathbb{Z}}(1) = \chi(\bar{1})$, by (5.1), which in turn equals 1.

One final property of $\chi_{\mathbb{Z}}$ that we need is:

(D5) $|\chi(a)| = 1$ for all $a \in U_m$; hence $|\chi_{\mathbb{Z}}(n)| \leq 1$ for all $n \in \mathbb{Z}$.

The proof of (D5) makes use of a little theorem in group theory which says that if G is a finite group with $|G|$ elements, then $a^{|G|} = 1$ for every $a \in G$. Now, given $a \in U_m$, we can write (as just seen) $1 = \chi(\bar{1}) = \chi(a^{|U_m|}) = \chi(a)^{|U_m|}$ (since χ is a group homomorphism), which shows that $\chi(a)$ is a $|U_m|$-th root of unity: $|\chi(a)| = 1$ for all $a \in U_m$. Hence $|\chi_{\mathbb{Z}}(n)| \leq 1$ for all $n \in \mathbb{Z}$, by definition (5.1).

Given a character χ modulo m, it follows that we can form the zeta function, or Dirichlet series

$$L(s,\chi) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\chi_{\mathbb{Z}}(n)}{n^s} = \sum_{(n,m)=1} \frac{\chi(\bar{n})}{n^s}, \tag{5.2}$$

called a *Dirichlet L-function*, which converges for $\operatorname{Re} s > 1$, by (D5). $L(s,\chi)$ is holomorphic on the domain $\operatorname{Re} s > 1$, by the same argument given for the Riemann zeta function $\zeta(s)$. Since $\chi_{\mathbb{Z}} \not\equiv 0$ ($\chi_{\mathbb{Z}}(1) = 1$), and since $\chi_{\mathbb{Z}}$ is completely multiplicative, formula (3.19) implies:

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THEOREM 5.3 (EULER PRODUCT FOR DIRICHLET L-FUNCTIONS). *Assume* $\operatorname{Re} s > 1$. *Then*

$$L(s,\chi) = \frac{1}{\prod\limits_{p\in P}\left(1-\chi_{\mathbb{Z}}(p)p^{-s}\right)} = \frac{1}{\prod\limits_{\substack{p\in P\\ p\nmid m}}\left(1-\chi_{\mathbb{Z}}(p)p^{-s}\right)}. \tag{5.4}$$

The second statement of equality follows by (D1), since for a prime $p \in P$, saying that $(p,m) \neq 1$ is the same as saying that $p \mid m$.

As an example, define $\chi_0 : \mathbb{Z} \to \mathbb{C}$ by

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,m) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for n$\in \mathbb{Z}$. Then χ_0 satisfies (D1). If $n_1, n_2, l \in \mathbb{Z}$ such that $n_1 = n_2 + lm$ (i.e., $\bar{n}_1 = \bar{n}_2$), then $(n_1,m) = 1 \iff (n_2,m) = 1$, so χ_0 satisfies (D2). If $(n_1,m) = 1$ and $(n_2,m) = 1$, then $(n_1n_2,m) = 1$, so χ_0 also satisfies (D3), and χ_0 therefore defines a Dirichlet character modulo m. We call χ_0 (or the induced character $U_m \to \mathbb{C}^*$) the *principal character* modulo m. Again since p is a prime, we see by equation (5.4) that for $\operatorname{Re} s > 1$

$$L(s,\chi_0) = \frac{1}{\prod\limits_{\substack{p\in P\\ p\nmid m}}(1-p^{-s})}. \tag{5.5}$$

Then for $\operatorname{Re} s > 1$

$$L(s,\chi_0)\frac{1}{\prod\limits_{\substack{p\in P\\ p\mid m}}(1-p^{-s})} = \frac{1}{\prod\limits_{p\in P}(1-p^{-s})} = \zeta(s) \tag{5.6}$$

by formula (0.2). That is,

$$L(s,\chi_0) = \zeta(s)\prod_{\substack{p\in P\\ p\mid m}}(1-p^{-s}) \tag{5.7}$$

for $\operatorname{Re} s > 1$.

If $\chi_0 : U_m \to \mathbb{C}^*$ also denotes the character modulo m induced by $\chi_0 : \mathbb{Z} \to \mathbb{C}$, then $\chi_0(\bar{n}) = 1$ for every $\bar{n} \in U_m$ (by (5.1)) since $(n,m) = 1$.

As another simple example, take $m = 5$: $Z_{(5)} = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$, and it is easily checked that $U_5 = \{\bar{1},\bar{2},\bar{3},\bar{4}\}$. Moreover the equations $\chi(\bar{1}) \stackrel{\text{def}}{=} \chi(\bar{4}) \stackrel{\text{def}}{=} 1$, $\chi(\bar{2}) \stackrel{\text{def}}{=} \chi(\bar{3}) \stackrel{\text{def}}{=} -1$ define a character $\chi^{(5)} = \chi : U_5 \to \mathbb{C}^*$ modulo 5. The induced map $\chi_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ in definition (5.1) is given by $\chi_{\mathbb{Z}}(1) = 1$, $\chi_{\mathbb{Z}}(2) = -1$, $\chi_{\mathbb{Z}}(3) = -1$, $\chi_{\mathbb{Z}}(4) = 1$, $\chi_{\mathbb{Z}}(5) = 0$ (since $(5,5) \neq 1$), $\chi_{\mathbb{Z}}(6) = 1$, $\chi_{\mathbb{Z}}(7) = -1$, $\chi_{\mathbb{Z}}(8) = -1$, $\chi_{\mathbb{Z}}(9) = 1, \ldots$ (since $\bar{6} = \bar{1}$, $\bar{7} = \bar{2}$, $\bar{8} = \bar{3}$, $\bar{9} = \bar{4}$, with $(n,5) = 1$

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for $n = 6, 7, 8, 9$). The corresponding Dirichlet L-function is therefore given, for $\operatorname{Re} s > 1$, by

$$L(s, \chi^{(5)}) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s} \pm \cdots,$$

Next take $m = 8 : Z_{(8)} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$, $U_8 = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$. The map $\chi^{(8)} = \chi : U_8 \to \mathbb{C}^*$ given by $\chi(\bar{1}) \overset{\text{def}}{=} \chi(\bar{7}) \overset{\text{def}}{=} 1$, $\chi(\bar{3}) \overset{\text{def}}{=} \chi(\bar{5}) \overset{\text{def}}{=} -1$ is a character modulo 8, with $\chi_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ in definition (5.1) given by $\chi_{\mathbb{Z}}(1) = 1$, $\chi_{\mathbb{Z}}(2) = 0$, $\chi_{\mathbb{Z}}(3) = -1$, $\chi_{\mathbb{Z}}(4) = 0$, $\chi_{\mathbb{Z}}(5) = -1$, $\chi_{\mathbb{Z}}(6) = 0$, $\chi_{\mathbb{Z}}(7) = 1$, $\chi_{\mathbb{Z}}(8) = 0$, $\chi_{\mathbb{Z}}(9) = 1$, $\chi_{\mathbb{Z}}(10) = 0$, $\chi_{\mathbb{Z}}(11) = -1$, $\chi_{\mathbb{Z}}(12) = 0$, $\chi_{\mathbb{Z}}(13) = -1$, $\chi_{\mathbb{Z}}(14) = 0$, $\chi_{\mathbb{Z}}(15) = 1$, $\chi_{\mathbb{Z}}(16) = 0$, $\chi_{\mathbb{Z}}(17) = 1, \ldots$. Then

$$L(s, \chi^{(8)}) = 1 - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s} + \frac{1}{17^s} \pm \cdots.$$

From formula (5.7) it follows that the L-function $L(s, \chi_0)$ admits a meromorphic continuation to the full complex plane, with $s = 1$ as its only singularity — a simple pole with residue $\prod_{p \in P,\ p \mid m} \left(1 - \frac{1}{p}\right)$. If $\chi \neq \chi_0$ it is known that $L(s, \chi)$ at least extends to $\operatorname{Re} s > 0$ and, moreover, that $L(1, \chi) \neq 0$. For example, for the characters $\chi^{(5)}$, $\chi^{(8)}$ modulo 5 and 8, respectively, constructed in the previous examples, one has

$$L(1, \chi^{(5)}) = \frac{1}{\sqrt{5}} \log\left(\frac{3 + \sqrt{5}}{2}\right), \quad L(1, \chi^{(8)}) = \frac{1}{\sqrt{8}} \log(3 + 2\sqrt{2}).$$

If $\chi \neq \chi_0$ is a *primitive* character modulo m, a notion that we shall define presently, then $L(s, \chi)$ does continue meromorphically to $\mathbb{C}$, and it has a decent functional equation.

First we define the notion of an imprimitive character. Suppose $k > 0$ is a divisor of m. Then there is a natural (well-defined) map $q : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$, given by $q(n + m\mathbb{Z}) \overset{\text{def}}{=} n + k\mathbb{Z}$ for $n \in \mathbb{Z}$. If $(n, m) = 1$ then $(n, k) = 1$ since $k \mid m$. Therefore the restriction $q^* \overset{\text{def}}{=} q|_{U_m}$ maps U_m to U_k, and is a homomorphism between these two groups. Let $\psi^{(k)} : U_k \to \mathbb{C}^*$ be a character modulo k. By definition, $\psi^{(k)}$ is a homomorphism and hence so is $\psi^{(k)} \circ q^* : U_m \to \mathbb{C}^*$. That is, given a positive divisor k of m we have an induced character $\chi \overset{\text{def}}{=} \psi^{(k)} \circ q^*$ modulo m. Characters χ modulo m that are induced this way, say for $k \neq m$, are called *imprimitive*. χ is called a *primitive* character if it is not imprimitive, in which case m is also called the *conductor* of χ. Thus for a primitive character χ modulo m, the L-function $L(s, \chi)$ satisfies a theory similar to (but a bit more complicated than) that of the Riemann zeta function $\zeta(s)$.

In the Introduction we referred to the prime number theorem, expressed in equation (0.3), as a *monumental result*, and we noted quite briefly the role of $\zeta(s)$ in its proof. Similarly, the study of the L-functions $L(s, \chi)$ leads to a

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monumental result regarding primes in an arithmetic progression. Namely, in 1837 Dirichlet proved that there are *infinitely many primes* in any arithmetic progression $n, n+m, n+2m, n+3m, \ldots$, where n, m are positive, relatively prime integers — a key aspect of the proof being the fact (pointed out earlier) that $L(1,\chi) \neq 0$ if $\chi \neq \chi_0$. Dirichlet's proof relates, moreover, $L(1,\chi)$ to a Gaussian *class number* — an invariant in the study of binary quadratic forms. One can obtain, also, a *prime number theorem for arithmetic progressions* (from the *Siegel–Walfisz theorem*), where the counting function $\pi(x)$ in (0.3) is replaced by the function $\pi(x;m,n) \stackrel{\text{def}}{=}$ the number of primes $p \leq x$, with $p \equiv n (\operatorname{mod} m)$, for n, m relatively prime. One can also formulate and prove a prime number theorem for *graphs*. This is discussed in section 3.3 of the lectures of Audrey Terras.

Lecture 6. Radiation density integral, free energy, and a finite-temperature zeta function

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Theorems 1.13 and 1.18 provide for integral representations of $\zeta(s)$, for $\operatorname{Re} s > 1$, that serve as starting points for its analytic continuation. The following, nice integral representation also serves as a starting point. We apply it to compute Planck's radiation density integral. We also consider a free energy – zeta function connection.

THEOREM 6.1. *For* $\operatorname{Re} s > 1$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\,dt}{e^t - 1}. \tag{6.2}$$

We can regard the integral on the right as the sum

$$\int_0^1 \frac{t^{s-1}\,dt}{e^t - 1} + \int_1^\infty \frac{t^{s-1}\,dt}{e^t - 1},$$

where the second integral converges absolutely for all $s \in \mathbb{C}$, *and the first integral, understood as the limit* $\lim_{\alpha \to 0+} \int_\alpha^1 t^{s-1}\,dt/(e^t - 1)$, *exists for* $\operatorname{Re} s > 1$.

The proof of Theorem 6.1 is developed in two stages. First, for $\alpha, \beta, a \in \mathbb{R}$ and $s \in \mathbb{C}$, with $\alpha < \beta$ and $a > 0$, write $\int_\alpha^\beta e^{-at} t^{s-1}\,dt = a^{-s} \int_{a\alpha}^{a\beta} e^{-v} v^{s-1}\,dv$, by the change of variables $v = at$. In particular,

$$\int_1^\beta e^{-at} t^{s-1}\,dt = a^{-s} \int_a^{a\beta} e^{-t} t^{s-1}\,dt \quad \text{for } \beta > 1, \tag{6.3}$$

$$\int_\alpha^1 e^{-at} t^{s-1}\,dt = a^{-s} \int_{a\alpha}^{a} e^{-t} t^{s-1}\,dt \quad \text{for } \alpha < 1. \tag{6.4}$$

For $s \in \mathbb{C}, m > 0$, consider the integral

$$I_m(s) \stackrel{\text{def}}{=} \int_0^\infty \frac{t^{s-1}e^{-mt}e^{-t}}{1-e^{-t}}\,dt = \int_0^\infty \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt \tag{6.5}$$

which we check does converge for $\operatorname{Re} s > 1$. For $t > 0$, we have $e^t > 1+t$, so $1/(e^t-1) < 1/t$, and hence

$$\left|\frac{t^{s-1}e^{-mt}}{e^t-1}\right| < \frac{t^{\sigma-1}e^{-mt}}{t} = t^{\sigma-2}e^{-mt} \tag{6.6}$$

for $\sigma = \operatorname{Re} s$. By (6.3), $\int_1^\beta t^{\sigma-2}e^{-mt}\,dt = m^{-(\sigma-1)}\int_m^{m\beta} e^{-t}t^{\sigma-2}\,dt$ for $\beta > 1$. Let $\beta \to \infty$: then $\int_1^\infty t^{\sigma-2}e^{-mt}\,dt$ exists and

$$\int_1^\infty t^{\sigma-2}e^{-mt}\,dt = \frac{1}{m^{\sigma-1}}\int_m^\infty e^{-t}t^{\sigma-2}\,dt.$$

In view of (6.6), therefore, $\int_1^\infty \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt$ converges absolutely for every $s \in \mathbb{C}$ and $m > 0$, and

$$\left|\int_1^\infty \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt\right| \le \int_1^\infty \left|\frac{t^{s-1}e^{-mt}}{e^t-1}\right|dt \le \frac{1}{m^{\sigma-1}}\int_m^\infty e^{-t}t^{\sigma-2}\,dt. \tag{6.7}$$

By the change of variables $v = 1/t$ for $t > 0$,

$$\int_\alpha^1 \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt = \int_1^{1/\alpha} \frac{\left(\frac{1}{v}\right)^{s-1}e^{-m(1/v)}}{(e^{1/v}-1)v^2}\,dv \tag{6.8}$$

for $0 < \alpha < 1$. Here, by the inequality in (6.6), we can write

$$\left|\left(\frac{1}{v}\right)^{s-1}\frac{e^{-m(1/v)}}{(e^{1/v}-1)v^2}\right| < \left(\frac{1}{v}\right)^{\sigma-2}\frac{e^{-m/v}}{v^2} = v^{-\sigma}e^{-m/v} \le v^{-\sigma}. \tag{6.9}$$

But $\displaystyle\int_1^{1/\alpha} v^{-\sigma}\,dv = \frac{(1/\alpha)^{1-\sigma}-1}{1-\sigma}$, so $\displaystyle\lim_{\alpha\to 0+}\int_1^{1/\alpha} v^{-\sigma}\,dv = \frac{1}{\sigma-1}$ for $\sigma > 1$, i.e.,

$$\int_1^\infty \frac{\left(\frac{1}{v}\right)^{s-1}e^{-m(1/v)}}{(e^{1/v}-1)v^2}\,dv$$

converges absolutely for $\operatorname{Re} s > 1$ and

$$\left|\int_1^\infty \frac{\left(\frac{1}{v}\right)^{s-1}e^{-m(1/v)}}{(e^{1/v}-1)v^2}\,dv\right| \le \int_1^\infty \left|\frac{\left(\frac{1}{v}\right)^{s-1}e^{-m(1/v)}}{(e^{1/v}-1)v^2}\right|dv \le \int_1^\infty v^{-\sigma}e^{-m/v}\,dv,$$

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by the inequality in (6.9). The right-hand side equals $\lim_{\beta\to\infty}\int_1^\beta v^{-\sigma}e^{-m/v}dv$, or, by the change of variables $t = 1/v$,

$$\lim_{\beta\to\infty}\int_{1/\beta}^{1} t^{\sigma-2}e^{-mt}\,dt = \lim_{\beta\to\infty} m^{-(\sigma-1)}\int_{m/\beta}^{m} e^{-t}t^{\sigma-2}\,dt$$
$$= \frac{1}{m^{\sigma-1}}\int_0^m e^{-t}t^{\sigma-2}\,dt.$$

That is, by (6.8), $\displaystyle\lim_{\alpha\to 0+}\int_\alpha^1 \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt$ exists for $\operatorname{Re} s > 1$, and, with $\sigma = \operatorname{Re} s$,

$$\left|\lim_{\alpha\to 0+}\int_\alpha^1 \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt\right| \le \frac{1}{m^{\sigma-1}}\int_0^m e^{-t}t^{\sigma-2}\,dt. \tag{6.10}$$

We have therefore checked that the integral $I_m(s)$ defined by (6.5) converges for $\operatorname{Re} s > 1$ (and in fact the portion $\int_1^\infty t^{s-1}e^{-mt}/(e^t-1)\,dt$ converges absolutely for all $s \in \mathbb{C}$), and that, moreover, for $\sigma = \operatorname{Re} s$, we have

$$\left|\int_0^1 \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt\right| \le \frac{1}{m^{\sigma-1}}\int_0^m e^{-t}t^{\sigma-2}\,dt,$$
$$\left|\int_1^\infty \frac{t^{s-1}e^{-mt}}{e^t-1}\,dt\right| \le \frac{1}{m^{\sigma-1}}\int_m^\infty e^{-t}t^{\sigma-2}\,dt,$$

by the inequalities (6.7) and (6.10). This says that

$$\left|I_m(s)\right| \le \frac{1}{m^{\sigma-1}}\left(\int_0^m e^{-t}t^{\sigma-2}\,dt + \int_m^\infty e^{-t}t^{\sigma-2}\,dt\right)$$
$$= \frac{1}{m^{\sigma-1}}\int_0^\infty e^{-t}t^{\sigma-2}\,dt = \frac{1}{m^{\sigma-1}}\Gamma(\sigma-1),$$

by definition (1.6). Since $m^{\sigma-1}\to 0$ as $m\to\infty$ for $\sigma > 1$, we see that

$$\lim_{m\to\infty} I_m(s) = 0 \tag{6.11}$$

for $\operatorname{Re} s > 1$!

We move now to the second stage of the proof of Theorem 6.1, which is quite brief. Fix integers n, m with $1 \le n \le m$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, and set $v = nt$. Again by (1.6), $\int_0^\infty e^{-nt}t^{s-1}\,dt = n^{-s}\int_0^\infty e^{-v}v^{s-1}dv = \Gamma(s)/n^s$, so

$$\Gamma(s)\sum_{n=1}^{m}\frac{1}{n^s} = \int_0^\infty t^{s-1}\left(\sum_{n=1}^{m} e^{-nt}\right)dt = \int_0^\infty t^{s-1}\frac{e^{-t}(1-e^{-mt})}{1-e^{-t}}\,dt$$
$$= \int_0^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt + I_m(s).$$

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(Here the the last equality comes from (6.5) and the last but one from the formula for the partial sum of the geometric series.) Therefore

$$\Gamma(s)\sum_{n=1}^{m}\frac{1}{n^s}=\int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt+I_m(s). \tag{6.12}$$

We check the existence of the integral on the right-hand side for $\operatorname{Re} s>1$: write

$$\frac{t^{s-1}}{e^t-1}=t^{s-1}e^{-t}+\frac{t^{s-1}e^{-t}}{e^t-1}$$

for $t>0$; the integral $\int_0^\infty t^{s-1}e^{-t}\,dt=\Gamma(s)$ converges for $\operatorname{Re} s>0$ and the integral of the last term, which equals $I_1(s)$ by (6.5), converges for $\operatorname{Re} s>1$, as established in the first stage of the proof.

Let $m\to\infty$ in equation (6.12): then $\Gamma(s)\zeta(s)=\int_0^\infty \frac{t^{s-1}\,dt}{e^t-1}$ by (6.11) for $\operatorname{Re} s>1$, concluding the proof of Theorem 6.1.

COROLLARY 6.13. *For* $a>0$, $\operatorname{Re} s>1$,

$$\int_0^\infty \frac{t^{s-1}\,dt}{e^{at}-1}=\frac{\Gamma(s)\zeta(s)}{a^s}. \tag{6.14}$$

In particular, for $a>0$, $n=1,2,3,4,\ldots$,

$$\int_0^\infty \frac{t^{2n-1}}{e^{at}-1}\,dt=\frac{(-1)^{n+1}\,(2\pi/a)^{2n}\,B_{2n}}{4n}. \tag{6.15}$$

PROOF. This follows from (6.2) once the obvious change of variables $v=at$ is executed. By formula (6.14),

$$\int_0^\infty \frac{t^{2n-1}}{e^{at}-1}\,dt=\frac{\Gamma(2n)\zeta(2n)}{a^{2n}}$$

for $n\in\mathbb{Z}$, $n\ge 1$. Since $\Gamma(2n)=(2n-1)!$ (because $\Gamma(m)=(m-1)!$ for $m\in\mathbb{Z}, m\ge 1$), one can now appeal to formula (2.1) to conclude the proof of equation (6.15). □

As an application of formula (6.15) we shall compute Planck's *radiation density integral*. But first we provide some background.

On 14 December 1900, a paper written by Max Karl Ernst Ludwig Planck and entitled "On the theory of the energy distribution law of the normal spectrum" was presented to the German Physical Society. That date is considered to be the birthday of quantum mechanics, as that paper set forth for the first time the hypothesis that the energy of emitted radiation is *quantized*. Namely, the energy cannot assume arbitrary values but only integral multiples $0, h\nu, 2h\nu, 3h\nu, \ldots$ of the basic energy value $E=h\nu$, where ν is the frequency of the radiation and h is what is now called *Planck's constant*. We borrow a quotation from Hermann

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Weyl's notable book [37]: "The magic formula $E = h\nu$ from which the whole of quantum theory is developed, establishes a universal relation between the frequency ν of an oscillatory process and the energy E associated with such a process". The quantization of energy has profound consequences regarding the structure of matter.

Planck was led to his startling hypothesis while searching for a theoretical justification for his newly proposed formula for the energy density of thermal (or "blackbody") radiation. Lord Rayleigh had proposed earlier that year a theoretical explanation for the experimental observation that the rate of energy emission $f(\nu;T)$ by a body at temperature T in the form of electromagnetic radiation of frequency ν grows, under certain conditions, with the square of ν, and the *total* energy emitted grows with the fourth power of T. In the quantitative form derived by James Jeans a few years later, Rayleigh's formula reads

$$f(\nu;T) = \frac{8\pi\nu^2}{c^3}kT, \tag{6.16}$$

where c is the speed of light and k is Boltzmann's constant. As ν grows, however, this formula was known to fail. Wilhelm Wien had already proposed, in 1896, the empirically more accurate formula

$$f(\nu;T) = a\nu^3 e^{-b\nu/T}. \tag{6.17}$$

Unlike the Rayleigh–Jeans formula (6.16), Wien's avoids the "ultraviolet catastrophe". (This colorful name was coined later by Paul Ehrenfest for the notion that a functional form for $f(\nu;T)$ might yield an *infinite value* for the total energy, $\int_0^\infty f(\nu;T)d\nu = \infty$ — "ultraviolet" because the divergence sets in at high frequencies.) However, the lack of a theoretical explanation for Wien's law, and its wrong prediction for the asymptotic limit at low frequencies — proportional to ν^3 rather than ν^2 — made it unsatisfactory as well.

By October 1900 Planck had come up with a formula that had the right asymptotic behavior in both directions and was soon found to be very accurate:

$$f(\nu;T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}, \tag{6.18}$$

where the new constant h was introduced. In his December paper, already mentioned, he provides a justification for this formula using the earlier notions of electromagnetic oscillators and statistical-mechanical entropy, but invoking the additional assumption that the energy of the oscillators is restricted to multiples of $E_\nu = h\nu$. This then is the genesis of quantization.

Note that given the approximation $e^x \simeq 1+x$ for a very small value of x, one has the low frequency approximation $e^{h\nu/kT} - 1 \simeq h\nu/kT$, which when used

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in formula (6.18) gives $f(\nu;T) \simeq 8\pi\nu^2 kT/c^3$ — the Rayleigh–Jeans result (6.16), as expected by our previous remarks.

We now check that in contrast to an infinite total energy value implied by formula (6.16), integration over the full frequency spectrum via Planck's law (6.18) does yield a *finite* value. The result is:

PROPOSITION 6.19. *Planck's radiation density integral*

$$I(T) \stackrel{\text{def}}{=} \int_0^\infty f(\nu;T)d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT}-1}$$

(see (6.18)) *has the finite value* $8\pi^5 k^4 T^4/15c^3h^3$.

The proof is quite immediate. In formula (6.15) choose $n = 2$, $a = h/kT$; then

$$I(T) = \frac{8\pi h}{c^3}(-1)^3\left(\frac{2\pi kT}{h}\right)^4 \frac{B_4}{8}.$$

Since $B_4 = -1/30$ by (2.2), the desired value of $I(T)$ is achieved.

The radiation energy density $f(\nu;T)$ in (6.18) is related to the thermodynamics of the quantized harmonic oscillator; namely, it is related to the thermodynamic *internal energy* $U(T)$. We mention this because $U(T)$, in turn, is related to the *Helmholtz free energy* $F(T)$ which has, in fact, a zeta function connection. A quick sketch of this mix of ideas is as follows, where proofs and details can be found in my book on quantum mechanics [42] (along with some historic remarks).

One of the most basic, elementary facts of quantum mechanics is that the quantized harmonic oscillator of frequency ν has the sequence

$$\left\{E_n \stackrel{\text{def}}{=} \left(n+\tfrac{1}{2}\right)h\nu\right\}_{n=0}^{\infty}$$

as its energy levels. The corresponding *partition function* $Z(T)$ is given by

$$Z(T) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \exp\left(-\frac{E_n}{kT}\right), \tag{6.20}$$

where again T denotes temperature and k denotes Boltzmann's constant. This sum is easily computed:

$$Z(T) = \sum_{n=0}^{\infty} e^{-h\nu/2kT}\left(e^{-h\nu/kT}\right)^n = e^{-h\nu/2kT}\frac{1}{1-e^{-h\nu/kT}};$$

i.e.,

$$Z(T) = \frac{1}{2\sinh\dfrac{h\nu}{2kT}}. \tag{6.21}$$

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The importance of the partition function $Z(T)$ is that from it one derives basic thermodynamic quantities such as

the Hemholtz free energy $F(T) \overset{\text{def}}{=} -kT\log Z(T)$,
the entropy $S(T) \overset{\text{def}}{=} -\partial F/\partial T$,
the internal energy $U(T) \overset{\text{def}}{=} F(T)+TS(T) = F(T) - T\dfrac{\partial F}{\partial T}$.

Using (6.21), one computes that by these definitions

$$F(T) = \frac{h\nu}{2} + kT\log\Big(1-\exp\Big(-\frac{h\nu}{kT}\Big)\Big),$$
$$S(T) = k\Big(\frac{h\nu/kT}{\exp(h\nu/kT)-1} - \log\Big(1-\exp\Big(-\frac{h\nu}{kT}\Big)\Big)\Big), \tag{6.22}$$
$$U(T) = \frac{h\nu}{2} + \frac{h\nu}{\exp(h\nu/kT)-1},$$

which means that the factor $h\nu/(e^{h\nu/kT}-1)$ of $f(\nu;T)$ in equation (6.18) differs from the internal energy $U(T)$ *exactly* by the quantity $E_0 = \frac{1}{2}h\nu$, which is the *ground state* energy (also called the *zero-point* energy) of the quantized harmonic oscillator (since $E_n = (n+\frac{1}{2})h\nu$). However, our main interest is in setting up a free energy – zeta function connection. Here's how it goes.

For convenience let $\beta \overset{\text{def}}{=} 1/(kT)$ denote the inverse temperature. Form the *finite temperature zeta function*

$$\zeta(s;T) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} \frac{1}{(4\pi^2n^2 + h^2\nu^2\beta^2)^s}, \tag{6.23}$$

which turns out to be well-defined and holomorphic for $\operatorname{Re} s > \frac{1}{2}$. In [42] we show that $\zeta(s;T)$ has a meromorphic continuation to $\operatorname{Re} s < 1$ given by

$$\zeta(s;T) = \frac{\Gamma(s-\frac{1}{2})}{\sqrt{4\pi}\,\Gamma(s)a^{s-1/2}} + 2(\sqrt{a})^{1-2a}\frac{\sin\pi s}{\pi}\int_1^\infty \frac{(x^2-1)^{-s}}{\exp(\sqrt{a}x)-1}dx \tag{6.24}$$

for $a \overset{\text{def}}{=} h^2\nu^2\beta^2$. Moreover $\zeta(s;T)$ is holomorphic at $s=0$ and

$$\zeta'(0;T) = -\sqrt{a} - 2\log\big(1-\exp(-\sqrt{a})\big); \tag{6.25}$$

see Theorem 14.4 and Corollary 14.2 of [42]. By definition, $\sqrt{a} = h\nu\beta = h\nu/kT$. Therefore by formula (6.25) (which does require some work to derive from (6.24)), and by the first formula in (6.22), one discovers that

$$F(T) = -\frac{kT}{2}\zeta'(0;T), \tag{6.26}$$

which is the free energy – zeta function connection.

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We will meet the zero-point energy $E_0 = \frac{1}{2}h\nu$ again in the next lecture regarding the discussion of Casimir energy. In chapter 16 of [42] another finite temperature zeta function is set up in the context of Kaluza–Klein space-times with spatial sector $\mathbb{R}^m \times \Gamma\backslash G/K$, where Γ is a discrete group of isometries of the rank 1 symmetric space G/K; here K is a maximal compact subgroup of the semisimple Lie group G. In this broad context a partition function $Z(T)$ and free energy-zeta function connection still exist.

Lecture 7. Zeta regularization, spectral zeta functions, Eisenstein series, and Casimir energy

Zeta regularization is a powerful, elegant procedure that allows one to assign to a manifestly infinite quantitiy a *finite* value by providing it a special value zeta interpretation. Such a procedure is therefore of enormous importance in physics, for example, where infinities are prolific. As a simple example, we consider the sum $S = 1 + 2 + 3 + 4 + \cdots = \sum_{n=1}^{\infty} n$, which is obviously infinite. This sum arises naturally in string theory — in the discussion of *transverse Virasoro operators*, for example. A *string* (which replaces the notion of a *particle* in quantum theory, at the Plancktian scale 10^{-33} cm) sweeps out a surface called a *world-sheet* as it moves in d-dimensional space-time $\mathbb{R}^d = \mathbb{R}^1 \times \mathbb{R}^{d-1}$ - in contrast to a world-line of a point-particle. For Bosonic string theory (where there are no *fermions*, but only *bosons*) certain Virasoro constraints force d to assume a specific value. Namely, the condition

$$1 = -\left(\frac{d-2}{2}\right) S \tag{7.1}$$

arises, which as we shall see forces the critical dimension $d = 26$, 1 being the value of a certion *normal ordering* constant. In fact, we write $S = \sum_{n=1}^{\infty} 1/n^s$, where $s = -1$, which means that it is natural to reinterpret S as the special zeta value $\zeta(-1)$. Thus we *zeta regularize* the infinite quantity S by assigning to it the value $-\frac{1}{12}$, according to (2.14). Then by condition (7.1), indeed we must have $d = 26$.

Interestingly enough, the "strange" equation

$$1 + 2 + 3 + 4 + \cdots = -\tfrac{1}{12}, \tag{7.2}$$

which we now understand to be perfectly meaningful, appears in a paper of Ramanujan — though he had no knowledge of the zeta function. It was initially dismissed, of course, as ridiculous and meaningless.

As another simple example we consider "$\infty!$", that is, the product $P = 1 \cdot 2 \cdot 3 \cdot 4 \cdots$, which is also infinite. To zeta regularize P, we consider first $\log P = \sum_{n=1}^{\infty} \log n$ (which is still infinite), and we note that since $\zeta'(s) =$

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$-\sum_{n=1}^{\infty}(\log n)n^{-s}$ for $\operatorname{Re} s > 1$, by equation (1.1), if we *illegally* take $s = 0$ the *false* result $-\zeta'(0) = \sum_{n=1}^{\infty} \log n = \log P$ follows. However the left-hand side here is well-defined and in fact it has the value $\frac{1}{2}\log 2\pi$ by equation (2.6). The finite value $\frac{1}{2}\log 2\pi$, therefore, is naturally assigned to $\log P$ and, consequently, we define

$$P = \prod_{n=1}^{\infty} n = \infty! \stackrel{\text{def}}{=} e^{-\zeta'(0)} = e^{\frac{1}{2}\log 2\pi} = \sqrt{2\pi}. \tag{7.3}$$

More complicated products can be regularized in a somewhat similar manner. A typical set-up for this is as follows. One has a compact smooth manifold M with a Riemannian metric g, and therefore a corresponding *Laplace–Beltrami operator* $\Delta = \Delta(g)$ where $-\Delta$ has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \qquad \lim_{j\to\infty} \lambda_j = \infty. \tag{7.4}$$

If n_j denotes the (finite) multiplicity of the j-th eigenvalue λ_j of $-\Delta$, then one can form the corresponding *spectral zeta function* (cf. definition (0.1))

$$\zeta_M(s) = \sum_{j=1}^{\infty} \frac{n_j}{\lambda_j^s} \tag{7.5}$$

which is well-defined for $\operatorname{Re} s > \frac{1}{2}\dim M$, due to the discovery by H. Weyl of the asymptotic result $\lambda_j \sim j^{2/\dim M}$, as $j \to \infty$. S. Minakshisundaram and A. Pleijel [26] showed that $\zeta_M(s)$ admits a meromorphic continuation to the complex plane and that, in particular, $\zeta_M(s)$ is holomorphic at $s = 0$. Thus $e^{-\zeta_M(0)}$ is well-defined and, as in definition (7.3), we set

$$\det{}' -\Delta = \prod_{j=1}^{\infty} \lambda_j^{n_j} \stackrel{\text{def}}{=} e^{-\zeta_M'(0)}, \tag{7.6}$$

where the prime $'$ here indicates that the product of eigenvalues (which in finite dimensions corresponds to the determinant of an operator) is taken over the nonzero ones. Indeed, similar to the preceding example with the infinite product $P = \prod_{j=1}^{\infty} j$, the formal, illegal computation

$$\begin{aligned}\exp\left(-\frac{d}{ds}\sum_{j=1}^{\infty}\frac{n_j}{\lambda_j^s}\bigg|_{s=0}\right) &= \exp\left(\sum_{j=1}^{\infty}\frac{n_j}{\lambda_j^s}\log\lambda_j\bigg|_{s=0}\right) = \exp\left(\sum_{j=1}^{\infty} n_j \log\lambda_j\right) \\ &= \prod_{j=1}^{\infty} e^{n_j \log\lambda_j} = \prod_{j=1}^{\infty}\lambda_j^{n_j}\end{aligned} \tag{7.7}$$

serves as the motivation for definition (7.6). Clearly this definition of determinant makes sense for more general operators (with a discrete spectrum) on other

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infinite-dimensional spaces. It is useful, moreover, for Laplace-type operators on smooth sections of a vector bundle over M.

The following example is more involved, where M is a complex torus (in fact M is assumed to be the world-sheet of a bosonic string; see Appendix C of [42], for example). For a fixed complex number $\tau = \tau_1 + i\tau_2$ in the upper half-plane ($\tau_2 > 0$) and for the corresponding integral lattice,

$$L_\tau \overset{\text{def}}{=} \{a + b\tau \mid a, b \in \mathbb{Z}\}, \qquad M \overset{\text{def}}{=} \mathbb{C} \backslash L_\tau. \tag{7.8}$$

In this case it is known that $-\Delta$ has a multiplicity free spectrum (i.e., every $n_j = 1$) given by

$$\left\{\lambda_{mn} \overset{\text{def}}{=} \frac{4\pi^2}{\tau_2^2}|m + n\tau|^2\right\}_{m,n\in\mathbb{Z}},$$

and consequently the corresponding spectral zeta function of (7.5) is given by

$$\zeta_M(s) = \frac{\tau_2^s}{(4\pi^2)^s} E^*(s, \tau) \tag{7.9}$$

for Re $s > 1$, where

$$E^*(s,\tau) \overset{\text{def}}{=} \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{\tau_2^s}{|m + n\tau|^{2s}} \tag{7.10}$$

(with $\mathbb{Z}_*^2 = \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}$ as before) is a standard *nonholomorphic Eisenstein series*. That is, in contrast to the series $G_k(\tau)$ in definition (4.4), $E^*(s, \tau)$ is not a holomorphic function of τ. As a function of s, it is a standard fact that $E^*(s, \tau)$, which is holomorphic for Re $s > 1$, admits a meromorphic continuation to the full complex plane, with a simple pole at $s = 1$ as its only singularity. Hence, by equation (7.9), the same assertion holds for $\zeta_M(s)$. By [7; 14; 35; 38], for example, the continuation of $E^*(s, \tau)$ is given by

$$\begin{aligned} E^*(s,\tau) &= 2\zeta(2s)\tau_2^s + 2\zeta(2s-1)\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\tau_2^{-s+1} \\ &\quad + \frac{4\pi^s}{\Gamma(s)}\tau_2^{1/2}\sum_{m=1}^\infty\sum_{n=1}^\infty e^{-2\pi i m n\tau_1}\left(\frac{n}{m}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi m n\tau_2) \\ &\qquad + \frac{4\pi^s}{\Gamma(s)}\tau_2^{1/2}\sum_{m=1}^\infty\sum_{n=1}^\infty e^{2\pi i m n\tau_1}\left(\frac{n}{m}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi m n\tau_2) \end{aligned} \tag{7.11}$$

for Re $s > 1$, where

$$K_\nu(x) \overset{\text{def}}{=} \frac{1}{2}\int_0^\infty \exp\left(-\frac{x}{2}\left(t + \frac{1}{t}\right)\right) t^{\nu-1}\, dt \tag{7.12}$$

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is the Macdonald–Bessel (or K-Bessel) function for $\nu \in \mathbb{C}$, $x > 0$. Introducing the divisor function $\sigma_\nu(n) \overset{\text{def}}{=} \sum_{0<d,\ d\,|\,n} d^\nu$, and using forumla (D.11) on page 92 on the entire functions of s appearing in the last two sums, we can rewrite (7.11) as

$$
\begin{aligned}
&= 2\zeta(2s)\tau_2^s + 2\zeta(2s-1)\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\tau_2^{-s+1} \\
&\quad + \frac{4\pi^s}{\Gamma(s)}\tau_2^{1/2}\sum_{n=1}^{\infty}\sigma_{-2s+1}(n)e^{-2\pi n\tau_1 i}K_{s-\frac{1}{2}}(2\pi n\tau_2)n^{s-\frac{1}{2}} \\
&\qquad + \frac{4\pi^s}{\Gamma(s)}\tau_2^{1/2}\sum_{n=1}^{\infty}\sigma_{-2s+1}(n)e^{2\pi n\tau_1 i}K_{s-\frac{1}{2}}(2\pi n\tau_2)n^{s-\frac{1}{2}}. \qquad (7.13)
\end{aligned}
$$

The sum of the first two terms in (7.13) has $s = \frac{1}{2}$ as a removable singularity. To see this, note first that since $\zeta(s)$ has residue $= 1$ at $s = 1$,

$$
\lim_{s\to\frac{1}{2}}\left(s-\tfrac{1}{2}\right)\zeta(2s)\tau_2^s = \tfrac{1}{2}\lim_{z\to 1}(z-1)\zeta(z)\tau_2^{z/2}
$$

(for $z = 2s$), and this equals $\tau_2^{1/2}/2$. We also have $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\zeta(0) = -\frac{1}{2}$ (by (2.5)), and $w\Gamma(w) = \Gamma(w+1)$. Thus

$$
\left(\frac{z-1}{2}\right)\Gamma\left(\frac{z-1}{2}\right) = \Gamma\left(\frac{z+1}{2}\right);
$$

hence

$$
\begin{aligned}
&\lim_{s\to\frac{1}{2}}\left(s-\tfrac{1}{2}\right)\zeta(2s-1)\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\tau_2^{-s+1} \\
&\qquad = \lim_{z\to 1}\zeta(z-1)\sqrt{\pi}\left(\frac{z-1}{2}\right)\frac{\Gamma\left(\frac{z-1}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}\tau_2^{-z/2+1} \\
&\qquad = \zeta(0)\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2})}\lim_{z\to 1}\Gamma\left(\frac{z+1}{2}\right)\tau_2^{1/2} = -\tau_2^{1/2}/2.
\end{aligned}
$$

It follows that the limit as $s \to \frac{1}{2}$ of $(s-\frac{1}{2}) \times$ the first two terms in (7.13) vanishes, as desired. This proves our claim that $s = 1$ is the only singularity of $E^*(s,\tau)$, which arises as a simple pole from the second term, $2\zeta(2s-1) \times \sqrt{\pi}\Gamma(s-\frac{1}{2})/\Gamma(s)\tau_2^{-s+1}$, in (7.13), due to the factor $\zeta(2s-1)$. Moreover the

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residue at $s = 1$ can be easily evaluated setting $z = 2s - 1$:

$$\lim_{s\to 1}(s-1)2\zeta(2s-1)\sqrt{\pi}\,\Gamma(s-\tfrac{1}{2})/\Gamma(s)\tau_2^{-s+1}$$
$$= \lim_{z\to 1}(z-1)\zeta(z)\sqrt{\pi}\Big(\Gamma\Big(\frac{z}{2}\Big)/\Gamma\Big(\frac{z+1}{2}\Big)\Big)\tau_2^{-z/2+1/2}$$
$$= \sqrt{\pi}(\sqrt{\pi}/1) = \pi.$$

From (7.13) we also get $E^*(0,\tau) = 2\zeta(2s)\tau_2^s\big|_{s=0}$:

$$E^*(0,\tau) = -1. \tag{7.14}$$

Moreover, the functional equation

$$\frac{E^*(1-s,\tau)}{\Gamma(s)} = \frac{\pi^{1-2s}E^*(s,\tau)}{\Gamma(1-s)}, \tag{7.15}$$

say for $s \neq 0, 1$, follows since $\sigma_\nu(n)$, $K_\nu(x)$ satisfy the functional equations

$$\sigma_\nu(n) = n^\nu\sigma_{-\nu}(n), \quad K_\nu(x) = K_{-\nu}(x), \tag{7.16}$$

and since $\zeta(s)$ satisfies the functional equation (1.16). The second equation in (7.16) follows by the change of variables $u = 1/t$ in definition (7.12), and the first equation is the formula

$$\sigma_\nu(n) = n^\nu \sum_{\substack{0<d\\ d\,|\,n}} \frac{1}{d^\nu},$$

which we checked in remarks following the statement of Theorem 4.23. Namely, $d > 0$ runs through the divisors of n as $\frac{n}{d}$ does. We check equation (7.15) by replacing s by $1-s$ in equation (7.13). By (7.16), $\sigma_{-2(1-s)+1}(n)n^{1-s-\frac{1}{2}} = \sigma_{-2s+1}n^{s-\frac{1}{2}}$ and $K_{1-s-\frac{1}{2}}(2\pi n\tau_2) = K_{s-\frac{1}{2}}(2\pi n\tau_2)$; hence

$$\frac{4\pi^{1-s}}{\Gamma(1-s)}\tau_2^{1/2}\sum_{n=1}^{\infty}\sigma_{-2(1-s)+1}(n)e^{\pm 2\pi n\tau_1 i}K_{1-s-\frac{1}{2}}(2\pi n\tau_2)n^{1-s-\frac{1}{2}} =$$
$$\frac{4\pi^{1-s}}{\Gamma(1-s)}\tau_2^{1/2}\sum_{n=1}^{\infty}\sigma_{-2s+1}(n)e^{\pm 2\pi n\tau_1 i}K_{s-\frac{1}{2}}(2\pi n\tau_2)n^{s-\frac{1}{2}}. \tag{7.17}$$

In equation (1.16) replace s by $2s$ and $2s-1$ separately, to obtain

$$\zeta(1-2s) = \frac{\pi^{-2s+\frac{1}{2}}\Gamma(s)\zeta(2s)}{\Gamma(\frac{1}{2}-s)},$$
$$\zeta(2-2s) = \zeta(1-(2s-1)) = \pi^{1/2}\pi^{1-2s}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(1-s)}$$

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say for $2s, 2s-1 \neq 0, 1$ (i.e., $s = 0, \frac{1}{2}, 1$). This gives

$$2\zeta(2(1-s))\tau_2^{1-s} + 2\zeta(2(1-s)-1)\sqrt{\pi}\frac{\Gamma(1-s-\frac{1}{2})}{\Gamma(1-s)}\tau_2^{-(1-s)+1}$$
$$= 2\sqrt{\pi}\pi^{1-2s}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(1-s)}\tau_2^{-s+1} + \frac{2\pi^{-2s+\frac{1}{2}}\Gamma(s)\zeta(2s)\sqrt{\pi}\Gamma(\frac{1}{2}-s)}{\Gamma(\frac{1}{2}-s)\Gamma(1-s)}\tau_2^s. \tag{7.18}$$

By (7.13), (7.17), (7.18), we have for $\dfrac{E^*(1-s,\tau)}{\Gamma(s)}$ the value

$$\frac{\pi^{1-2s}}{\Gamma(1-s)}\left(2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)\tau_2^{-s+1} + 2\zeta(2s)\tau_2^s\right)$$
$$+ \frac{\pi^{1-2s}}{\Gamma(1-s)}\left(\frac{4\pi^s}{\Gamma(s)}\tau_2^{1/2}\sum_{n=1}^{\infty}\sigma_{-2s+1}(n)e^{-2\pi n\tau_1 i}K_{s-\frac{1}{2}}(2\pi n\tau_2)n^{s-\frac{1}{2}} + \frac{4\pi^s}{\Gamma(s)}\tau_2^{1/2}\sum_{n=1}^{\infty}\sigma_{-2s+1}(n)e^{2\pi n\tau_1 i}K_{s-\frac{1}{2}}(2\pi n\tau_2)n^{s-\frac{1}{2}}\right)$$
$$= \frac{\pi^{1-2s}}{\Gamma(1-s)}E^*(s,\tau)$$

which gives (7.15), as desired. (Note that we have stayed away from $s = 0, \frac{1}{2}, 1$; but equation (7.15) clearly holds for $s = \frac{1}{2}$.)

One other result is needed in order to compute $\zeta_M'(0)$:

THEOREM 7.19 (KRONECKER'S FIRST LIMIT FORMULA).

$$\lim_{s\to 1}\left(E^*(s,\tau) - \frac{\pi}{s-1}\right) = 2\pi\left(\gamma - \log 2 - \log \tau_2^{1/2}\,|\eta(\tau)|^2\right). \tag{7.20}$$

where γ is the Euler–Mascheroni constant in definition (1.26), *and where $\eta(\tau)$ is the Dedekind eta function in definition* (3.27).

Formula (7.20) compares with the limit result (1.25), though it is a more involved result; see [35].

For $f(z) \overset{\text{def}}{=} \Gamma(z)\pi^{2-2z}/\Gamma(2-z)$

$$f'(1) = -2\log\pi + 2\Gamma'(1) = -2\log\pi - 2\gamma \tag{7.21}$$

by the quotient rule. By equations (7.14), (7.15), (7.21) and Theorem 7.19, and the fact that $(1-z)\Gamma(1-z) = \Gamma(2-z)$ (i.e., $w\Gamma(w) = \Gamma(w+1)$), we have,

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with $z = 1 - s$,

$$
\begin{aligned}
\frac{\partial E^*}{\partial s}(0,\tau) &\overset{\text{def}}{=} \lim_{s\to 0} \frac{E^*(s,\tau)+1}{s} = \lim_{z\to 1} \frac{E^*(1-z,\tau)+1}{1-z} \\
&= \lim_{z\to 1}\left(\frac{E^*(1-z,\tau)\Gamma(z)}{\Gamma(z)(1-z)} - \frac{1}{z-1}\right) \\
&= \lim_{z\to 1}\left(\frac{\Gamma(z)}{(1-z)}\pi^{1-2z}\frac{E^*(z,\tau)}{\Gamma(1-z)} - \frac{1}{z-1}\right) \\
&= \lim_{z\to 1}\left(\frac{\Gamma(z)\pi^{1-2z}}{\Gamma(2-z)}\left(E^*(z,\tau)-\frac{\pi}{z-1}\right) + \frac{\Gamma(z)\pi^{1-2z}}{\Gamma(2-z)}\frac{\pi}{z-1} - \frac{1}{z-1}\right) \\
&\overset{\text{def}}{=} 2[\gamma - \log 2 - \log \tau_2^{1/2}\,|\eta(\tau)|^2] + \left[\lim_{z\to 1}\frac{f(z)-f(1)}{z-1} = f'(1)\right] \\
&= -2\log 2 - 2\log \tau_2^{1/2}\,|\eta(\tau)|^2 - 2\log\pi \\
&= -\log 4\pi^2\tau_2\,|\eta(\tau)|^4\,,
\end{aligned}
$$

which, with equations (7.9), (7.14) gives (finally)

$$\zeta_M'(0) = -\log \tau_2^2\,|\eta(\tau)|^4\,. \tag{7.22}$$

Hence, by definition (7.6), the regularized determinant is given by

$$\det{}' -\Delta = \tau_2^2\,|\eta(\tau)|^4\,. \tag{7.23}$$

One is actually interested in the power $(\det' -\Delta)^{-d/2} = \tau_2^{-d}\,|\eta(\tau)|^{-2d}$, where $d = 26$ is the critical dimension mentioned above. This power represents a *one-loop* contribution to the "sum of embeddings" of the string world sheet (the complex torus in definition (7.8)) into the target space $\mathbb{R}^{26}$.

In the next lecture we shall make use, similarly, of the meromorphic continuation of the generalized Epstein zeta function

$$E(s,m;\vec{a},\vec{b}) \overset{\text{def}}{=} \sum_{\vec{n}=(n_1,n_2,\dots,n_d)\in\mathbb{Z}^d} \left(a_1(n_1-b_1)^2+\cdots+a_d(n_d-b_d)^2+m^2\right)^{-s} \tag{7.24}$$

for $\operatorname{Re} s > d/2$, $m > 0$, $\vec{a} = (a_1,\dots,a_d)$, $\vec{b} = (b_1,\dots,b_d) \in \mathbb{R}^d$, $a_i > 0$. The result is, setting $\mathbb{Z}_*^d = \mathbb{Z}^d - \{0\}$,

$$
\begin{aligned}
E(s,m;\vec{a},\vec{b}) &= \frac{\pi^{d/2}\Gamma(s-\frac{d}{2})m^{d-2s}}{\sqrt{a_1a_2\cdots a_d}\;\Gamma(s)} + \frac{2\pi^s m^{d/2-s}}{\sqrt{a_1a_2\cdots a_d}\;\Gamma(s)} \\
&\quad\times \sum_{\vec{n}\in\mathbb{Z}_*^d} e^{2\pi i\sum_{j=1}^d n_jb_j}\left(\sum_{j=1}^d \frac{n_j^2}{a_j}\right)^{\frac{s-d/2}{2}} K_{\frac{d}{2}-s}\left(2\pi m\left(\sum_{j=1}^d\frac{n_j^2}{a_j}\right)^{\frac{1}{2}}\right)
\end{aligned} \tag{7.25}
$$

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for Re $s > d/2$. In particular, we shall need the special value $E(\frac{d+1}{2}, m; \vec{a}, \vec{b})$, which we now compute. For $s = \frac{d+1}{2}$ the first term on the right in (7.25) is

$$\frac{\pi^{\frac{d+1}{2}}}{m\left(\prod_{j=1}^{d} a_j\right)^{1/2} \Gamma\left(\frac{d+1}{2}\right)},$$

since $\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$. Also

$$K_{\frac{d}{2}-s}(x) = K_{-\frac{1}{2}}(x) = K_{\frac{1}{2}}(x),$$

by (7.16). This equals $\sqrt{\pi/2x}\, e^{-x}$ for $x > 0$; hence

$$\left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{\frac{s-d/2}{2}} K_{\frac{d}{2}-s}\left(2\pi m \left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{\frac{1}{2}}\right)$$
$$= \left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{1/4} \sqrt{\frac{\pi}{2}} (2\pi m)^{-\frac{1}{2}} \left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{-\frac{1}{4}} \exp\left(-2\pi m \left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{\frac{1}{2}}\right).$$

Therefore the second term on the right-hand side of (7.25) is

$$\frac{2\pi^{\frac{d+1}{2}} m^{-1/2}}{\left(\prod_{j=1}^{d} a_j\right)^{1/2} \Gamma\left(\frac{d+1}{2}\right)} \sum_{\vec{n} \in \mathbb{Z}_*^d} \frac{1}{2\sqrt{m}} \exp\left(2\pi i \sum_{j=1}^{d} n_j b_j\right) \exp\left(-2\pi m \left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{\frac{1}{2}}\right).$$

That is:

$$E\left(\frac{d+1}{2}, m; \vec{a}, \vec{b}\right) = \frac{\pi^{\frac{d+1}{2}}}{m\left(\prod_{j=1}^{d} a_j\right)^{1/2} \Gamma\left(\frac{d+1}{2}\right)}$$
$$+ \frac{\pi^{\frac{d+1}{2}}}{m\left(\prod_{j=1}^{d} a_j\right)^{1/2} \Gamma\left(\frac{d+1}{2}\right)} \sum_{\vec{n} \in \mathbb{Z}_*^d} e^{2\pi i \vec{n}\cdot\vec{b}} \exp\left(-2\pi m \left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{\frac{1}{2}}\right)$$
$$= \frac{\pi^{\frac{d+1}{2}}}{m\left(\prod_{j=1}^{d} a_j\right)^{1/2} \Gamma\left(\frac{d+1}{2}\right)} \sum_{\vec{n} \in \mathbb{Z}^d} e^{2\pi i \vec{n}\cdot\vec{b}} \exp\left(-2\pi m \left(\sum_{j=1}^{d} \frac{n_j^2}{a_j}\right)^{\frac{1}{2}}\right). \tag{7.26}$$

As a final example we consider the zeta regularization of Casimir energy, after a few general remarks.

In Lecture 6 it was observed that the sequence $\{E_n \overset{\text{def}}{=} (n + \frac{1}{2})h\nu\}$ is the sequence of energy levels of the quantized harmonic oscillator of frequency ν, where h denotes Planck's constant. In particular there exists a nonvanishing ground state energy (also called the *zero-point energy*) given by $E_0 = \frac{1}{2}h\nu$. Zero-point energy is a prevalent notion in physics, from quantum field theory (QFT), where it is also referred to as *vacuum energy*, to cosmology (concerning

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issues regarding the cosmological constant, for example), and in between. Based on Planck's radiation density formula (6.18), A. Einstein and O. Stern concluded (in 1913) that even at zero absolute temperature, atomic systems maintain an energy of the amount $E_0 = \frac{1}{2}h\nu$. It is quite well experimentally established that a vacuum (empty space) contains a large supply of residual energy (zero-point energy). Vacuum fluctuations is a large scale study. The energy due to vacuum distortion (*Casimir energy*), for example, was considered by H. Casimir and D. Polder in a 1948 ground-breaking study. Here the vacuum energy was modified by the introduction of a pair of uncharged, parallel, conducting metal plates. A striking prediction emerged: the prediction of the existence of a force of a purely quantum mechanical origin — one arising from zero-point energy changes of harmonic oscillators that make up the normal modes of the electromagnetic field. This force, which has now been measured experimentally by M. Spaarnay, S. Lamoreaux, and others, is called the *Casimir force*.

Casimir energy in various contexts has been computed by many Physicists, including some notable calculations by the co-editor Klaus Kirsten. We refer to his book [22] for much more information on this, and on related matters - a book with 424 references. The author has used the *Selberg trace formula* for general compact space forms $\Gamma \backslash G/K$, mentioned in Lecture 6, of rank-one symmetric spaces G/K to compute the Casimir energy in terms of the *Selberg zeta function* [40; 39]. This was done by Kirsten and others in some special cases.

Consider again a compact smooth Riemannian manifold (M, g) with discrete spectrum of its Laplacian $-\Delta(g)$ given by (7.4). In practice, M is the spatial sector of a space-time manifold $\mathbb{R} \times M$ with metric $-dt^2 + g$. Formally, the Casimir energy in this context is given by the infinite quantity

$$E_C = \frac{1}{2}\sum_{j=1}^{\infty} n_j \lambda_j^{1/2}, \tag{7.27}$$

up to some omitted factors like h. It is quite clear then how to regularize E_C. Namely, consider $\lambda_j^{1/2}$ as $1/\lambda_j^s$ for $s = -\frac{1}{2}$ and therefore assign to E_C the meaning

$$E_C = \tfrac{1}{2}\zeta_M(-1/2), \tag{7.28}$$

where $\zeta_M(s)$ is the spectral zeta function of definition (7.5), meromorphically continued. If dim M is even, for example, the poles of $\zeta_M(s)$ are simple, finite in number, and can occur only at one of points $s = 1, 2, 3, \ldots, d/2$ (see [26]), in which case E_C in (7.28) is surely a well-defined, finite quantity. However if dim M is odd, $\zeta_M(s)$ will generally have infinitely many simple poles — at the points $s = \frac{1}{2}\dim M - n$, for $0 \le n \in \mathbb{Z}$. This would include the point $s = -\frac{1}{2}$ if dim $M = 5$ and $n = 3$, for example. Assume therefore that dim M is even.

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When M is one of the above compact space forms, for example, then (based on the results in [41]) E_C can be expressed explicitly in terms of the structure of Γ and the spherical harmonic analysis of G/K — and in terms of the Selberg zeta function attached to $\Gamma\backslash G/K$, as just mentioned. Details of this are a bit too technical to mention here; we have already listed some references. We point out only that by our assumptions on M, the corresponding Lie group pairs (G, K) are given by

$$\begin{aligned}
&G = \mathrm{SO}_1(m,1), && K = \mathrm{SO}(m), && m \geq 2,\\
&G = \mathrm{SU}(m,1), && K = U(m), && m \geq 2,\\
&G = \mathrm{SP}(m,1), && K = \mathrm{SP}(m) \times \mathrm{SP}(1), && m \geq 2,\\
&G = F_{4(-20)}, && K = \mathrm{Spin}(9), &&
\end{aligned}$$

where $F_{4(-20)}$ is a real form of the complex Lie group with exceptional Lie algebra F_4 with Dynkin diagram 0—0 = 0—0. More specifically, $F_{4(-20)}$ is the unique real form for which the difference $\dim G/K - \dim K$ assumes the value -20.

In addition to the reference [22], the books [13; 12] are a good source for information on and examples of Casimir energy, and for applications in general of zeta regularization.

Lecture 8. Epstein zeta meets gravity in extra dimensions

We compute the Kaluza–Klein modes of the 4-dimensional gravitational potential V_{4+d} in the presence of d extra dimensions compactified on a d-torus. The result is known of course [3; 21], but we present here an argument based on the special value $E\big(\frac{d+1}{2}, m; \vec{a}, \vec{b}\big)$ computed in equation (7.26) of the generalized Epstein zeta function $E(s, m; \vec{a}, \vec{b})$ defined in (7.24).

G. Nordström in 1914 and T. Kaluza (independently) in 1921 were the first to unify Einstein's 4-dimensional theory of gravity with Maxwell's theory of electromagnetism. They showed that 5-dimensional general relativity contained both theories, but under an assumption that was somewhat artificial - the so-called "cylinder condition" that in essence restricted physicality of the fifth dimension. O. Klein's idea was to compactify that dimension and thus to render a plausible physical basis for the cylinder assumption.

Consider, for example, the fifth dimension (the "extra dimension") compactified on a circle Γ. This means that instead of considering the Einstein gravitational field equations

$$R_{ij}(g) - \frac{g_{ij}}{2} R(g) - \Lambda g_{ij} = -\frac{8\pi G}{c^4} T_{ij} \tag{8.1}$$

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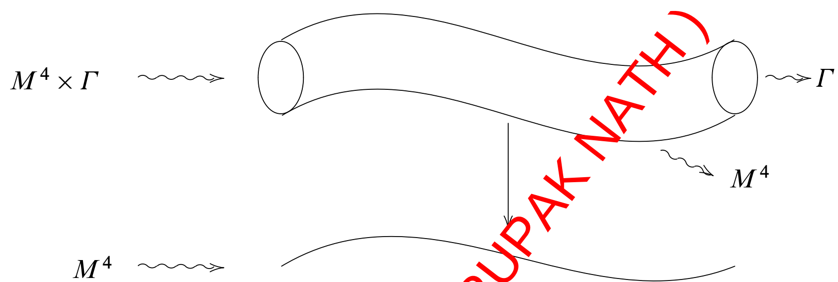
on a 4-dimensional space-time M^4 [8; 11], one considers these equations on the 5-dimensional product $M^4 \times \Gamma$. In (8.1), $g = [g_{ij}]$ is a Riemannian metric (the solution of the Einstein equations) with Ricci tensor $R_{ij}(g)$ and scalar curvature $R(g)$, Λ is a cosmological constant, and T_{ij} is an energy momentum tensor which describes the matter content of space-time — the left-hand side of (8.1) being pure geometry; G is the Newton constant and c is the speed of light. Given the non-observability of the fifth dimension, however, one takes Γ to be extremely small, say with an extremely small radius $R > 0$. Geometrically we have a fiber bundle $M^4 \times \Gamma \to M^4$ with structure group Γ.



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On all "fields" $F(x, \theta) : M^4 \times \mathbb{R} \to \mathbb{C}$ on $M^4 \times \mathbb{R}$ there is imposed, moreover, periodicity in the second variable:

$$F(x, \theta + 2\pi R) = F(x, \theta) \tag{8.2}$$

for $(x, \theta) \in M^4 \times \mathbb{R}$.

For $n \in \mathbb{Z}$ and $f(x)$ on M^4 fixed, the function $F_{n,f}(x, \theta) \stackrel{\text{def}}{=} f(x)e^{in\theta/R}$ is an example of a field on $M^4 \times \mathbb{R}$ that satisfies equation (8.2). For a general field $F(x, \theta)$, subject to reasonable conditions, and the periodicity condition (8.2), one would have a Fourier series expansion

$$F(x, \theta) = \sum_{n \in \mathbb{Z}} F_{n, f_n} \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} f_n(x) e^{in\theta/R} \tag{8.3}$$

in which case the functions $f_n(x)$ are called *Kaluza–Klein modes* of $F(x, \theta)$.

Next we consider d extra dimensions compactified on a d-torus

$$\Gamma^d \stackrel{\text{def}}{=} \Gamma_1 \times \cdots \times \Gamma_d,$$

where the Γ_j are circles with extremely small radii $R_j > 0$, and we consider a field $V_{4+d}(x, y, z, x_1, \ldots, x_d)$ on $(\mathbb{R}^3 - \{0\}) \times \mathbb{R}^d$. Thus $\mathbb{R}^3 - \{0\}$ replaces M^4,

Γ^d replaces Γ, and $\vec{x} \overset{\text{def}}{=} (x_1, \ldots, x_d) \in \mathbb{R}^d$ replaces θ in the previous discussion. The field is given by

$$V_{4+d}(x, y, z, x_1, \ldots, x_d) \overset{\text{def}}{=} -MG_{4+d} \sum_{\vec{n}=(n_1,\ldots,n_d)\in\mathbb{Z}^d} \frac{1}{\left(r^2 + \sum_{j=1}^{d} (x_j - 2\pi n_j R_j)^2\right)^{\frac{d+1}{2}}}, \tag{8.4}$$

which is the gravitational potential due to extra dimensions of an object of mass M at a distance $\left(r^2 + \sum_{j=1}^{d} x_j^2\right)^{1/2}$ for $r^2 \overset{\text{def}}{=} x^2 + y^2 + z^2$; here G_{4+d} is the $(4+d)$-dimensional Newton constant. Note that, analogously to equation (8.2),

$$\begin{aligned} V_{4+d}&(x, y, z, x_1 + 2\pi R_1, \ldots, x_d + 2\pi R_d) \\ &\overset{\text{def}}{=} -MG_{4+d} \sum_{(n_1,\ldots,n_d)\in\mathbb{Z}^d} \frac{1}{\left(r^2 + \sum_{j=1}^{d} (x_j - 2\pi(n-1)jR_j)^2\right)^{\frac{d+1}{2}}} \\ &= -MG_{4+d} \sum_{(n_1,\ldots,n_d)\in\mathbb{Z}^d} \frac{1}{\left(r^2 + \sum_{j=1}^{d} (x_j - 2\pi n_j R_j)^2\right)^{\frac{d+1}{2}}} \\ &= V_{4+d}(x, y, z, x_1, \ldots, x_d), \end{aligned} \tag{8.5}$$

where of course we have used that $n_j - 1$ varies over $\mathbb{Z}$ as n_j does. Thus, analogously to equation (8.3), we look for a Fourier series expansion

$$V_{4+d}(x, y, z, x_1, \ldots, x_d) = \sum_{\vec{n}\in\mathbb{Z}^d} f_{\vec{n}}(x, y, z) \exp\left(i\vec{n} \cdot \overrightarrow{\left(\frac{x_1}{R_1}, \ldots, \frac{x_d}{R_d}\right)}\right), \tag{8.6}$$

where the functions $f_{\vec{n}}(x, y, z)$ on $\mathbb{R}^3 - \{0\}$ would be called the Kaluza–Klein modes of $V_{4+d}(x, y, z, \vec{x} = (x_1, \ldots, x_d))$.

It is easy, in fact, to establish the expansion (8.6) and to compute the modes $f_{\vec{n}}(x, y, z)$ explicitly. For this, define

$$a_j \overset{\text{def}}{=} (2\pi R_j)^2 > 0, \qquad b_j \overset{\text{def}}{=} \frac{x_j}{2\pi R_j},$$

and note that since $(x_j - 2\pi n_j R_j)^2 = \left(2\pi R_j\left(\frac{x_j}{2\pi R_j} - n_j\right)\right)^2 = a_j(b_j - n_j)^2$ we can write, by definition (8.4),

$$\begin{aligned} V_{4+d}(x, y, z, \vec{x}) &= -MG_{4+d} \sum_{\vec{n}\in\mathbb{Z}^d} \frac{1}{\left(\sum_{j=1}^{d} a_j(n_j - b_j)^2 + r^2\right)^{\frac{d+1}{2}}} \\ &= -MG_{4+d} E\left(\frac{d+1}{2}, r; \vec{a}, \vec{b}\right), \end{aligned} \tag{8.7}$$

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by definition (7.24). Thus we are in a pleasant position to apply formula (7.26): For

$$\Sigma_d \stackrel{\text{def}}{=} (2\pi)^d \prod_{j=1}^{d} R_j, \quad \Omega_d \stackrel{\text{def}}{=} \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \tag{8.8}$$

we have $\left(\prod_{j=1}^{d} a_j\right)^{1/2} = \Sigma_d$, and we see that

$$V_{4+d}(x,y,z,\vec{x}) = -\frac{MG_{4+d}\Omega_d}{2r\Sigma_d} \sum_{\vec{n}\in\mathbb{Z}^d} \exp\left(i\vec{n}\cdot\overrightarrow{\left(\frac{x_1}{R_1},\ldots,\frac{x_d}{R_d}\right)}\right) \exp\left(-r\left(\sum_{j=1}^{d}\frac{n_j^2}{R_j^2}\right)^{\frac{1}{2}}\right) \tag{8.9}$$

by definition of a_j and b_j, for $r^2 \stackrel{\text{def}}{=} x^2+y^2+z^2$. This proves the Fourier series expansion (8.6), where we see that the Kaluza–Klein modes $f_{\vec{n}}(x,y,z)$ are in fact given by

$$f_{\vec{n}}(x,y,z) = -\frac{MG_{4+d}\Omega_d}{2r\Sigma_d} \exp\left(-r\left(\sum_{j=1}^{d}\frac{n_j^2}{R_j^2}\right)^{\frac{1}{2}}\right) \tag{8.10}$$

for $\vec{n} = (n_1,\ldots,n_d) \in \mathbb{Z}^d, (x,y,z) \in \mathbb{R}^3 - \{0\}$. Since $V_{4+d}(x,y,z,\vec{x})$ is actually real-valued, we write equation (8.9) as

$$V_{4+d}(x,y,z,x_1,\ldots,x_d) = -\frac{MG_{4+d}\Omega_d}{2r\Sigma_d} \sum_{\vec{n}\in\mathbb{Z}^d} \exp\left(-r\left(\sum_{j=1}^{d}\frac{n_j^2}{R_j^2}\right)^{\frac{1}{2}}\right) \cos\left(\vec{n}\cdot\overrightarrow{\left(\frac{x_1}{R_1},\ldots,\frac{x_d}{R_d}\right)}\right). \tag{8.11}$$

Since $2\pi R_i$ is the length of Γ_i, $\Gamma^d \stackrel{\text{def}}{=} \prod_{i=1}^{d} \Gamma_i$ has volume $\prod_{i=1}^{d} 2\pi R_i = (2\pi)^d \prod_{i=1}^{d} R_i$. That is, Σ_d in definition (8.8) (or in formula (8.11)) is the volume of the compactifying d-torus Γ^d. Similarly Ω_d in (8.8) or in (8.11), one knows, is the surface area of the unit sphere $\{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$ in $\mathbb{R}^{d+1}$.

In [21], for example, the choice $x_1 = \cdots = x_d = 0$ is made. Going back to the compactification on a circle, $d = 1$, $R_1 = R$ for example, we can write the sum in (8.11) as

$$\begin{aligned} 1 + \sum_{n\in\mathbb{Z}-\{0\}} e^{-r|n|/R} &= 1 + 2\sum_{n=1}^{\infty}\left(e^{-r/R}\right)^n \\ &= 1 + \frac{2e^{-r/R}}{1-e^{-r/R}} \simeq 1 + 2e^{-r/R} \end{aligned} \tag{8.12}$$

for $x_1 = 0$, where we keep in mind that R is extremely small. Thus in (8.12), r/R is extremely large; i.e., $e^{-r/R}$ is extremely small. For

$$K_d \stackrel{\text{def}}{=} MG_{4+d}\Omega_d/2\Sigma_d,$$

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we get by (8.11) and (8.12)

$$V_5(x, y, z, x_1) \simeq -\frac{K_1}{r}(1 + 2e^{-r/R}), \tag{8.13}$$

which is a correction to the Newtonian potential $V = -K_1/r$ due to an extra dimension.

The approximation (8.13) compares with the general deviations from the Newtonian inverse square law that are known to assume the form

$$V = -\frac{K}{r}(1 + \alpha e^{-r/\lambda})$$

for suitable parameters α, λ. Apart from the toroidal compactification that we have considered, other compactifications are important as well [21] — especially Calabi–Yau compactifications. Thus the d-torus Γ^d is replaced by a Calabi–Yau manifold — a compact Kähler manifold whose first Chern class is zero.

Lecture 9. Modular forms of nonpositive weight, the entropy of a zero weight form, and an abstract Cardy formula

A famous formula of John Cardy [9] computes the asymptotic density of states $\rho(L_0)$ (the number of states at level L_0) for a general two-dimensional conformal field theory (CFT): For the holomorphic sector

$$\rho(L_0) = e^{2\pi\sqrt{cL_0/6}}, \tag{9.1}$$

where the Hilbert space of the theory carries a representation of the Virasoro algebra Vir with generators $\{L_n\}_{n\in\mathbb{Z}}$ and central charge c. Vir has Lie algebra structure given by the usual commutation rule

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{9.2}$$

for $n, m \in \mathbb{Z}$. The CFT entropy S is given by

$$S = \log \rho(H_0) = 2\pi\sqrt{\frac{cL_0}{6}}. \tag{9.3}$$

From the Cardy formula one can derive, for example, the Bekenstein–Hawking formula for BTZ black hole entropy [10]; see also my Speaker's Lecture presented later. More generally, the entropy of black holes in string theory can be derived — the derivation being statistical in nature, and microscopically based [34].

For a CFT on the two-torus with partition function

$$Z(\tau) = trace\ e^{2\pi i(L_0 - \frac{c}{24})\tau} \tag{9.4}$$

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on the upper half-plane π^+ [5], the entropy S can be obtained as follows. Regarding $Z(\tau)$ as a modular form with Fourier expansion

$$Z(\tau) = \sum_{n \geq 0} c_n e^{2\pi i (n - c/24)\tau}, \tag{9.5}$$

one takes

$$S = \log c_n \tag{9.6}$$

for large n. In [5], for example, (also see [4]) the Rademacher–Zuckerman exact formula for c_n is applied, where $Z(\tau)$ is assumed to be modular of weight $w = 0$. This is problematic however since the proof of that exact formula works only for modular forms of *negative* weight. In this lecture we indicate how to resolve this contradiction (thanks to some nice work of N. Brisebarre and G. Philibert), and we present what we call an abstract Cardy formula (with logarithmic correction) for holomorphic modular forms of zero weight. In particular we formulate, abstractly, the sub-leading corrections to Bekenstein–Hawking entropy that appear in formula (14) of [5].

The discussion in Lecture 4 was confined to holomorphic modular forms of non-negative integral weight. We consider now forms of *negative* weight $w = -r$ for $r > 0$, where r need not be an integer. The prototypic example will be the function $F_0(z) \overset{\text{def}}{=} 1/\eta(z)$, where $\eta(z)$ is the Dedekind eta function defined in (3.27), and where it will turn out that $w = -\frac{1}{2}$. We will use, in fact, the basic properties of $F_0(z)$ to serve as motivation for the general definition of a form of negative weight.

We begin with the *partition function* $p(n)$ on $\mathbb{Z}^+$. For n a positive integer, define $p(n)$ as the number of ways of writing n as an (orderless) sum of positive integers. For example, 3 is expressible as $3 = 1 + 2 = 1 + 1 + 1$, so $p(3) = 3$; $4 = 4 = 1 + 3 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1$, so $p(4) = 5$;

$$5 = 5 = 2+3 = 1+4 = 1+1+3 = 1+2+2 = 1+1+1+2 = 1+1+1+1+1,$$

so $p(5) = 7$; similarly $p(2) = 2$, $p(1) = 1$. We set $p(0) \overset{\text{def}}{=} 1$. Clearly $p(n)$ grows quite quickly with n. A precise asymptotic formula for $p(n)$ was found by G. Hardy and S. Ramanujan in 1918, and independently by J. Uspensky in 1920:

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \quad \text{as } n \to \infty \tag{9.7}$$

(the notation means that the ratio between the two sides of the relation (9.7) tends to 1 as $n \to \infty$). For example, it is known that

$$\begin{aligned} p(1000) &= 24{,}061{,}467{,}864{,}032{,}622{,}473{,}692{,}149{,}727{,}991 \\ &\simeq 2.4061 \times 10^{31}, \end{aligned} \tag{9.8}$$

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whereas for $n = 1000$ in (9.7)

$$e^{\pi\sqrt{\frac{2n}{3}}}/4n\sqrt{3} \simeq 2.4402 \times 10^{31}, \tag{9.9}$$

which shows that the asymptotic formula is quite good.

L. Euler found the generating function for $p(n)$. Namely, he showed that

$$\sum_{n=0}^{\infty} p(n)z^n = \frac{1}{\prod_{n=1}^{\infty}(1-z^n)} \tag{9.10}$$

for $z \in \mathbb{C}$ with $|z| < 1$. By this formula and the definition of $\eta(z)$, we see immediately that

$$F_0(z) \stackrel{\text{def}}{=} \frac{1}{\eta(z)} = e^{-\pi i\tau/12} \sum_{n=0}^{\infty} p(n)e^{2\pi i n\tau} \tag{9.11}$$

on π^+.

The following profound result is due to R. Dedekind. To prepare the ground, for $x \in \mathbb{R}$ define

$$((x)) \stackrel{\text{def}}{=} \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$$

where, as before, $[x]$ denotes the largest integer not exceeding x.

THEOREM 9.12. *Fix* $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma \stackrel{\text{def}}{=} SL(2,\mathbb{Z})$, *with* $c > 0$, *and define*

$$S(\gamma) = \frac{a+d}{12c} - \frac{1}{4} - s(d,c),$$

where $s(d,c)$ (called a Dedekind sum) *is given by*

$$s(d,c) \stackrel{\text{def}}{=} \sum_{\mu \in \mathbb{Z}/c\mathbb{Z}} \left(\left(\frac{\mu}{c}\right)\right)\left(\left(\frac{d\mu}{c}\right)\right). \tag{9.13}$$

Then, for $z \in \pi^+$

$$F_0(\gamma \cdot z) = e^{-i\pi\left(S(\gamma)+\frac{1}{4}\right)}\big(-i(cz+d)\big)^{-\frac{1}{2}} F_0(z) \tag{9.14}$$

for $-\pi/2 < \arg\big(-i(cz+d)\big) < \pi/2$, *where* $\gamma \cdot z$ *is defined in equation* (4.3).

The sum in definition (9.13) is over a complete set of coset representatives μ in $\mathbb{Z}$. The case $c = 0$ is much less profound; then

$$\gamma = \begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}$$

(since $1 = \det\gamma = ad$), and

$$F_0(z \pm b) = F_0(\gamma \cdot z) = e^{\mp\pi i b/12} F_0(z). \tag{9.15}$$

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In particular we can write $F_0(z+1) = e^{-\pi i/12} F_0(z) = e^{-2\pi i/24+2\pi i} F_0(z) = e^{2\pi i(1-\frac{1}{24})} F_0(z) = e^{2\pi i\alpha} F_0(z)$ for $\alpha \overset{\text{def}}{=} 1 - \frac{1}{24} = \frac{23}{24}$.

In summary, $F_0(z) = 1/\eta(z)$ satisfies the following conditions:

(i) $F_0(z)$ is holomorphic on π^+. (This follows from Lecture 3).)

(ii) $F_0(z+1) = e^{2\pi i\alpha} F_0(z)$ for some real $\alpha \in [0,1)$ (indeed, with $\alpha = \frac{23}{24}$).

(iii) $F_0(\gamma \cdot z) = \varepsilon(a,b,c,d)\big(-i(cz+d)\big)^{-r} F_0(z)$ for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ with $c > 0$, for some $r > 0$, $-\pi/2 < \arg\big(-i(cz+d)\big) < \pi/2$, and for a function $\varepsilon(\gamma) = \varepsilon(a,b,c,d)$ on Γ with $|\varepsilon(\gamma)| = 1$ (indeed, for $r = \frac{1}{2}$ and $\varepsilon(a,b,c,d) = \exp\big(-i\pi\big(\frac{a+d}{12c} - s(d,c)\big)\big)$, by Theorem 9.12).

(iv) $F_0(z) = e^{2\pi i\alpha z} \sum_{n=-\mu}^{\infty} a_n e^{2\pi i n z}$ on π^+ for some integer $\mu \geq 1$ (indeed, for $\mu = 1$, $a_n = p(n+1)$ for $n \geq -1$, and $a_n = 0$ for $n \leq -2$, by Euler's formula (9.11)).

Note that by conditions (i) and (ii), the function $f(z) \overset{\text{def}}{=} e^{-2\pi i\alpha z} F_0(z)$ is holomorphic on π^+, and it satisfies $f(z+1) = f(z)$. Thus, again by equation (4.1), $f(z)$ has a Fourier expansion $f(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}$ on π^+. That is, conditions (i) and (ii) imply that $F_0(z)$ has a Fourier expansion

$$F_0(z) = e^{2\pi i\alpha z} \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}$$

on π^+, and condition (iv) means that we require that $a_{-n} = 0$ for $n > \mu$, for some positive integer μ.

We abstract these properties of $F_0(z)$ and, in general, we define a *modular form of negative weight* $-r$, for $r > 0$, with *multiplier* $\varepsilon : \Gamma \to \{z \in \mathbb{C} \mid |z| = 1\}$ to be a function $F(z)$ on π^+ that satisfies conditions (i), (ii), (iii), and (iv) for some α and μ with $0 \leq \alpha < 1$, $\mu \in \mathbb{Z}$, $\mu \geq 1$. Thus $\eta(z)^{-1}$ is a modular form of weight $-\frac{1}{2}$ and multiplier $\varepsilon(a,b,c,d) = \exp\big(-i\pi\big(\frac{a+d}{12c} - s(d,c)\big)\big)$, with $\alpha = \frac{23}{24}$, $\mu = 1$, and with Fourier coefficients $a_n = p(n+1)$, as we note again.

For modular forms of positive integral weight, there are no general formulas available that explicitly compute their Fourier coefficients — apart from Theorem 4.23 for holomorphic Eisenstein series. For forms of negative weight however, there is a remarkable, explicit (but complicated) formula for their Fourier coefficients, due to H. Rademacher and H. Zuckerman [31]; also see [29; 30].

Before stating this formula we consider some of its ingredients. First, we have the modified Bessel function

$$I_\nu(t) \overset{\text{def}}{=} \left(\frac{t}{2}\right)^{\nu} \sum_{m=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2m}}{m!\,\Gamma(\nu+m+1)} \tag{9.16}$$

for $t > 0$, $\nu \in \mathbb{C}$; the series here converges absolutely by the ratio test. Next, for $k, h \in \mathbb{Z}$ with $k \geq 1$, $h \geq 0$, $(h,k) = 1$, and $h < k$ choose a solution h' of the

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congruence $hh' \equiv -1 (\text{mod } k)$. For example, $(h, k) = 1$ means that the equation $xh + yk = 1$ has a solution $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. Then $-xh = -1 + yk$ means that $h' \overset{\text{def}}{=} -x$ is a solution. Since $hh' = -1 + lk$ for some $l \in \mathbb{Z}$, we see that $(hh' + 1)/k = l$ is an integer and

$$\det \begin{bmatrix} h' & -(hh'+1)/k \\ k & -h \end{bmatrix} = 1, \quad \text{so} \quad \gamma \overset{\text{def}}{=} \begin{bmatrix} h' & -(hh'+1)/k \\ k & -h \end{bmatrix} \in \Gamma.$$

Hence

$$\varepsilon(\gamma) = \varepsilon\Big(h', -\frac{hh'+1}{k}, k, -h\Big) \tag{9.17}$$

is well-defined. Finally, for $u, v \in \mathbb{C}$ we define the *generalized Kloosterman sum*

$$A_{k,u}(v) = A_k(v, u) \overset{\text{def}}{=} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \varepsilon(\gamma)^{-1} e^{-\frac{2\pi i}{k}((u-\alpha)h' + (v+\alpha)h)} \tag{9.18}$$

for $\varepsilon(\gamma)$ given in (9.17), and for $0 \le \alpha < 1$ above. If $k = 1$, for example, then $0 \le h < k$, so $h = 0$, and we can take $h' = 0$:

$$A_{1,\mu}(v) = A_1(v, u) = \varepsilon(0, -1, 1, 0)^{-1} = \varepsilon(0, -1, 1, 0) \overset{\text{def}}{=} \varepsilon_0. \tag{9.19}$$

The desired formula expresses the coefficients a_n for $n \ge 0$ in terms of the finitely many coefficients $a_{-\mu}, a_{-\mu+1}, a_{-\mu+2}, \ldots, a_{-2}, a_{-1}$ as follows:

THEOREM 9.20 (H. RADEMACHER AND H. ZUCKERMAN). *Let $F(z)$ be a modular form of negative weight $-r$, $r > 0$, with multiplier ε, and with Fourier expansion $F(z) = e^{2\pi i \alpha z} \sum_{n=-\mu}^{\infty} a_n e^{2\pi i n z}$ on π^+ given by condition* (iv) *above, where $0 \le \alpha < 1 \le \mu \in \mathbb{Z}$. Then for $n \ge 0$ with not both $n, \alpha = 0$,*

$$a_n = 2\pi \sum_{j=1}^{\mu} a_{-j} \sum_{k=1}^{\infty} \frac{A_{k,j}(n)}{k} \Big(\frac{j-\alpha}{n+\alpha}\Big)^{\frac{r+1}{2}} I_{r+1}\Big(\frac{4\pi}{k}(j-\alpha)^{1/2}(n+\alpha)^{1/2}\Big), \tag{9.21}$$

where $A_{k,j}(n)$ is defined in (9.18) *and $I_\nu(t)$ is the modified Bessel function in* (9.16).

Note that for $1 \le j \le \mu$, $j \ge 1 > \alpha \Rightarrow j - \alpha > 0$ in equation (9.21). Also $n + \alpha > 0$ there since $n, \alpha \ge 0$ with not both $n, \alpha = 0$.

Using the asymptotic result

$$\lim_{t \to \infty} \sqrt{2\pi t}\, I_\nu(t) e^{-t} = 1 \tag{9.22}$$

for the modified Bessel function $I_\nu(t)$ in (9.16), and also the trivial estimate $|A_{k,j}(n)| \le k$ that follows from (9.18), one can obtain from the explicit formula

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(9.21) the following asymptotic behavior of a_n as $n \to \infty$. Assume that $a_{-\mu} \neq 0$ and define

$$a^{\infty}(n) \stackrel{\text{def}}{=} \frac{a_{-\mu}}{\sqrt{2}} \varepsilon_0 \frac{(\mu-\alpha)^{\frac{r}{2}+\frac{1}{4}}}{(n+\alpha)^{\frac{r}{2}+\frac{3}{4}}} \exp\bigl(4\pi(\mu-\alpha)^{1/2}(n+\alpha)^{1/2}\bigr), \tag{9.23}$$

say for $n \geq 1$, for $\varepsilon_0 \stackrel{\text{def}}{=} \varepsilon\bigl(\bigl[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\bigr]\bigr)$ in (9.19). Then in [23], for example, it is shown that

$$a_n \sim a^{\infty}(n) \quad \text{as } n \to \infty \tag{9.24}$$

which gives the asymptotic behavior of the Fourier coefficients of a modular form $F(z)$ of negative weight $-r$ with Fourier expansion as in the statement of Theorem 9.20. For forms of *zero weight* a quite similar result is given in equation (9.30) below. The asymptotic result (9.7) follows from (9.24) applied to $F_0(z)$, in which case formula (9.21) provides an *exact* formula (due to Rademacher) for $p(n)$ [2; 29; 30].

For $a, b, k \in \mathbb{Z}$ with $k \geq 1$ the *classical Kloosterman sum* $S(a, b; k)$ is defined by

$$S(a, b; k) = \sum_{\substack{h \in \mathbb{Z}/k\mathbb{Z} \\ (h,k)=1}} e^{\frac{2\pi i}{k}(ah+b\bar{h})} \tag{9.25}$$

where $h\bar{h} \equiv 1 (\text{mod } k)$. These sums will appear in the next theorem (Theorem 9.27) that is a companion result of Theorem 9.20.

We consider next modular forms $F(z)$ of *weight zero*. That is, $F(z)$ is a holomorphic function on π^+ such that $F(\gamma \cdot z) = F(z)$ for $\gamma \in \Gamma$, and with Fourier expansion

$$F(z) = \sum_{n=-\mu}^{\infty} a_n e^{2\pi i n z} \tag{9.26}$$

on π^+, for some positive integer μ. In case $F(z)$ is the modular invariant $j(z)$, for example, this expansion is that given in equation (4.44) with $\mu = 1$, in which case the a_n there are computed explicitly by H. Petersson and H. Rademacher [27; 28], independently - by a formula similar in structure to that given in (9.21). For the general case in equation (9.26) the following extension of the Petersson–Rademacher formula is available [6]:

THEOREM 9.27 (N. BRISEBARRE AND G. PHILIBERT). *For a modular form $F(z)$ of weight zero with Fourier expansion given by equation* (9.26), *its n-th Fourier coefficient a_n is given by*

$$a_n = 2\pi \sum_{j=1}^{\mu} a_{-j} \sqrt{\frac{j}{n}} \sum_{k=1}^{\infty} \frac{S(n, -j; k)}{k} I_1\left(\frac{4\pi\sqrt{nj}}{k}\right) \tag{9.28}$$

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for $n \geq 1$, *where* $S(n, -j; k)$ *is defined in* (9.25) *and* I_1 $(t > 0)$ *in* (9.16).

M. Knopp's asymptotic argument in [23] also works for a weight zero form (as he shows), provided the trivial estimate $|A_{k,j}(n)| \leq k$ used above is replaced by the less trivial Weil estimate $|S(a, b; k)| \leq C(\varepsilon)(a, b, k)^{1/2} k^{1/2+\varepsilon}$, $\forall \varepsilon > 0$. The conclusion is that if $a_{-\mu} \neq 0$, and if

$$a^{\infty}(n) \stackrel{\text{def}}{=} \frac{a_{-\mu}}{\sqrt{2}} \frac{\mu^{1/4}}{n^{3/4}} e^{4\pi\sqrt{\mu n}}, n \geq 1, \tag{9.29}$$

then

$$a_n \sim a^{\infty}(n) : \lim_{n \to \infty} \frac{a_n}{a^{\infty}(n)} = 1. \tag{9.30}$$

We see that, *formally*, definition (9.29) is obtained by taking $\varepsilon_0 = 1, r = 0$, and $\alpha = 0$ in definition (9.23) - in which case formulas (9.21) and (9.28) are also formally the same. Going back to the Fourier expansion of the modular invariant $j(z)$ given in equation (4.44), where $a_{-\mu} = a_{-1} = 1$, we obtain from (9.30) that ([27; 28])

$$a_n \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2} n^{3/4}} \text{ as } n \to \infty. \tag{9.31}$$

A stronger result than (9.31), namely that

$$a_n = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2} n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} + \varepsilon_n\right), |\varepsilon_n| \leq \frac{.055}{n} \tag{9.32}$$

(also due to Brisebarre and Philibert [6]) plays a key role in my study of the asymptotics of the Fourier coefficients of extremal partition functions of certain conformal field theories; see Theorem 5-16 of my Speaker's Lecture (page 345), and the remark that follows it.

Motivated by physical considerations, and by equation (9.5) in particular, we consider a modular form of weight zero with Fourier expansion

$$f(z) = e^{2\pi i \Delta z} \sum_{n \geq 0} c_n e^{2\pi i n z} \tag{9.33}$$

on π^+, where we assume that Δ is a negative integer. Δ corresponds to $-c/24$ in (9.5), say for a positive central charge c; thus $c = 24(-\Delta)$, a case considered in my Speaker's Lecture. $\mu \stackrel{\text{def}}{=} -\Delta$ is a positive integer such that for $a_n \stackrel{\text{def}}{=} c_{n+\mu}$, we have (taking $c_n = 0$ for $n \leq -1$) $a_{-n} = 0$ for $n > \mu$. Moreover, since

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$\sum_{n=0}^{\infty} d_{n-\mu} = \sum_{n=-\mu}^{\infty} d_n$, we see that for $d_n \overset{\text{def}}{=} a_n e^{2\pi i n z}$ we have

$$\sum_{n=-\mu}^{\infty} a_n e^{2\pi i n z} = \sum_{n=0}^{\infty} a_{n-\mu} e^{2\pi i (n-\mu) z}$$

$$\overset{\text{def}}{=} \sum_{n=0}^{\infty} c_n e^{2\pi i (n+\Delta) z} = f(z), \tag{9.34}$$

by (9.33). That is, $f(z)$ has the form (9.26), which means that we can apply formula (9.28), and the asymptotic result (9.30).

Assume that $c_0 \neq 0$, and define

$$c^{\infty}(n) \overset{\text{def}}{=} \frac{c_0}{\sqrt{2}} \frac{|\Delta|^{1/4}}{(n+\Delta)^{3/4}} e^{4\pi |\Delta|^{1/2} (n+\Delta)^{1/2}} \tag{9.35}$$

for $n + \Delta \geq 1$. By definition (9.29), for $n - \mu \overset{\text{def}}{=} n + \Delta \geq 1$,

$$a^{\infty}(n-\mu) = \frac{a_{-\mu} \mu^{1/4}}{\sqrt{2}(n-\mu)^{3/4}} e^{4\pi \sqrt{\mu(n-\mu)}} \overset{\text{def}}{=} c^{\infty}(n),$$

as $a_{-\mu} \overset{\text{def}}{=} c_0 \neq 0$. Therefore by (9.30)

$$1 = \lim_{n\to\infty} \frac{a_{n-\mu}}{a^{\infty}(n-\mu)} = \lim_{n\to\infty} \frac{c_n}{c^{\infty}(n)} : c_n \sim c^{\infty}(n) \text{ as } n \to \infty, \tag{9.36}$$

for $c^{\infty}(n)$ defined in (9.35). Thus (9.36) gives the asymptotic behavior of the Fourier coefficients c_n of the modular form $f(z)$ of weight zero in (9.33).

Motivated by equation (9.6), and given the result (9.36) we define *entropy function* $S(n)$ associated to $f(z)$ by

$$S(n) \overset{\text{def}}{=} \log c^{\infty}(n) \tag{9.37}$$

for $n + \Delta \geq 1$, in case $c_0 > 0$. Also we set

$$S_0(n) \overset{\text{def}}{=} 2\pi \sqrt{4|\Delta|(n+\Delta)}, \tag{9.38}$$

for $n + \Delta \geq 1$. Then for $c = 24(-\Delta) = 24|\Delta|$, as considered above, (i.e., for $4|\Delta| = c/6$) $S_0(n)$ corresponds to the CFT entropy in equation (9.3), where $n + \Delta$ corresponds to the L_0 there. Moreover, by definition (9.37) we obtain

$$S(n) = S_0(n) + \left(\tfrac{1}{4}\log|\Delta| - \tfrac{3}{4}\log(n+\Delta) - \tfrac{1}{2}\log 2 + \log c_0\right), \tag{9.39}$$

which we can regard as an *abstract Cardy formula with logarithmic correction*, given by the four terms parenthesized. Note that, apart from, the term $\log c_0$, equation (9.39) bears an exact resemblance to equation (5.22) of my Speaker's Lecture. We regard $S_0(n)$ in definition (9.38), of course, as an abstract *Bekenstein–Hawking function* associated to the modular form $f(z)$ in (9.33) of zero weight. Equation (9.39) also corresponds to equation (14) of [5].

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To close things out, we also apply formula (9.28) to $f(z)$. For $n-\mu \geq 1$,

$$c_n = a_{n-\mu} = 2\pi \sum_{j=1}^{\mu} a_{-j} \sqrt{\frac{j}{n-\mu}} \frac{S(n-\mu, -j; k)}{k} I_1\left(\frac{4\pi\sqrt{(n-\mu)j}}{k}\right). \quad (9.40)$$

Use $\sum_{j=1}^{\mu} d_j = d_\mu + d_{\mu-1} + \ldots + d_2 + d_1 = \sum_{j=0}^{\mu-1} d_{\mu-j}$ and $a_{-(\mu-j)} \stackrel{\text{def}}{=} c_j$ to write equation (9.40) as

$$c_n = 2\pi \sum_{j=0}^{\mu-1} c_j \sqrt{\frac{\mu-j}{n-\mu}} \frac{S(n-\mu, -(\mu-j); k)}{k} I_1\left(\frac{4\pi\sqrt{(n-\mu)(\mu-j)}}{k}\right) \quad (9.41)$$

where $\mu \stackrel{\text{def}}{=} -\Delta$. For $0 \leq j \leq \mu - 1 = -\Delta - 1$, $j \in \mathbb{Z}$, we have $j \geq 0$ and $j + \Delta \leq -1 < 0$. Conversely if $j \geq 0$, $j \in \mathbb{Z}$, and $j + \Delta < 0$, then as $\Delta \in \mathbb{Z}$ we have $j + \Delta \leq -1$, so $0 \leq j \leq -\Delta - 1 = \mu - 1$. Of course $j + \Delta < 0$ also means that $\mu - j = -\Delta - j = |j + \Delta|$. Thus we can write equation (9.41) as

$$c_n = 2\pi \sum_{\substack{j \geq 0 \\ j+\Delta<0}} c_j \sqrt{\frac{|j+\Delta|}{n+\Delta}} \frac{S(n+\Delta, j+\Delta; k)}{k} I_1\left(\frac{4\pi\sqrt{(n+\Delta)|j+\Delta|}}{k}\right) \quad (9.42)$$

for $n + \Delta \, (= n - \mu) \geq 1$:

THEOREM 9.43 (A REFORMULATION OF THEOREM 9.27). *For a modular form $f(z)$ of weight zero with Fourier expansion given by equation* (9.33), *its n-th Fourier coefficient c_n is given by equation* (9.42), *for $n + \Delta \geq 1$. Here Δ is assumed to be a negative integer.*

Instead of applying Theorem 9.20 and taking $r = 0$ there, without justification, physicists can now use Theorem 9.43 for a CFT modular invariant partition function, such as that of equation (9.4), and therefore stand on steady mathematical ground.

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Appendix

A. Uniform convergence of improper integrals. For the reader's convenience we review the conditions under which an improper integral $f(s) = \int_a^\infty F(t,s)\,dt$ defines a holomorphic function $f(s)$. In particular, a verification of the entirety of the function $J(s)$ in equation (1.4) is provided.

The function $F(t,s)$ is defined on a product $[a,\infty) \times D$ with $D \subset \mathbb{C}$ some open subset, where it is assumed that $\int_a^\infty F(t,s)\,dt$ exists for each $s \in D$ — say $t \mapsto F(t,s)$ is integrable on $[a,b]$ for every $b > a$. Thus $f(s)$ is well-defined on D. By definition, the integral $f(s)$ is *uniformly convergent* on some subset $D_0 \subset D$ if to each $\varepsilon > 0$ there corresponds a number $B(\varepsilon) > a$ such that for $b > B(\varepsilon)$ one has $\left|\int_a^b F(t,s)\,dt - f(s)\right| < \varepsilon$ for all $s \in D_0$. An equivalent definition is given by the following *Cauchy criterion*: $f(s)$ is uniformly convergent on D_0 if and only if to each $\varepsilon > 0$ there corresponds a number $B(\varepsilon) > a$ such that $\left|\int_{b_1}^{b_2} F(t,s)\,dt\right| < \varepsilon$ for all $b_2 > b_1 > B(\varepsilon)$ and all $s \in D_0$. For clearly if $f(s)$ is uniformly convergent on D_0 and $\varepsilon > 0$ is given, we can choose $B(\varepsilon) > a$ such that $\left|\int_a^b F(t,s)\,dt - f(s)\right| < \varepsilon/2$ for $b > B(\varepsilon)$, $s \in D_0$. Then for $b_2 > b_1 > B(\varepsilon)$ and $s \in D_0$, we have

$$\int_{b_1}^{b_2} F(t,s)\,dt = \int_a^{b_2} F(t,s)\,dt - f(s) - \left(\int_a^{b_1} F(t,s)\,dt - f(s)\right),$$

which implies $\left|\int_{b_1}^{b_2} F(t,s)\,dt\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Conversely, assume the alternate condition. Define the sequence $\{f_n(s)\}_{n>a}$ of functions on D_0 by

$$f_n(s) \stackrel{\text{def}}{=} \int_a^n F(t,s)\,dt.$$

Given $\varepsilon > 0$ we can choose, by hypothesis, $B(\varepsilon) > a$ such that $\left|\int_{b_1}^{b_2} F(t,s)\,dt\right| < \varepsilon$ for $b_2 > b_1 > B(\varepsilon)$ and $s \in D_0$. Let $N(\varepsilon)$ be an integer $> B(\varepsilon)$. Then for integers $n > m \geq N(\varepsilon)$ and for all $s \in D_0$ we see that $\left|f_n(s) - f_m(s)\right| = \left|\int_m^n F(t,s)\,dt\right| < \varepsilon$. Therefore, by the standard Cauchy criterion, the sequence $\{f_n(s)\}_{n>a}$ converges uniformly on D_0 to a function $g(s)$ on D_0: For any $\varepsilon_1 > 0$, there exists an integer $N(\varepsilon_1) > a$ such that for an integer $n \geq N(\varepsilon_1)$, one has $\varepsilon_1 > \left|f_n(s) - g(s)\right| = \left|\int_a^n F(t,s)\,dt - f(s)\right|$ for all $s \in D_0$, since necessarily $g(s) = f(s)$. Now let $\varepsilon > 0$ be given. Again, by hypothesis, we can choose $B(\varepsilon) > a$ such that for $b_2 > b_1 > B(\varepsilon)$ one has that $\left|\int_{b_1}^{b_2} F(t,s)\,dt\right| < \varepsilon/2$ for all $s \in D_0$. Taking the quantity ε_1 considered a few lines above equal to $\varepsilon/2$, we can find an integer $N(\varepsilon_1) > B(\varepsilon_1)$ such that for an integer $n \geq N(\varepsilon_1)$, $\varepsilon/2 = \varepsilon_1 > \left|\int_a^n F(t,s)\,dt - f(s)\right|$ for all $s \in D_0$. Thus suppose $b > N(\varepsilon_1)$. Then, for all $s \in D_0$,

$$\left|\int_a^b F(t,s)\,dt - f(s)\right| = \left|\int_a^{N(\varepsilon_1)} F(t,s)\,dt - f(s) + \int_{N(\varepsilon_1)}^b F(t,s)\,dt\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where $N(\varepsilon_1) > a$ since $B(\varepsilon_1) > a$, with $N(\varepsilon_1)$ dependent only on ε. This shows that $f(s)$ is uniformly convergent on D_0. The Cauchy criterion is therefore validated.

As an example, we use the Cauchy criterion to prove the following, very useful result:

THEOREM A.1 (WEIERSTRASS M-TEST). *Let $M(t) \geq 0$ be a function on $[a,\infty)$ that is integrable on each $[a,b]$ with $b > a$. Assume also that $I \stackrel{\text{def}}{=}$*

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$\int_a^\infty M(t)\,dt$ exists. If $|F(t,s)| \le M(t)$ on $[a,\infty)\times D_0$, then $f(s) = \int_a^\infty F(t,s)\,dt$ converges uniformly on D_0. Again D_0 is any subset of D.

PROOF. Let $\varepsilon > 0$ be assigned. That $I = \lim_{b\to\infty} \int_a^b M(t)\,dt$ implies there exists a number $B(\varepsilon) > a$ such that $\left|I - \int_a^b M(t)\,dt\right| < \varepsilon/2$ for $b > B(\varepsilon)$. If $b_2 > b_1 > B(\varepsilon)$, then

$$\left|\int_{b_1}^{b_2} M(t)\,dt\right| = \left|\int_a^{b_2} M(t)\,dt - I + I - \int_a^{b_1} M(t)\,dt\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;$$

hence $\left|\int_{b_1}^{b_2} F(t,s)\,dt\right| \le \int_{b_1}^{b_2} |F(t,s)|\,dt \le \int_{b_1}^{b_2} M(t)\,dt = \left|\int_{b_1}^{b_2} M(t)\,dt\right|$ (since $M(t) \ge 0$, $b_2 > b_1$). But this is less than ε, for all $s \in D_0$. Theorem A.1 follows, therefore, by the Cauchy criterion. □

The question of the holomorphicity of $f(s)$ is settled by the following theorem.

THEOREM A.2. *Again let $F(t,s)$ be defined on $[a,\infty)\times D$ with $D \subset \mathbb{C}$ an open subset. Assume*

(i) *$F(t,s)$ is continuous on $[a,\infty)\times D$ (in particular for each $s \in D$, $t \mapsto F(t,s)$ is integrable on $[a,b]$ for every $b > a$);*
(ii) *for every $t \ge a$ fixed, $s \mapsto F(t,s)$ is holomorphic on D;*
(iii) *for every $s \in D$ fixed, $t \mapsto \partial F(t,s)/\partial s$ is continuous on $[a,\infty)$;*
(iv) *$f(s) \overset{\text{def}}{=} \int_a^\infty F(t,s)\,dt$ converges for every $s \in D$; and*
(v) *$f(s)$ converges uniformly on compact subsets of D.*

Then $f(s)$ is holomorphic on D, and $f'(s) = \int_a^\infty \partial F(t,s)/\partial s\,dt$ for every $s \in D$. Implied here is the existence of the improper integral $\int_a^\infty \partial F(t,s)/\partial s\,dt$ on D.

The idea of the proof is to reduce matters to a situation where the integration $\int_a^\infty$ over an infinite range is replaced by that over a finite range $\int_a^n$, where holomorphicity is known to follow. This is easily done by considering again the sequence $\{f_n(s)\}_{n>a}$ discussed earlier: $f_n(s) \overset{\text{def}}{=} \int_a^n F(t,s)\,dt$ on D, which is well-defined by (i). If $K \subset D$ is compact, then given (v), the above argument with D_0 now taken to be K shows exactly (by way of the Cauchy criterion) that $\{f_n(s)\}_{n>a}$ converges uniformly on K (to $f(s)$ by (iv)). On the other hand, by (i), (ii), (iii) we have that (i)$'$ $F(t,s)$ is continuous on $[a,n]\times D$, (ii)$'$ for every $t \in [a,n]$ fixed, $s \mapsto F(t,s)$ is holomorphic on D, and (iii)$'$ for every $s \in D$ fixed, $t \mapsto \partial F(t,s)/\partial s$ is continuous on $[a,n]$; here $a < n \in \mathbb{Z}$. Given (i)$'$, (ii)$'$ and (iii)$'$, it is standard in complex variables texts that $f_n(s) = \int_a^n F(t,s)\,dt$ is holomorphic on D and that $f_n'(s) = \int_a^n \partial F/\partial s(t,s)\,dt$. Since we have noted that the sequence $\{f_n(s)\}_{n>a}$ converges uniformly to $f(s)$ on compact subsets K of D, it follows by the Weierstrass theorem that $f(s)$ is holomorphic on D, and that $f_n'(s) \mapsto f'(s)$ pointwise on D — with uniform convergence on compact subsets

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of D, in fact. That is, $f'(s) = \lim_{n\to\infty} f_n'(s) = \lim_{n\to\infty} \int_a^n \partial F(t,s)/\partial s\, dt = \int_a^\infty \partial F(t,s)/\partial s\, dt$ on D, which proves Theorem A.2.

As an application, we check that the function $J(s)$ in definition (1.4) is an entire function. First we claim that the function $\theta_0(t) \overset{\text{def}}{=} \sum_{n=1}^\infty e^{-\pi n^2 t}$, for $t > 0$, converges uniformly on $[1,\infty)$. This is clear, by the Weierstrass M-test, since for $n, t \geq 1$, we have $n^2 \geq n$, hence $\pi n^2 t \geq \pi n t \geq \pi n$, hence $e^{-\pi n^2 t} \leq e^{-\pi n}$, and moreover $\sum_{n=1}^\infty e^{-\pi n}$ is a convergent geometric series. Therefore $\theta_0(t)$ is continuous on $[1,\infty)$, since the terms $e^{-\pi n^2 t}$ are continuous in t on $[1,\infty)$. By definitions (1.2), (1.4), $J(s) = \int_1^\infty F(t,s)\, dt$ for $F(t,s) \overset{\text{def}}{=} \theta_0(t) t^s$ on $[1,\infty)\times\mathbb{C}$, where $F(t,s)$ therefore is also continuous. Again for $n, t \geq 1$, $\pi n^2 t \geq \pi n t$ and also $\pi t \geq \pi$, so $e^{-\pi n^2 t} \leq e^{-\pi n t}$ and $e^{-\pi t} \leq e^{-\pi}$, so $1 - e^{-\pi t} \geq 1 - e^{-\pi}$, so

$$\frac{1}{1-e^{-\pi t}} \leq \frac{1}{1-e^{-\pi}} = \frac{e^\pi}{e^\pi - 1} \overset{\text{def}}{=} C.$$

That is,

$$\theta_0(t) = \sum_{n=1}^\infty e^{-\pi n^2 t} \leq \sum_{n=1}^\infty e^{-\pi n t} = \sum_{n=1}^\infty (e^{-\pi t})^n = \frac{e^{-\pi t}}{1-e^{-\pi t}} \leq C e^{-\pi t}$$

for $t \geq 1$, so $\left|F(t,s)\right| \leq C e^{-\pi t} t^{\operatorname{Re} s}$ on $[1,\infty)\times\mathbb{C}$, where $\int_1^\infty e^{-bt} t^a\, dt$ converges for $b > 0, a \in \mathbb{R}$. Thus $J(s)$ converges absolutely for every $s \in \mathbb{C}$. We see that conditions (i) and (iv) of Theorem A.2 hold. Conditions (ii) and (iii) certainly hold. To check condition (v), let $K \subset \mathbb{C}$ be any compact subset. The continuous function $s \to \operatorname{Re} s$ on K has an upper bound $\sigma : \operatorname{Re} s \leq \sigma$ on $K \Rightarrow t^{\operatorname{Re} s} \leq t^\sigma$ on $[1,\infty)\times K$ (since $\log t \geq 0$ for $t \geq 1$). That is, on $[1,\infty)\times K$ the estimate $|F(t,s)| \leq C e^{-\pi t} t^\sigma$ holds where $\int_1^\infty e^{-\pi t} t^\sigma\, dt < \infty$, implying that $J(s)$ converges uniformly on K, by Theorem A.1. Therefore $J(s)$ is holomorphic on $\mathbb{C}$ by Theorem A.2.

B. A Fourier expansion (or q-expansion). The function $q(z) \overset{\text{def}}{=} e^{2\pi i z}$ is holomorphic and it satisfies the periodicity condition $q(z+1) = q(z)$. Suppose $f(z)$ is an arbitrary holomorphic function defined on an open horizontal strip

$$S_{b_1,b_2} \overset{\text{def}}{=} \{z \in \mathbb{C} \mid b_1 < \operatorname{Im} z < b_2\} \tag{B.1}$$

as indicated in the figure at the top of the next page, where $b_1, b_2 \in \mathbb{R}$, $b_1 < b_2$.

Suppose also that $f(z)$ satisfies the periodicity condition $f(z+1) = f(z)$ on S_{b_1,b_2}; clearly $z \in S_{b_1,b_2}$ implies $z + r \in S_{b_1,b_2}$ for all $r \in \mathbb{R}$. Then $f(z)$ has a *Fourier expansion* (also called a q-*expansion*)

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q(z)^n = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z} \tag{B.2}$$

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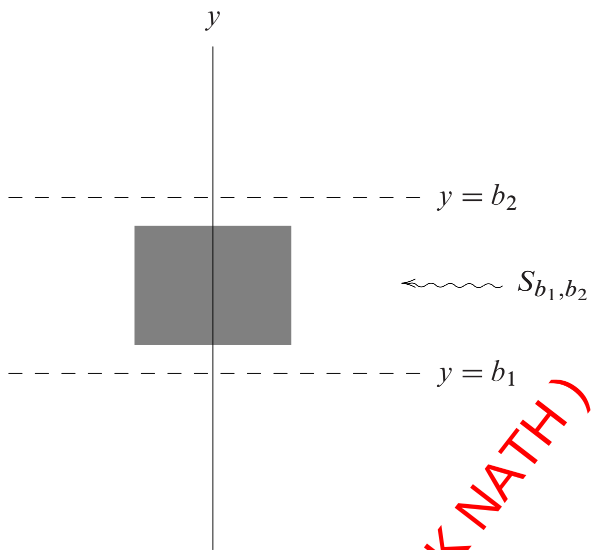
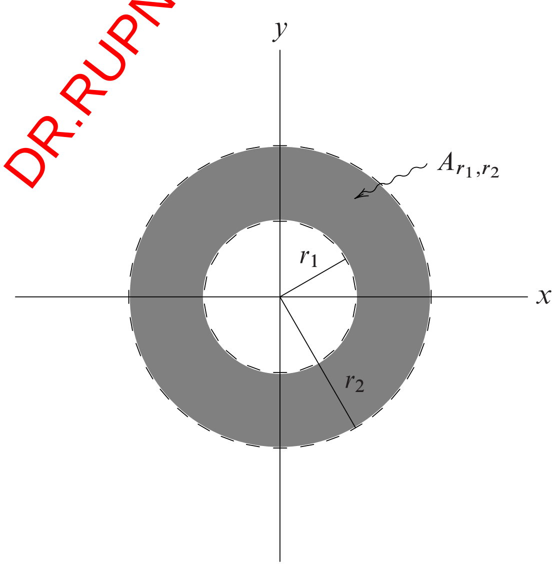


on S_{b_1,b_2}, for suitable coefficients $a_n \in \mathbb{C}$; see Theorem B.7 and equation (B.6) below for an expression of the a_n. The finiteness of b_2 is *not* essential for the validity of equation (B.2). In fact, one of its most useful applications is in case when S_{b_1,b_2} is the upper half plane: $b_1 = 0$, $b_2 = \infty$. The Fourier expansion of $f(z)$ follows from the local invertibility of the function $q(z)$ and the Laurent expansion of the function $(f \circ q^{-1})(z)$. We fill in the details of the proof.

Note first that $q(z)$ is a surjective map of the strip S_{b_1,b_2} onto the annulus $A_{r_1,r_2} \stackrel{\text{def}}{=} \{w \in \mathbb{C} \mid r_1 < |w| < r_2\}$ for $r_1 \stackrel{\text{def}}{=} e^{-2\pi b_2} > 0$, $r_2 \stackrel{\text{def}}{=} e^{-2\pi b_1} > 0$ (see figure below).



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For if $z = x + iy \in S_{b_1,b_2}$, we have $|q(z)| = e^{-2\pi y}$ and $b_1 < y < b_2$, so $e^{-2\pi b_1} > e^{-2\pi y} > e^{-2\pi b_2}$, and $w = q(z) \in A_{r_1,r_2}$. On the other hand, if $w \in A_{r_1,r_2}$ is given choose $t \in \mathbb{R}$ such that $e^{it} = w/|w|$ (since $w \neq 0$), and define $r = \log|w|$. Then one quickly checks that $z \overset{\text{def}}{=} t/2\pi + ir/(-2\pi) \in S_{b_1,b_2}$ such that $q(z) = w$, as desired.

From $f(z+1) = f(z)$, it follows by induction that $f(z+n) = f(z)$ for every positive integer n, and therefore for every negative integer n, $f(z) = f(z+n+(-n)) = f(z+n)$; i.e. $f(z+n) = f(z)$ for every $n \in \mathbb{Z}$. Also since $q(z_1) = q(z_2) \iff e^{2\pi i z_1} = e^{2\pi i z_2} \iff e^{2\pi i(z_1 - z_2)} = 1 \iff z_1 = z_2 + n$ for some $n \in \mathbb{Z}$, the surjectivitiy of $q(z)$ implies that the equation $F(q(z)) = f(z)$ provides for a well-defined function $F(w)$ on the annulus A_{r_1,r_2}. To check that $F(w)$ is holomorphic, given that $f(z)$ is holomorphic, take any $w_0 \in A_{r_1,r_2}$ and choose $z_0 \in S_{b_1,b_2}$ such that $q(z_0) = w_0$, again by the surjectivity of $q(z)$. Since $q'(z) = 2\pi i e^{2\pi i z}$ implies in particular that $q'(z_0) \neq 0$, one can conclude that $q(z)$ is locally invertible at z_0: there exist $\varepsilon > 0$ and a neighborhood N of z_0, $N \subset S_{b_1,b_2}$, on which q is injective with

$$q(N) = N_\varepsilon(q(z_0)) = N_\varepsilon(w_0) \overset{\text{i.e.}}{=} \{w \in \mathbb{C} \mid |w - w_0| < \varepsilon\} \subset A_{r_1,r_2},$$

and with q^{-1} holomorphic on $N_\varepsilon(w_0)$. Thus, on $N_\varepsilon(w_0)$,

$$F(w) = F\big(q(q^{-1}(w))\big) = (f \circ q^{-1})(w),$$

which shows that F is holomorphic on $N_\varepsilon(w_0)$ and thus is holomorphic on A_{r_1,r_2}, as $w_0 \in A_{r_1,r_2}$ is arbitrary.

Now $F(w)$ has a Laurent expansion

$$F(w) = \sum_{n=0}^{\infty} \tilde{a}_n w^n + \sum_{m=1}^{\infty} \frac{\tilde{b}_m}{w^m}$$

on the annulus A_{r_1,r_2} where the coefficients $\tilde{a}_n, \tilde{b}_m$ are given by

$$\tilde{a}_n = \frac{1}{2\pi i} \int_\Gamma \frac{F(w)\, dw}{w^{n+1}}, \quad \tilde{b}_m = \frac{1}{2\pi i} \int_\Gamma \frac{F(w)}{w^{-m+1}} dw,$$

for any circle Γ in A_{r_1,r_2} that separates the circles $|w| = r_1$, $|w| = r_2$. We choose Γ to be the circle centered at $w = 0$ with radius $R \overset{\text{def}}{=} e^{-2\pi b}$, given any b with $b_1 < b < b_2$; $r_1 < R < r_2$. For a continuous function $\psi(w)$ on Γ the change of variables $v(t) = 2\pi t$ on $[0, 1]$ permits the expression

$$\int_\Gamma \psi(w)\, dw = \int_0^{2\pi} \psi(\mathrm{Re}^{iv}) R i e^{iv} dv = 2\pi i \int_0^1 \psi(\mathrm{Re}^{2\pi i t})\, \mathrm{Re}^{2\pi i t}\, dt$$
$$= 2\pi i \int_0^1 \psi(e^{2\pi i(t+ib)}) e^{2\pi i(t+ib)}\, dt,$$

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by definition of R. For the choices $\psi(w) = F(w)/w^{n+1}$, $F(w)/w^{-m+1}$, respectively, one finds that

$$\begin{aligned} \tilde{a}_n &= \int_0^1 F(e^{2\pi i(t+ib)})e^{-2\pi i n(t+ib)}\,dt, \\ \tilde{b}_m &= \int_0^1 F(e^{2\pi i(t+ib)})e^{2\pi i m(t+ib)}\,dt, \end{aligned} \tag{B.3}$$

for $n \geq 0, m \geq 1$. However $t + ib \in S_{b_1,b_2}$ since $b_1 < b < b_2$ so by definition of $F(w)$ the equations in (B.3) are

$$\begin{aligned} \tilde{a}_n &= \int_0^1 f(t+ib)e^{-2\pi i n(t+ib)}\,dt, \\ \tilde{b}_m &= \int_0^1 f(t+ib)e^{2\pi i m(t+ib)}\,dt, \end{aligned} \tag{B.4}$$

for $n \geq 0, m \geq 1$, and moreover the Laurent expansion of $F(w)$ has a restatement

$$f(z) = \sum_{n=0}^{\infty} \tilde{a}_n q(z)^n + \sum_{m=1}^{\infty} \frac{\tilde{b}_m}{q(z)^m} \tag{B.5}$$

on S_{b_1,b_2}. One can codify the preceding formulas by defining

$$a_n \stackrel{\text{def}}{=} \int_0^1 f(t+ib)e^{-2\pi i n(t+ib)}\,dt \tag{B.6}$$

for $n \in \mathbb{Z}$, again for $b_1 < b < b_2$. Then $a_n = \tilde{a}_n$ for $n \geq 0$ and $a_{-n} = \tilde{b}_n$ for $n \geq 1$. By equation (B.5) we have therefore completed the proof of equation (B.2):

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THEOREM B.7 (A FOURIER EXPANSION). *Let $f(z)$ be holomorphic on the open strip S_{b_1,b_2} defined in equation* (B.1), *and assume that $f(z)$ satisfies the periodicity condition $f(z+1) = f(z)$ on S_{b_1,b_2}. Then $f(z)$ has a Fourier expansion on S_{b_1,b_2} given by equation* (B.2), *where the a_n are given by equation* (B.6) *for $n \in \mathbb{Z}$, for arbitrary b subject to $b_1 < b < b_2$.*

Theorem B.7 is valid if S_{b_1,b_2} is replaced by the upper half-plane π^+ (with $b_1 = 0$, $b_2 = \infty$), for example, as we have indicated. For clearly the preceding arguments hold for $b_2 = \infty$. Here, in place of the statement that $q : S_{b_1,b_2} \to A_{r_1,r_2}$ is surjective (again for $r_1 \stackrel{\text{def}}{=} e^{-2\pi b_2}$, $r_2 \stackrel{\text{def}}{=} e^{-2\pi b_1}$, $b_2 < \infty$), one simply employs the statement that $q : \pi^+ \to \{w \in \mathbb{C} \mid 0 < |w| < 1\}$ is surjective.

C. Poisson summation and Jacobi inversion. The Jacobi inversion formula (1.3) can be proved by a special application of the *Poisson summation formula* (PSF). The latter formula, in essence, is the statement

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n), \tag{C.1}$$

for a suitable class of functions $f(x)$ and a suitable normalization of the Fourier transform $\hat{f}(x)$ of $f(x)$. The purpose here is to prove a slightly more general version of the PSF, which applied in a special case, coupled with a Fourier transform computation, indeed does provide for a proof of equation (1.3).

For a function $h(x)$ on $\mathbb{R}$, the definition

$$\hat{h}(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(t)e^{-2\pi ixt}\, dt \tag{C.2}$$

will serve as our normalization of its *Fourier transform*. Here's what we aim to establish:

THEOREM C.3 (POISSON SUMMATION). *Let $f(z)$ be a holomorphic function on an open horizontal strip $S_\delta \overset{\text{def}}{=} \{z \in \mathbb{C} \mid -\delta < \operatorname{Im} z < \delta\}, \delta > 0$, say with $f|_{\mathbb{R}} \in L^1(\mathbb{R}, dx)$. Assume that the series $\sum_{n=0}^{\infty} f(z+n)$, $\sum_{n=1}^{\infty} f(z-n)$ converge uniformly on compact subsets of S_δ. Then for any $z \in S_\delta$*

$$\sum_{n\in\mathbb{Z}} f(z+n) = \sum_{n\in\mathbb{Z}} e^{2\pi inz} \hat{f}(n). \tag{C.4}$$

In particular for $z = 0$ we obtain equation (C.1).

PROOF. By the Weierstrass theorem, the uniform convergence of the series $\sum_{n=0}^{\infty} f(z+n)$ and $\sum_{n=1}^{\infty} f(z-n)$ on compact subsets of S_δ means that the function

$$F(z) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} f(z+n) = \sum_{n=0}^{\infty} f(z+n) + \sum_{n=1}^{\infty} f(z-n)$$

on S_δ is holomorphic. $F(z)$ satisfies $F(z+1) = \sum_{n\in\mathbb{Z}} f(z+n+1) = \sum_{n\in\mathbb{Z}} f(z+n) = F(z)$ on S_δ. Therefore, Theorem B.7 of Appendix B is applicable, where the choice $b = 0$ is made ($b_1 = -\delta, b_2 = \delta$): $F(z)$ has a Fourier expansion

$$F(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi inz} \tag{C.5}$$

on S_δ, where $a_n \overset{\text{def}}{=} \int_0^1 F(t)e^{-2\pi int}\, dt$ for $n \in \mathbb{Z}$. Since $[0,1] \subset S_\delta$ is compact, for $n \in \mathbb{Z}$ fixed the series $\sum_{l=0}^{\infty} f(t+l)e^{-2\pi int}$ and $\sum_{l=1}^{\infty} f(t-l)e^{-2\pi int}$ (whose sum is $F(t)e^{-2\pi int}$) converge uniformly on $[0,1]$ (by hypothesis, given

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of course that $|e^{-2\pi int}| = 1$). Therefore a_n can be obtained by termwise integration; we start by writing

$$\begin{aligned} a_n &= \int_0^1 \Big(\sum_{l=0}^{\infty} f(t+l)e^{-2\pi int} + \sum_{l=1}^{\infty} f(t-l)e^{-2\pi int} \Big)\, dt \\ &= \sum_{l=0}^{\infty} \int_0^1 f(t+l)e^{-2\pi int}\, dt + \sum_{l=1}^{\infty} \int_0^1 f(t-l)e^{-2\pi int}\, dt \\ &= \sum_{l=0}^{\infty} \int_l^{l+1} f(v)e^{-2\pi in(v-l)} dv + \sum_{l=1}^{\infty} \int_{-l}^{-l+1} f(v)e^{-2\pi in(v+l)} dv, \end{aligned}$$

by the change of variables $v(t) = t + l$ and $v(t) = t - l$ on $[l, l+1]$, $[-l, -l+1]$, respectively (with $l \in \mathbb{Z}$), This is further equal to $\sum_{l=0}^{\infty} \int_l^{l+1} f(t)e^{-2\pi int}\, dt + \sum_{l=1}^{\infty} \int_{-l}^{-l+1} f(t)e^{-2\pi int}\, dt = \int_0^{\infty} f(t)e^{-2\pi int}\, dt + \int_{-\infty}^{0} f(t)e^{-2\pi int}\, dt = \int_{-\infty}^{\infty} f(t)e^{-2\pi int}\, dt = \hat{f}(n)$, by definition (C.2). That is, by (C.5), for $z \in S_\delta$ $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inz} = F(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} f(z+n)$, which concludes the proof of Theorem C.3. □

Other proofs of the PSF exist. In contrast to the complex-analytic one just presented, a real-analytic proof (due to Bochner) is given in Chapter 14 of [38], for example, based on *Fejér's Theorem*, which states that the Fourier series of a continuous, 2π-periodic function $\psi(x)$ on $\mathbb{R}$ is *Cesàro summable* to $\psi(x)$.

As an example, choose $f(z) = f_t(z) \stackrel{\text{def}}{=} e^{-\pi z^2 t}$ for $t > 0$ fixed. In this case $f(z)$ is an entire function whose restriction to $\mathbb{R}$ is Lebesgue integrable; the restriction is in fact a Schwartz function. We claim that the series

$$\sum_{n=0}^{\infty} f(z+n) \quad \text{and} \quad \sum_{n=1}^{\infty} f(z-n)$$

converge uniformly on compact subsets K of the plane. Since K is compact the continuous functions $z \mapsto e^{-\pi z^2 t}$ and $z \mapsto \operatorname{Re} z$ on $\mathbb{C}$ are bounded on K: $\left|e^{-\pi z^2 t}\right| \le M_1$, $\left|\operatorname{Re} z\right| \le M_2$ on K for some positive numbers M_1, M_2. Let n_0 be an integer $> 1 + 2M_2$. Then for $n \in \mathbb{Z}$ with $n \ge n_0$ one has $n^2 \ge n(1 + 2M_2)$, hence $n^2 - 2nM_2 \ge n$, so that $f(z+n) = e^{-\pi z^2 t} e^{-\pi(n^2+2nz)t}$ for $z \in K$. But

$$\left|e^{-\pi(n^2+2nz)t}\right| = e^{-\pi(n^2+2n\operatorname{Re} z)t} \le e^{-\pi(n^2-2nM_2)t} = e^{-\pi nt}$$

and $\left|e^{-\pi z^2 t}\right| \le M_1$, so $|f(z+n)| \le M_1 e^{-\pi nt}$ on K, with $\sum_{n=0}^{\infty} Me^{-\pi nt}$ clearly convergent for $t > 0$. Therefore, by the M-test, $\sum_{n=0}^{\infty} f(z+n)$ converges absolutely and uniformly on K. Similarly, for $n \in \mathbb{Z}$, we have $f(z-n) = e^{-\pi z^2 t} e^{-\pi(n^2-2nz)t}$, where for $n \ge n_0$ and $z \in K$ again $n^2 - 2nM_2 \ge n$, but where we now use the bound $\operatorname{Re} z \le M_2$: $n^2 - 2n\operatorname{Re} z \ge n^2 - 2nM_2 \ge n$

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$|f(z-n)| \le M_1 e^{-\pi t n}$ on K (for $n \ge n_0$), so $\sum_{n=1}^{\infty} f(z-n)$ converges absolutely and uniformly on K, which shows that $f_t(z) \overset{\text{def}}{=} e^{-\pi z^2 t}$, $t > 0$, satisfies the hypotheses of Theorem C.3. The conclusion

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} \hat{f}_t(n) \tag{C.6}$$

is therefore safe, and the left-hand side here is $\theta(t)$ by definition (1.2). One is therefore placed in the pleasant position of computing the Fourier transform

$$\hat{f}_t(x) \overset{\text{(C.2)}}{=} \int_{-\infty}^{\infty} e^{-\pi y^2 t} e^{-2\pi i x y}\, dy, \tag{C.7}$$

which is a classical computation that we turn to now (for the sake of completeness).

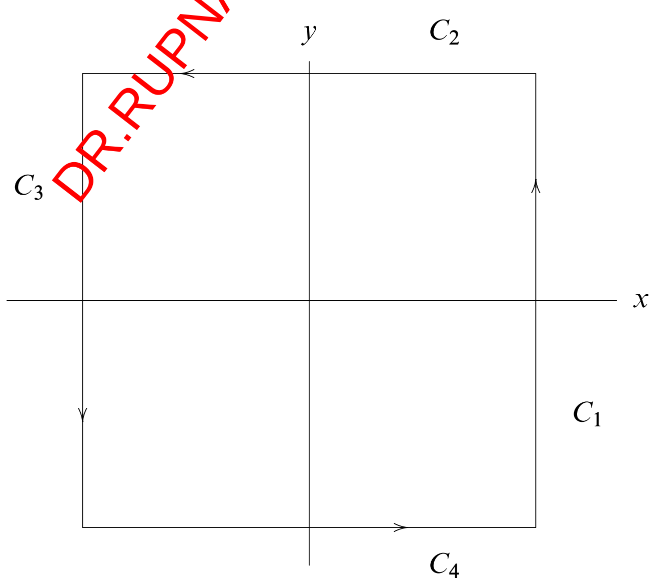
For real numbers a, b, c, t with $a < b$, $t > 0$, note that $e^{-v^2} e^{2\pi i c v/\sqrt{\pi t}} = e^{-v^2} e^{2icv\sqrt{\pi/t}} = e^{-\pi c^2/t} e^{-(v-ic\sqrt{\pi/t})^2}$. By the change of variables $v(x) = \sqrt{\pi t}\,x$ on $[a\sqrt{\pi t}, b\sqrt{\pi t}]$, therefore,

$$\int_a^b e^{-\pi x^2 t} e^{2\pi i c x}\, dx = \frac{e^{-\pi c^2/t}}{\sqrt{\pi t}} \int_{a\sqrt{\pi t}}^{b\sqrt{\pi t}} e^{-(v-ic\sqrt{\pi/t})^2}\, dv. \tag{C.8}$$

Next we show that for $b \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{-(x+ib)^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx. \tag{C.9}$$

To do this, assume first that $b > 0$ and consider the counterclockwise oriented rectangle C_R of height b and width $2R$: $C_R = C_1 + C_2 + C_3 + C_4$.



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By Cauchy's theorem, $0 = I_R \stackrel{\text{def}}{=} \int_{C_R} e^{-z^2}\,dz$. Now

$$\begin{aligned}
&\int_{C_1} e^{-z^2}\,dz = \int_{-R}^{R} e^{-x^2}\,dx,\\
&\int_{C_2} e^{-z^2}\,dz = i\int_0^b e^{-(R+ix)^2}\,dx = i e^{-R^2}\int_0^b e^{-2xRi}e^{x^2}\,dx,\\
&\int_{C_3} e^{-z^2}\,dz = -\int_{-R}^{R} e^{-(x+ib)^2}\,dx,\\
&\int_{C_4} e^{-z^2}\,dz = -\int_{C_2} e^{-z^2}\,dz
\end{aligned}.$$

Thus $\left|\int_{C_2} e^{-z^2}\,dz\right| \le e^{-R^2}\int_0^b e^{x^2}\,dx$, which tends to 0 as $R \to \infty$. That is, $0 = \lim_{R\to\infty} I_R = \int_{-\infty}^{\infty} e^{-x^2}\,dx - \int_{-\infty}^{\infty} e^{-(x+ib)^2}\,dx$, which proves equation (C.9) for $b > 0$. If $b < 0$, write $\int_{-\infty}^{\infty} e^{-(x+ib)^2}\,dx = \int_{-\infty}^{\infty} e^{-(-x+ib)^2}\,dx = \int_{-\infty}^{\infty} e^{-(x+i(-b))^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ by the previous case, since $-b > 0$. Thus (C.9) holds for *all* $b \in \mathbb{R}$ (since it clearly holds for $b = 0$). By (C.8) it then follows that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi x^2 t} e^{2\pi icx}\,dx &= \lim_{R\to\infty}\int_{-R}^{R} e^{-\pi x^2 t} e^{2\pi icx}\,dx\\
&= \frac{e^{-\pi c^2/t}}{\sqrt{\pi t}} \lim_{R\to\infty}\int_{-R\sqrt{\pi t}}^{R\sqrt{\pi t}} e^{-(x+i(-c)\sqrt{\pi/t})^2}\,dx\\
&= \frac{e^{-\pi c^2/t}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{e^{-\pi c^2/t}}{\sqrt{\pi t}}\sqrt{\pi}. \qquad \text{(C.10)}
\end{aligned}$$

PROPOSITION C.11. *For $c \in \mathbb{R}$ and $t > 0$, we have*

$$\int_{-\infty}^{\infty} e^{-\pi x^2 t} e^{-2\pi icx}\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2 t} e^{2\pi icx}\,dx = \frac{e^{-\pi c^2/t}}{\sqrt{t}}.$$

Hence equation (C.7) *is the statement that* $\hat{f}_t(x) = e^{-\pi x^2/t}/\sqrt{t}$.

Having noted that the left-hand side of equation (C.6) is $\theta(t)$, we see that (C.6) (by Proposition C.11) now reads $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}/\sqrt{t} \stackrel{\text{def}}{=} \theta(\frac{1}{t})/\sqrt{t}$, which proves the Jacobi inversion formula (1.3).

D. A divisor lemma and a scholium. The following discussion is taken, nearly word for word, from [38] and thus it has wider applications — for example, applications to the theory of Eisenstein series (see pages 274–276 of that reference). For integers d, n with $d \ne 0$ write $d \mid n$, as usual, if d divides n, and write $d \nmid n$ if d does not divide n. For $n \ge 1$, $\nu \in \mathbb{C}$ let $\sigma_\nu(n) \stackrel{\text{def}}{=} \sum_{0<d,\ d\mid n} d^\nu$ denote the *divisor function*, and for $k, n \ge 1$ in $\mathbb{Z}$ let

$$d(k,n) = \begin{cases} 1 & \text{if } k \mid n,\\ 0 & \text{if } k \nmid n.\end{cases}$$

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Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $a \overset{\text{def}}{=} \sum_{n=1}^{\infty} |a_n| < \infty$. We shall prove a lemma to the effect that

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} m^{\nu} a_{mn}\right) = \sum_{n=1}^{\infty} \sigma_{\nu}(n) a_n. \tag{D.1}$$

Before formulating a precise statement of (D.1) it is useful to consider some simple observations. For $m, k \geq 1$ in $\mathbb{Z}$ set

$$s_m^{(k)} \overset{\text{def}}{=} \sum_{j=1}^{m} a_{kj}, \qquad t_m^{(k)} \overset{\text{def}}{=} \sum_{l=1}^{km} d(k,l) a_l.$$

We first prove by induction that the m-th partial sum $s_m^{(k)}$ equals $t_m^{(k)}$ for every m. For $m = 1$, $s_1^{(k)} = a_k$. On the other hand, $t_1^{(k)} = a_k$ because an integer l in the range $1 \leq l \leq k$ is a multiple of k if and only if $l = k$. Proceeding inductively, one has $s_{m+1}^{(k)} = s_m^{(k)} + a_{k(m+1)} = t_m^{(k)} + a_{k(m+1)}$. On the other hand, $t_{m+1}^{(k)} = t_m^{(k)} + d(k, km+1) a_{km+1} + d(k, km+2) a_{km+2} + \cdots + d(k, km+k) a_{km+k}$. For $l \in \mathbb{Z}$ and $1 \leq l \leq k$, we have $k \mid km+l \iff k \mid l \iff k = l$ (again), so $t_{m+1}^{(k)} = t_m^{(k)} + a_{km+k} = s_{m+1}^{(k)}$, which completes the induction.

Now take $m \to \infty$ in the equality $s_m^{(k)} = t_m^{(k)}$ to conclude that $\sum_{j=1}^{\infty} a_{kj}$ exists and

$$\sum_{j=1}^{\infty} a_{kj} = \sum_{l=1}^{\infty} d(k,l) a_l. \tag{D.2}$$

If $\operatorname{Re}\nu < -1$, then since $|d(k,n) k^{\nu} a_n| \leq |a_n| / k^{-\operatorname{Re}\nu}$, we have

$$\sum_{k=1}^{\infty} |d(k,n) k^{\nu} a_n| \leq |a_n| \zeta(-\operatorname{Re}\nu)$$

(where $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $\operatorname{Re} s > 1$, is the Riemann zeta function) and moreover the iterated series $\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |d(k,n) k^{\nu} a_n|\right)$ converges:

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} |d(k,n) k^{\nu} a_n|\right) \leq a \zeta(-\operatorname{Re}\nu). \tag{D.3}$$

By elementary facts regarding double series (found in advanced calculus texts) it follows that one can conclude that the double series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} d(k,n) k^{\nu} a_n$ converges absolutely, and that

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} d(k,n) k^{\nu} a_n\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} d(k,n) k^{\nu} a_n\right). \tag{D.4}$$

Similarly for $k \geq 1$ fixed, equation (D.2) (with $\{a_n\}_{n=1}^{\infty}$ replaced by $\{|a_n|\}_{n=1}^{\infty}$) yields $\sum_{m=1}^{\infty} |k^{\nu} a_{km}| = k^{\operatorname{Re}\nu} \sum_{m=1}^{\infty} |a_{km}| = k^{\operatorname{Re}\nu} \sum_{l=1}^{\infty} d(k,l) |a_l|$. That is,

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$\sum_{m=1}^{\infty} |k^{\nu} a_{km}| < \infty$ and moreover the iterated series $\sum_{k=1}^{\infty} \left(\sum_{m=1}^{\infty} |k^{\nu} a_{km}| \right)$ converges, as it equals $\sum_{k=1}^{\infty} k^{\operatorname{Re}\nu} \sum_{l=1}^{\infty} d(k,l)|a_l| \le a\zeta(-\operatorname{Re}\nu)$. Thus, similarly to equation (D.4), one has that the double series $\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} k^{\nu} a_{km}$ converges absolutely, and equality of the corresponding iterated series prevails:

$$\sum_{k=1}^{\infty} \Big(\sum_{m=1}^{\infty} k^{\nu} a_{km} \Big) = \sum_{m=1}^{\infty} \Big(\sum_{k=1}^{\infty} k^{\nu} a_{km} \Big). \tag{D.5}$$

Given these observations, we can now state and prove the main lemma regarding the validity of equation (D.1):

DIVISOR LEMMA. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that the series $\sum_{n=1}^{\infty} |a_n|$ converges. Let $\sigma_{\nu}(n) = \sum_{0<d,\ d\,|\,n} d^{\nu}$ be the divisor function, as above, for $\nu \in \mathbb{C}, n \ge 1$ in $\mathbb{Z}$. If $\operatorname{Re}\nu < -1$, then the series $\sum_{n=1}^{\infty} \sigma_{\nu}(n) a_n$ converges absolutely, the iterated series $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} m^{\nu} a_{mn} \right)$ converges and formula* (D.1) *holds, i.e.*,

$$\sum_{n=1}^{\infty} \Big(\sum_{m=1}^{\infty} m^{\nu} a_{mn} \Big) = \sum_{n=1}^{\infty} \sigma_{\nu}(n) a_n.$$

The double series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\nu} a_{mn}$, in fact, converges absolutely and the corresponding iterated series $\sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} m^{\nu} a_{mn})$, $\sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} m^{\nu} a_{mn})$ coincide.

PROOF. $\sum_{k=1}^{n} d(k,n) k^{\nu} \overset{\text{def}}{=} \sum_{1 \le k \le n,\ k\,|\,n} k^{\nu} \overset{\text{def}}{=} \sigma_{\nu}(n)$, where $d(k,n) = 0$ for $k > n$. That is,

$$\sum_{k=1}^{\infty} d(k,n) k^{\nu} = \sigma_{\nu}(n) \tag{D.6}$$

for any $\nu \in \mathbb{C}$. The series $\sum_{n=1}^{\infty} \sigma_{\operatorname{Re}\nu}(n)|a_n|$ is, by (D.6), the iterated series $\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} d(k,n) k^{\operatorname{Re}\nu} |a_n| \right)$, which we have seen converges and is bounded above by $\zeta(-\operatorname{Re}\nu)$ according to (D.3). Clearly $|\sigma_{\nu}(n)| \le \sigma_{\operatorname{Re}\nu}(n)$, so that also $\sum_{n=1}^{\infty} |\sigma_{\nu}(n)||a_n|$ converges. Again by (D.6), we have $\sum_{n=1}^{\infty} \sigma_{\nu}(n) a_n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} d(k,n) k^{\nu} a_n \right)$. Now apply equations (D.4), (D.2), (D.5), successively, to express the latter iterated series as $\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} d(k,n) k^{\nu} a_n \right) = \sum_{k=1}^{\infty} \left(k^{\nu} \sum_{m=1}^{\infty} a_{km} \right) = \sum_{m=1}^{\infty} \left(\sum_{k=1}^{\infty} k^{\nu} a_{km} \right)$, which proves (D.1). We have already seen that the double series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\nu} a_{mn}$ (which equals $\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} k^{\nu} a_{km}$) converges absolutely. By equation (D.5), then one derives the equality of the corresponding iterated series $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} m^{\nu} a_{mn} \right)$ and $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} m^{\nu} a_{mn} \right)$. □

Going back to the equality $s_m^{(k)} = t_m^{(k)}$ of the previous page, we actually have the following fact, recorded for future application:

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SCHOLIUM. *Given a sequence of complex numbers* $\{a_n\}_{n=1}^{\infty}$ *and* $k \in \mathbb{Z}$ *with* $k \geq 1$, *the series* $\sum_{j=1}^{\infty} a_{kj}$ *converges if and only if the series* $\sum_{j=1}^{\infty} d(k, j)a_j$ *converges*, *in which case these series coincide*.

As an example, we use the Divisor Lemma to prove the next lemma, which is important for Lecture 4.

LEMMA D.7. *Fix* $z, k \in \mathbb{C}$ *with* $\operatorname{Im} z > 0$, $\operatorname{Re} k > 2$. *Then the iterated series* $\sum_{m=1}^{\infty}\bigl(\sum_{n=1}^{\infty} n^{k-1}e^{2\pi imn}\bigr)$ *exists*, *the series* $\sum_{n=1}^{\infty} \sigma_{1-k}(n)n^{k-1}e^{2\pi inz}$ *converges absolutely*, *and*

$$\sum_{m=1}^{\infty}\Bigl(\sum_{n=1}^{\infty} n^{k-1}e^{2\pi imn}\Bigr) = \sum_{n=1}^{\infty} \sigma_{1-k}(n)n^{k-1}e^{2\pi inz} = \sum_{n=1}^{\infty} \sigma_{k-1}(n)e^{2\pi inz}. \tag{D.8}$$

PROOF. The last equality comes from $\sigma_\nu(n) = n^\nu\sigma_{-\nu}(n)$. To show the first, set $a_n \overset{\text{def}}{=} n^{k-1}e^{2\pi inz}$. Then $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty} n^{\operatorname{Re} k-1}e^{-2\pi n \operatorname{Im} z}$ converges by the ratio test, since $\operatorname{Im} z > 0$. Also $m^{1-k}a_{mn} = m^{1-k}(mn)^{k-1}e^{2\pi imnz}$ $= n^{k-1}e^{2\pi imnz}$, where $\operatorname{Re} k > 2 \Rightarrow \operatorname{Re}(1-k) < -1$. By the Divisor Lemma (for $\nu = 1-k$), the series $\sum_{n=1}^{\infty} \sigma_{1-k}(n)a_n$ converges absolutely, the iterated series $\sum_{n=1}^{\infty}\bigl(\sum_{m=1}^{\infty} m^{1-k}a_{mn}\bigr)$ converges, and $\sum_{n=1}^{\infty}\bigl(\sum_{m=1}^{\infty} m^{1-k}a_{mn}\bigr) = \sum_{n=1}^{\infty} \sigma_{1-k}(n)a_n$. Substituting the value of a_n proves the desired equality. □

As another example, consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n \overset{\text{def}}{=} e^{\pm 2\pi nxi}K_{s-\frac{1}{2}}(2\pi ny)n^{s-\frac{1}{2}}$$

for $x, y \in \mathbb{R}$, $y > 0$, $s \in \mathbb{C}$ fixed, where

$$K_\nu(z) \overset{\text{def}}{=} \frac{1}{2}\int_0^\infty \exp\Bigl(-\frac{z}{2}\Bigl(t + \frac{1}{t}\Bigr)\Bigr)t^{\nu-1}\,dt \tag{D.9}$$

is the K-Bessel function for $\operatorname{Re} z > 0$, $\nu \in \mathbb{C}$. To see that $\sum_{n=1}^{\infty}|a_n| < \infty$, one applies the asymptotic result

$$\lim_{t\to\infty} \sqrt{t}\,K_\nu(t)e^t = \sqrt{\frac{\pi}{2}}. \tag{D.10}$$

In particular, choose $M_\nu > 0$ such that $\bigl|\sqrt{t}\,K_\nu(t)e^t - \sqrt{\pi/2}\bigr| < \sqrt{\pi/2}$ for $t > M_\nu$; that is, $|K_\nu(t)| < 2\sqrt{\pi/2t}\,e^{-t}$ for $t > M_\nu$. Then if $N_{s,y}$ is an integer with $N_{s,y} > M_{s-\frac{1}{2}}/2\pi y$ we see that for $n \geq N_{s,y}$, $2\pi ny > M_{s-\frac{1}{2}}$, so

$$\bigl|K_{s-\frac{1}{2}}(2\pi ny)\bigr| < 2\sqrt{\frac{\pi}{2\cdot 2\pi ny}}\,e^{-2\pi ny} = \frac{n^{-1/2}}{\sqrt{y}}e^{-2\pi ny};$$

therefore $|a_n| < (n^{\operatorname{Re} s-1}/\sqrt{y})\,e^{-2\pi ny}$, where $\sum_{n=1}^{\infty} n^{\operatorname{Re} s-1}e^{-2\pi ny}$ converges by the ratio test since $y > 0$.

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Now assume that $\operatorname{Re} s > 1$ so that $\nu \overset{\text{def}}{=} -2s+1$ satisfies $\operatorname{Re}\nu < -1$. Also $m^\nu a_{mn} \overset{\text{def}}{=} e^{\pm 2\pi mnxi} K_{s-\frac{1}{2}}(2\pi mny)\left(\frac{n}{m}\right)^{s-\frac{1}{2}}$ for $m, n \geq 1$. The Divisor Lemma gives

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} e^{\pm 2\pi mnxi} K_{s-\frac{1}{2}}(2\pi mny)\left(\frac{n}{m}\right)^{s-\frac{1}{2}}\right) = \sum_{n=1}^{\infty} \sigma_{-2s+1}(n) e^{\pm 2\pi nxi} K_{s-\frac{1}{2}}(2\pi ny) n^{s-\frac{1}{2}}, \quad \text{(D.11)}$$

with absolute convergence of the latter series, and convergence of the iterated series which coincides with the iterated series

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} e^{\pm 2\pi mnxi} K_{s-\frac{1}{2}}(2\pi mny)\left(\frac{n}{m}\right)^{s-\frac{1}{2}}\right).$$

E. Another summation formula and a proof of formula (2.4)**.** In addition to the useful Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \quad \text{(E.1)}$$

of Theorem C.3, there are other very useful, well known summation formulas. The one that we consider here assumes the form

$$\sum_{n\in\mathbb{Z}} f(n) = -\text{the sum of residues of } (\pi \cot \pi z) f(z) \text{ at the poles of } f(z), \quad \text{(E.2)}$$

for a suitable class of functions $f(z)$. As we applied formula (E.1) to a specific function (namely the function $f(z) = e^{-\pi z^2 t}$ for $t > 0$ fixed) to prove the Jacobi inversion formula (1.3), we will, similarly, apply formula (E.2) to a specific function (namely the function $f(z) = (z^2 + a^2)^{-1}$ for $a > 0$ fixed) to prove formula (2.4) of Lecture 2. The main observation towards the proof of formula (E.2) is that there is a nice bound for $|\cot \pi z|$ on a square C_N with side contours R_N, L_N and top and bottom contours T_N, B_N, as illustrated on the next page, for a fixed integer $N > 0$. The bound, in fact, is *independent* of N. Namely,

$$|\cot \pi z| \leq \max(1, B) < 2 \quad \text{(E.3)}$$

on C_N, for $B \overset{\text{def}}{=} (1+e^{-\pi})/(1-e^{-\pi})$. We begin by checking this known result.

For $z = x+iy$, $x, y \in \mathbb{R}$, we have $i\pi z = -\pi y + i\pi x$, and simple manipulations give

$$\cot \pi z = i\,\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = i\,\frac{e^{-\pi y}e^{i\pi x} + e^{\pi y}e^{-i\pi x}}{e^{-\pi y}e^{i\pi x} - e^{\pi y}e^{-i\pi x}}. \quad \text{(E.4)}$$

Hence

$$|\cot \pi z| \leq \frac{e^{-\pi y} + e^{\pi y}}{|e^{-\pi y}e^{i\pi x} - e^{\pi y}e^{-i\pi x}|} \leq \frac{e^{-\pi y} + e^{\pi y}}{|e^{-\pi y} - e^{\pi y}|}$$

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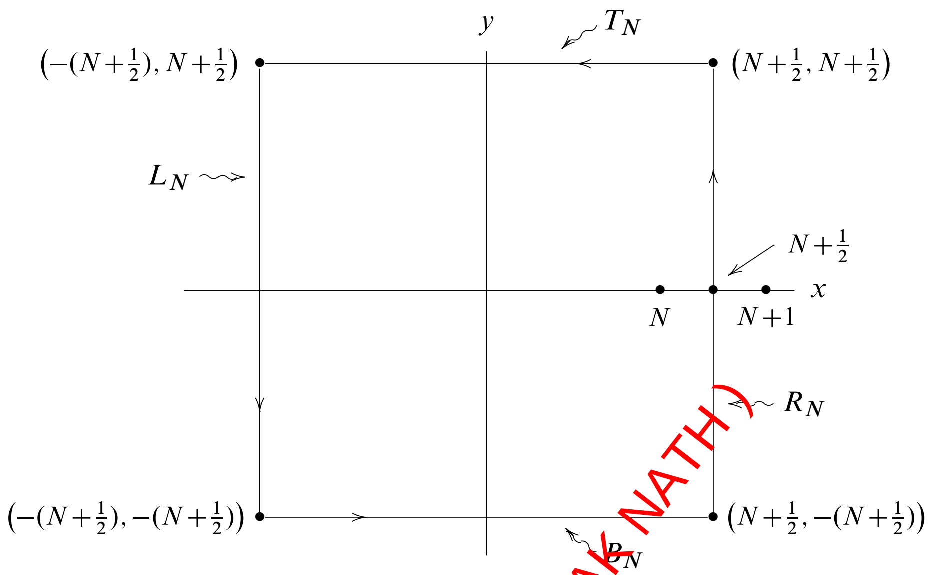

(since $|a-b| \geq \big||a|-|b|\big|$ for $a, b \in \mathbb{C}$). Since $|a| \geq \pm a$ for $a \in \mathbb{R}$, this becomes

$$|\cot \pi z| \leq \frac{e^{-\pi y}+e^{\pi y}}{\pm(e^{-\pi y}-e^{\pi y})}. \tag{E.5}$$

Suppose (in general) that $y > \frac{1}{2}$. Then $2\pi y > \pi$, so $e^{-2\pi y} < e^{-\pi}$ and $1-e^{-2\pi y} > 1-e^{-\pi}$, which, by the choice of the minus sign in (E.5), allows us to write

$$|\cot \pi z| \leq \left(\frac{e^{-\pi y}+e^{\pi y}}{e^{\pi y}-e^{-\pi y}}\right)\frac{e^{-\pi y}}{e^{-\pi y}} = \frac{e^{-2\pi y}+1}{1-e^{-2\pi y}} < \frac{e^{-\pi}+1}{1-e^{-\pi}} \stackrel{\text{def}}{=} B,$$

for $y > \frac{1}{2}$. Similarly if $y < -\frac{1}{2}$, then $2\pi y < -\pi$, so $e^{2\pi y} < e^{-\pi}$ and $1-e^{2\pi y} > 1-e^{-\pi}$, and by the choice of the plus sign in (E.5) we get

$$|\cot \pi z| \leq \left(\frac{e^{-\pi y}+e^{\pi y}}{e^{-\pi y}-e^{\pi y}}\right)\frac{e^{\pi y}}{e^{\pi y}} = \frac{1+e^{2\pi y}}{1-e^{2\pi y}} < \frac{1+e^{-\pi}}{1-e^{-\pi}} = B.$$

Thus we see that

$$|\cot \pi z| < B \stackrel{\text{def}}{=} \frac{1+e^{-\pi}}{1-e^{-\pi}} \tag{E.6}$$

for $z \in \mathbb{C}$ with either $\operatorname{Im} z > \frac{1}{2}$ or $\operatorname{Im} z < -\frac{1}{2}$. In particular on T_N, $\operatorname{Im} z = N+\frac{1}{2} > \frac{1}{2}$ and on B_N, $\operatorname{Im} z = -(N+\frac{1}{2}) < -\frac{1}{2}$ so by (E.6) the estimate

$$|\cot \pi z| < B \tag{E.7}$$

holds on both the contours T_N and B_N.

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Regarding the contours R_N and L_N we have $z = N + \frac{1}{2} + iy$ on R_N and $z = -(N + \frac{1}{2}) + iy$ on L_N. On R_N we have $e^{2\pi iz} = -e^{-2\pi y}$ (since $N \in \mathbb{Z}$) and, similarly, $e^{2\pi iz} = -e^{-2\pi y}$ on L_N.

Now consider the first equation in (E.4) and multiply the fraction by $1 = e^{i\pi z}/e^{i\pi z}$. On both R_N and L_N this leads to

$$\cot \pi z = i\,\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1} = i\,\frac{-e^{-2\pi y}+1}{-e^{-2\pi y}-1},$$

and we conclude that

$$|\cot \pi z| = \frac{|1-e^{-2\pi \operatorname{Im} z}|}{1+e^{-2\pi \operatorname{Im} z}} \le \frac{1+e^{-2\pi \operatorname{Im} z}}{1+e^{-2\pi \operatorname{Im} z}} = 1 \tag{E.8}$$

on both contours R_N and L_N. The inequalities (E.7), (E.8) therefore imply (E.3), as desired, where we note that $e^t \ge 1+t$ for $t \in \mathbb{R}$, so $e^{\pi} \ge 1+\pi > 3 \Rightarrow 2(e^{\pi}-1)-(e^{\pi}+1) = e^{\pi}-3 > 0$. That is, indeed

$$2 > \left(\frac{e^{\pi}+1}{e^{\pi}-1}\right)\frac{e^{-\pi}}{e^{-\pi}} = \frac{1+e^{-\pi}}{1-e^{-\pi}} \stackrel{\text{def}}{=} B.$$

We note also that $\sin \pi z = 0 \iff z = n \in \mathbb{Z}$. That is, since (again) $N \in \mathbb{Z}$ we cannot have $\sin \pi z = 0$ on C_N; in particular C_N avoids the poles of $\cot \pi z = \cos \pi z / \sin \pi z$, and $\cot \pi z$ is continuous on C_N.

Consider now a function $f(z)$ subject to the following two conditions:

C1. $f(z)$ is meromorphic on $\mathbb{C}$, with only finitely many poles $z_1, z_2, \dots, z_k$, none of which is an integer.

C2. There are numbers $M, \rho > 0$ such that $|f(z)| \le M/|z|^2$ holds for $|z| > \rho$.

Then:

THEOREM E.9. $\lim_{N\to\infty} \sum_{n=-N}^{N} f(n)$ *exists and equals minus the sum of the residues of the function* $f(z)\pi \cot \pi z$ *at the poles* $z_1, z_2, \dots, z_k$ *of* $f(z)$. *One can replace condition* C2, in fact, by the more general condition (E.11) below.

PROOF. Since the poles z_j are finite in number we can choose N sufficiently large that C_N encloses all of them. The function $\pi \cot \pi z$ has simple poles at the integers (again as $\sin \pi z = 0 \iff z = n \in \mathbb{Z}$) and the residue at $z = n \in \mathbb{Z}$ is immediately calculated to be 1. Therefore the residue of $\phi(z) \stackrel{\text{def}}{=} f(z)\pi \cot \pi z$ at $z = n \in \mathbb{Z}$ is $f(n)$. As none of the z_j are integers (by C1) the poles of $\phi(z)$ within C_N are given precisely by the set $\{z_j, n \mid 1 \le j \le k, -N \le n \le N, n \in \mathbb{Z}\}$. By the residue theorem, accordingly, we deduce that

$$\frac{1}{2\pi i}\int_{C_N} \phi(z)\,dz = \text{the sum of the residues of } \phi(z) \text{ at the } z_j + \sum_{n=-N}^{N} f(n). \tag{E.10}$$

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Now if

$$\lim_{N\to\infty}\int_{C_N}\phi(z)dz = 0, \tag{E.11}$$

we can let $N\to\infty$ in equation (E.10) and conclude the validity of Theorem E.9 (more generally, without condition C2).

We check that condition (E.11) is implied by condition C2. Since $|z|\geq N+\frac{1}{2}$ on C_N, we have for $N+\frac{1}{2}>\rho$ and z on C_N the bound

$$|f(z)|\leq \frac{M}{|z|^2}\leq \frac{M}{\left(N+\frac{1}{2}\right)^2}.$$

By the main inequality (E.3), we have $|\pi\cot\pi z| < 2\pi$ on C_N, so $|\phi(z)| = |f(z)\pi\cot\pi z| < 2\pi M/\left(N+\frac{1}{2}\right)^2$ on C_N. Given that the length of C_N is $4\left(2\left(N+\frac{1}{2}\right)\right)$ we therefore have the following estimate (for $N+\frac{1}{2}>\rho$):

$$\left|\int_{C_N}\phi(z)dz\right| \leq \frac{2\pi M}{\left(N+\frac{1}{2}\right)^2}8\left(N+\tfrac{1}{2}\right) = \frac{32\pi M}{2N+1}, \tag{E.12}$$

where we note that $\phi(z)$ is continuous on C_N, because, as seen, $\cot\pi z$ is continuous on C_N. The inequality (E.12) clearly establishes the condition (E.11), by which the proof of Theorem E.9 is concluded. □

As an example of Theorem E.9 we choose

$$f(z) = \frac{1}{z^2+a^2} = \frac{1}{(z-ai)(z+ai)}$$

for $a>0$ fixed. Hence f is meromorphic on $\mathbb{C}$ with exactly two simple poles $z_1 \overset{\text{def}}{=} ai$, $z_2 \overset{\text{def}}{=} -ai$. Suppose, for example, that $|z|>\sqrt{2}a$: Then

$$1-\frac{a^2}{|z|^2}>\frac{1}{2},\quad \text{so}\ \left|1+\frac{a^2}{z^2}\right|\geq 1-\frac{a^2}{|z|^2},\quad \text{so}\quad \left|\frac{z^2}{z^2+a^2}\right| = \frac{1}{\left|1+\frac{a^2}{z^2}\right|}<2.$$

Therefore $|f(z)|<2/|z|^2$; that is, $f(z)$ satisfies conditions C1, C2 with $M=2$, $\rho=\sqrt{2}a$. The residue of $\phi(z)\overset{\text{def}}{=} f(z)\pi\cot\pi z$ at z_1 is

$$\lim_{z\to z_1}(z-z_1)\phi(z) = \lim_{z\to ai}\frac{\pi\cot\pi z}{z+ai} = \frac{\pi\cot\pi ai}{2ai} = -\frac{\pi}{2a}\coth\pi a,$$

since $\cos iw = \cosh w$, $\sin iw = i\sinh w$. Similarly, the residue of $\phi(z)$ at z_2 is $-(\pi/2a)\coth\pi a$. As $f(-z)=f(z)$, $\sum_{n=-N}^{N} f(n) = \frac{1}{a^2}+2\sum_{n=1}^{N}\frac{1}{n^2+a^2}$. Theorem E.9 therefore gives

$$\frac{1}{a^2}+2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = -\left(-\frac{\pi}{2a}\coth\pi a-\frac{\pi}{2a}\coth\pi a\right) = \frac{\pi}{a}\coth\pi a,$$

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which proves the summation formula (2.4).

F. Generators of $\mathrm{SL}(2,\mathbb{Z})$**.** Let $\Gamma \overset{\text{def}}{=} \mathrm{SL}(2,\mathbb{Z})$. To prove that the elements $T \overset{\text{def}}{=} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S \overset{\text{def}}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \Gamma$ generate Γ, we start with a lemma.

LEMMA F.1. *Let* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ *with* $c \geq 1$. *Then* γ *is a finite product* $\gamma_1 \cdots \gamma_l$, *where each* $\gamma_j \in \Gamma$ *has the form* $\gamma_j = T^{n_j} S^{m_j}$ *for some* $n_j, m_j \in \mathbb{Z}$. *Here for a group element* g *and* $0 > n \in \mathbb{Z}$, *we have set* $g^n \overset{\text{def}}{=} (g^{-1})^{-n}$.

The proof is by induction on c. If $c = 1$, then $1 = \det\gamma = ad - bc = ad - b$, so $b = ad - 1$, so

$$\gamma = \begin{bmatrix} a & ad-1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \gamma_1\gamma_2$$

for $\gamma_1 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = T^a S^1$, $\gamma_2 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T^d S^0$. Proceeding by induction, we use the Euclidean algorithm to write $d = qc + r$ for $q, r \in \mathbb{Z}$ with $0 \leq r < c \geq 2$, say. If $r = 0$, then $1 = \det\gamma = ad - bc = (aq - b)c$, which shows that c is a positive divisor of 1. That is, the contradiction $c = 1$ implies that $r > 0$. Now

$$\gamma T^{-q} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -q \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -aq+b \\ c & -cq+d \end{bmatrix} = \begin{bmatrix} a & -aq+b \\ c & r \end{bmatrix},$$

so $\gamma T^{-q} S = \begin{bmatrix} a & -aq+b \\ c & r \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -aq+b & -a \\ r & -c \end{bmatrix}$, which equals $\gamma_1 \cdots \gamma_l$ by induction (since $1 \leq r < c$), where each γ_j has the form $\gamma_j = T^{n_j} S^{m_j}$ for some $n_j, m_j \in \mathbb{Z}$. Consequently,

$$\gamma = \gamma_1 \cdots \gamma_l S^{-1} T^q = (\gamma_1 \cdots \gamma_{l-1})(T^{n_l} S^{m_l - 1}) T^q S^0,$$

which has the desired form for γ and which therefore completes the induction and the proof of Lemma F.1.

THEOREM F.2. *The elements* T, S *generate* Γ: *Every* $\gamma \in \Gamma$ *is a finite product* $\gamma_1 \cdots \gamma_l$ *where each* $\gamma_j \in \Gamma$ *has the form* $\gamma_j = T^{n_j} S^{m_j}$ *for some* $n_j, m_j \in \mathbb{Z}$.

PROOF. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ be arbitrary. If $c = 0$, then $1 = \det\gamma = ad$, so $a = d = \pm 1$, so

$$\gamma = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = T^b S^0 \quad \text{or} \quad \gamma = \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = T^{-b} S^2.$$

Since the case $c \geq 1$ is already settled by Lemma F.1, there remains only the case $c \leq -1$. Then $\gamma S^2 = \gamma \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \gamma_1 \cdots \gamma_l$, by Lemma F.1 (since $-c \geq 1$), where the γ_j have the desired form. Thus $\gamma = \gamma_1 \cdots \gamma_l \cdot (T^0 S^{-2})$, as desired. □

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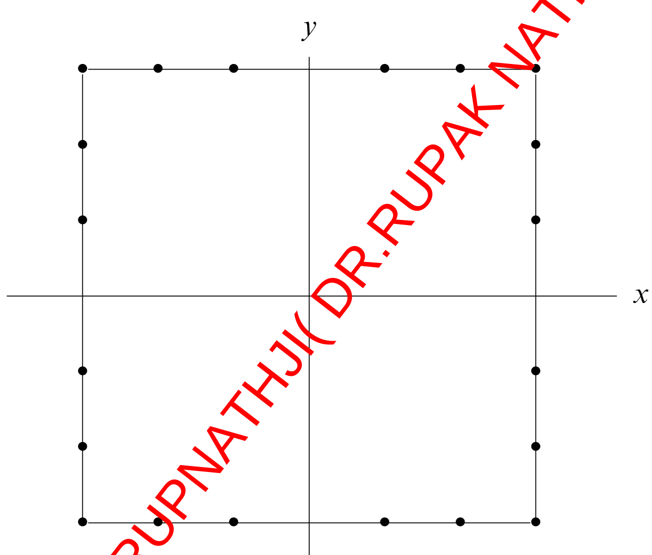
G. Convergence of the sum of $|m+ni|^{-\alpha}$ for $\alpha > 2$. To complete the argument that the Eisenstein series $G_k(z)$ converge absolutely and uniformly on each of the strips $S_{A,\delta}$ in definition (4.5) (for $k = 4, 6, 8, 10, 12, \dots$), we must show, according to the inequality (4.10), that the series

$$S(\alpha) \overset{\text{def}}{=} \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{1}{|m+ni|^\alpha}$$

converges for $\alpha > 2$, where $\mathbb{Z}_*^2 = \mathbb{Z}\times\mathbb{Z} - \{(0,0)\}$.

For $n \geq 1, n \in \mathbb{Z}$, let π_n denote the set of *integer* points on the boundary of the square with vertices (n,n), $(-n,n)$, $(-n,-n)$, $(n,-n)$. As an example, π_3 is illustrated here, with $24 = 8\cdot 3$ points.



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In general π_n has $|\pi_n| \overset{(i)}{=} 8n$ points. Also the π_n partition out all of the nonzero integer pairs:

$$\mathbb{Z}\times\mathbb{Z} - \{(0,0)\} = \bigcup_{n=1}^{\infty} \pi_n \tag{G.1}$$

is a disjoint union.

LEMMA G.2. $\displaystyle\sum_{(a,b)\in\pi_n} \frac{1}{|a+bi|^\alpha} \leq \frac{8}{n^{\alpha-1}}$ *for* $\alpha \geq 0, n \geq 1$.

PROOF. For $(a,b) \in \pi_n$, either $a = \pm n$ or $b = \pm n$, according to whether (a,b) lies on one of the vertical sides of the square (as illustrated above for π_3), or on one of the horizontal sides, respectively. Thus $a^2 + b^2 =$ either $n^2 + b^2$ or

$a^2+n^2 \Rightarrow a^2+b^2 \geq n^2$. That is, for $(a,b) \in \pi_n$ we have $|a+bi|^2 = a^2+b^2 \geq n^2$, so $|a+bi|^\alpha \geq n^\alpha$ (since $\alpha \geq 0$). Inverting and summing we get

$$\sum_{(a,b)\in\pi_n} \frac{1}{|a+bi|^\alpha} \leq \sum_{(a,b)\in\pi_n} \frac{1}{n^\alpha} \leq \frac{8n}{n^\alpha} \quad \text{(by (i))}$$
$$= \frac{8}{n^{\alpha-1}},$$

which proves Lemma G.2. □

Now use (G.1) and Lemma G.2 to write

$$S(\alpha) = \sum_{n=1}^{\infty} \sum_{(a,b)\in\pi_n} \frac{1}{|a+bi|^\alpha} \leq \sum_{n=1}^{\infty} \frac{8}{n^{\alpha-1}} \tag{G.3}$$

for $\alpha \geq 0$, which proves that $S(\alpha) < \infty$ for $\alpha - 1 > 1$, as desired.

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