Introduction

The lectures at Stanford which were the first version of this book were entitled 'Topological and Conformal Field Theory'. I have dropped conformal field theory from the title, as it plays a subordinate role in what follows, but I think it is worth beginning by explaining the contrasting positions which topological and conformal field theory occupy on the mathematical stage at present.

Conformal field theory — which by its nature is two-dimensional — is an elaborate and fairly well understood body of mathematical knowledge. It includes the representation theory of Kac-Moody algebras and the corresponding loop groups, and of the Virasoro algebra and the group of diffeomorphisms of the circle. From the point of view of physics, it is essentially the study of critical-point phenomena in two-dimensional statistical mechanics. No-one can doubt its mathematical depth and richness, or its potential for diverse applications. (A good example of an application quite far from the usual preoccupations of quantum field theory is Cardy's use of it in solving percolation problems [J]. But although conformal field theory is such a central and well-established part of mathematics it is still not altogether "assimilated" and it remains rather inaccessible to most pure mathematicians.

Topological field theory has a more dubious and almost opposite status. The idea of a topological field theory was introduced by Witten as a rudimentary structure to which, in principle, any quantum field theory reduces at very long distances and low energies. But the concept has a simple and attractive mathematical formalization, which was first written down by Atiyah [?]. Witten pointed out that, despite the simplicity of the idea, there are a number of examples of topological field theories which are very relevant in geometric topology. One of them provides a unified point of view on the knot invariants discovered by Vaughan Jones, and the associated invariants of 3-manifolds. Another encodes the Donaldson invariants of 4-manifolds, and the Floer cohomology groups of 3-manifolds. As far as I know, unfortunately, the ideas of topological field theory have not yet helped to solve any

DR.RUPNATHJI(DR.RUPAK NATH)

important problem in geometric topology, but — to me at least — the mathematical structures themselves are irresistibly fascinating. Low dimensional topology is dominated by the properties of surfaces, which are essentially combinatorial, and topological field theory seems to be telling us that the geometry of surfaces can be effectively described by familiar algebraic structures. Whether this is really an important discovery, rather than just an odd coincidence, it is too early to say, but there is some reason to hope that the formalism will be applicable in areas — algebraic number theory, for instance — not obviously related to its origins.

It seems likely that the mathematics of topological field theories will stand or fall with the success of string theory as a theory of gravitation and elementary particles. It would be very rash for me to try to say just what string theory is, but one possible way of looking at it is to say that it replaces the finite dimensional space-time manifolds of conventional quantum field theory by a completely new kind of "stringy" manifold. To give a conventional manifold is the same as to give the commutative algebra of smooth functions on it. A "stringy" manifold is described not by a commutative algebra but by a more sophisticated algebraic structure which is — from the point of view of topological field theory — a fairly natural generalization of a commutative algebra. My main object in these lectures is to explain this idea. It can perhaps be compared with the "non-commutative geometry" programme of Connes. The best prospect of a real mathematical success of the string programme seems to be the elucidation of mirror symmetry of Calabi-Yau manifolds, with which Kontsevich has recently made much progress.

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I shall outline the contents of the six lectures.

Lecture 1 gives the definition of a topological field theory, and describes the basic properties, emphasizing that a two-dimensional theory is the same thing as a commutative algebra. I also introduce two simple generalizations corresponding to non-commutative algebras (or linear categories), and to topological algebras. I have included a brief account of Witten's beautiful example of the description of the moduli spaces of bundles on curves by two-dimensional Yang-Mills theory.

Lecture 2 is a digression explaining how the index theory of the Dirac operator on a compact manifold can be described in the language of quantum field theory. It gives me the opportunity of introducing the basic concept of the fermionic Fock space in Dirac's original setting. There is also a general discussion of the determinant of a Fredholm operator, which is needed later in other situations, but is applied in this lecture to the η-invariant of the Dirac operator.

Lecture 3 introduces the idea of a two-dimensional topological field theory which takes values in additive categories rather than vector spaces. Such a theory is the same thing as a particular kind of braided tensor category, and it gives rise to a three-dimensional topological field theory, and hence to invariants of knots and 3-manifolds. The remarkable thing about these structures is that, though they crop up in many different guises and situations, there seems to be only one class of examples, which is obtained by deforming the category of representations of a Lie group. I think of them as the categories of representations of loop groups, but they are most often described as representations of "quantum groups at roots of unity".

Lecture 4 is a very brief account of conformal field theory, partly for its own sake, partly to describe some of the representation theory which exemplifies the ideas of the preceding lecture, and partly to introduce some ideas which are needed in the remaining two lectures, especially the BRST complex and the $N = 2$ supersymmetry algebra. The essential thing to understand about conformal field theory is its relation to the representation theory of the group of diffeomorphisms of the circle.

Lecture 5 is about two-dimensional topological field theories whose values are cochain complexes rather than vector spaces. These structures are sometimes called "string backgrounds", but I shall call them *string algebras*. The lecture begins with a short account of the conventional construction of an algebraic model of homotopy theory. I then consider why the infinite dimensional spaces which arise in quantum field theory need to be treated in a slightly different way. String algebras are designed to capture some of the essential features of loop spaces. There are two different kinds of examples: one kind models the algebra of functions on a loop space, and the other kind models the de Rham complex of a loop space. The second kind, though it arises in connection with $N = 2$ supersymmetric conformal field theories, probably has more to do with geometry and topology than with physics. It is the natural setting for Floer's homology theory for trajectories

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in a symplectic manifold, and for what has recently been called "quantum cohomology".

Lecture 6 continues the study of string algebras. I describe in more detail the relationship with the homotopy theory of the moduli spaces of surfaces, and also some connections with the theory of integrable systems. The true nature of these connections does not seem at all well understood. I discuss the WDVV equations and their relation to Frobenius manifolds in the sense of Dubrovin, and also the Witten-Kontsevich relation between the KdV equation and the characteristic numbers of the moduli spaces of surfaces.

DR.RUPNATHJI(DR.RUPAK NATH)