Lecture 2

The Index and Determinant of the Dirac Operator

This chapter is something of a digression, but is put in for two reasons. First, the patching-together properties of the index of the Dirac operator provide a good example of the formal structure of topological field theory, and will serve to motivate some of the abstract-looking category theory which we shall come to in the next chapter. Secondly, the construction of the determinant gives us our first example of a more-or-less "realistic" —not topological— field theory. It will be the basic tool when we turn to conformal field theory in Chapter Four.

An n-dimensional topological field theory gives us a number $F(X)$ for every closed oriented n-dimensional manifold X. Among the invariants of manifolds which arise in this way are the *signature* of a $4k$-manifold, i.e. the signature of the quadratic form on the middle-dimensional homology of X given by the intersection pairing, and the $\hat{A}$*-genus*, which is defined as the index of the Dirac operator on an even-dimensional spin manifold. I shall concentrate here on the $\hat{A}$-genus $\hat{A}(X)$, but the signature $\mathrm{sign}(X)$ is a closely related invariant, and I shall return to it in §2.?. (Both $\hat{A}(X)$ and $\mathrm{sign}(X)$ are additive, rather than multiplicative, for disjoint unions, so to fit them directly into the framework of Chapter 1 we should consider $e^{\hat{A}(X)}$ and $e^{\mathrm{sign(X)}}$.

Let us recall the most basic facts about Dirac operators and Fredholm operators.

2.1 The Dirac operator

Dirac defined his operator in Euclidean space $\mathbb{R}^n$ as

$$D = \sum \gamma_i \frac{\partial}{\partial x_i},$$

where $\gamma_1, \cdots, \gamma_n$ are $N \times N$ skew-hermitian matrices satisfying $\gamma_i^2 = -1$ and $\gamma_i\gamma_j = -\gamma_j\gamma_i$ when $i \neq j$. His idea was to find a first-order operator D whose square was the Laplacian:

$$D^2 = -\sum \left(\frac{\partial}{\partial x_i}\right)^2.$$

To give the matrices γ_i is the same as to give for each vector $\xi \in \mathbb{R}^n$ a matrix c_ξ depending linearly on ξ such that $c_\xi^2 = -||\xi||^2$, for if $\xi = (\xi_i)$ we can define $c_\xi = \sum \xi_i \gamma_i$. In other words, we are giving a matrix representation of the Clifford algebra $C(\mathbb{R}^n)$.

To make sense of this on a general Riemannian manifold M we must give for each $x \in M$ a complex vector space Δ_x with an inner product, called the space of *spinors* at x, and for each cotangent vector ξ at x a skew transformation

$$c_\xi : \Delta_x \to \Delta_x$$

such that $c_\xi^2 = -||\xi||^2$. The spaces Δ_x must fit together to form a vector bundle Δ on M. If each space Δ_x is an *irreducible* representation of the Clifford algebra $C(T_x^*)$ then Δ is called a *spin bundle*. A choice of such a bundle is traditionally called a *spin^c-structure* on M. To define the Dirac operator we also need to choose a connection on Δ which is compatible with the Levi-Civita connection of M. A connection in Δ is a rule which enables us to differentiate any spinor field s —i.e. a section s of Δ— along any tangent vector field ξ to M. I shall write the derivative $\nabla_\xi s$. Compatibility with the Levi-Civita connection means that

$$\nabla_\xi(c_\eta s) = c_{\nabla_\xi \eta} s \; + \; c_\eta \nabla_\xi s.$$

When we have a spin bundle with a connection, we define the Dirac operator D_M as the operator given locally by

(2.1.1) $$D_M = \sum c_{\xi_i} \nabla_{\xi_i}$$

where $\{\xi_i\}$ is a set of tangent vector fields which form an orthonormal basis of the tangent space at each point. (Of course D_M is independent of the choice of the ξ_i). Solutions s of the equation $D_M s = 0$ are called *harmonic* spinor fields.

We must distinguish two cases. If M is *even-dimensional* —say of dimension $n = 2k$— then the spin bundle Δ is of dimension 2^k, and it automatically splits as a sum

$$\Delta \; = \; \Delta^{even} \; \oplus \; \Delta^{odd},$$

where Δ^{even} and Δ^{odd} are the (± 1)-eigenspaces of the operator

$$\omega \; = \; i^{\frac{1}{2}n(n+1)} c_{\xi_1} c_{\xi_2} \dots c_{\xi_n},$$

which has square 1. The operator ω depends on the orientation of M, and changes sign if it is reversed. Reversing the orientation therefore interchanges Δ^{even} and Δ^{odd}. The Dirac operator takes sections of Δ^{even} to sections of Δ^{odd}, and vice versa. We shall write it

$$D_M \;=\; D_M^{even} \;\oplus\; D_M^{odd},$$

where $D_M^{even} \;:\; \Gamma(\Delta^{even}) \to \Gamma(\Delta^{odd})$ is the adjoint of $D_M^{odd} \;:\; \Gamma(\Delta^{odd}) \to \Gamma(\Delta^{even})$. When one speaks of the index of the Dirac operator, one always means the index of D_M^{even}, as the self-adjoint operator D_M has index zero.

If M is of *odd dimension* $n = 2k+1$, on the other hand, then the spin bundle Δ has dimension 2^k, and does not split into even and odd parts. In this case the operator ω acts as the scalar ± 1 on each Δ_x, and, by replacing c_ξ by $-c_\xi$ if necessary, we can assume ω acts as $+1$. Reversing the orientation of M therefore changes D_M to $-D_M$.

If M is a manifold with a boundary ∂M, then at each point $x \in \partial M$ we have the map $c_{v(x)} \;:\; \Delta_x \to \Delta_x$ such that $c_{v(x)}^2 = -1$, where $v(x)$ is the unit inward normal vector to ∂M at x. If M is even dimensional, these maps define an isomorphism

$$\Delta_M^{even}|\partial M \;\cong\; \Delta_M^{odd}|\partial M,$$

and either of these bundles can be identified with $\Delta_{\partial M}$. If M is odd-dimensional, then the $(\pm i)$-eigenspaces of $c_{v(x)}$ for $x \in \partial M$ split $\Delta_M|\partial M$ as the sum of two bundles which can be identified with $\Delta_{\partial M}^{even}$ and $\Delta_{\partial M}^{odd}$.

2.2 Fredholm operators

If E and F are topological vector spaces, a *Fredholm operator* $T : E \to F$ is a continuous linear map which has an inverse, or "parametrix", modulo operators of finite rank, i.e. for which there is a continuous $P \;:\; F \to E$ such that $P \circ T$ and $T \circ P$ differ from the identity by finite rank operators. If T is Fredholm it is easy to see that the kernel and cokernel

$$\begin{aligned} \ker(T) &= \{\xi \,\epsilon\, E \;:\; T\xi = 0\} \\ \operatorname{coker}(T) &= F/T(E) \end{aligned}$$

are finite dimensional, and that the image $T(E)$ is a closed subspace of F. The converse is true, too, if E and F are Fréchet spaces. If $T \;:\; E \to F$ is Fredholm, its *index* $\chi(T)$ is defined by

$$\chi(T) = \dim(\ker(T)) - \dim(\operatorname{coker}(T)).$$

The important property of the index is that in many situations it does not change when T is deformed continuously. For example, if E and F are Banach spaces, and the space $\mathrm{Fred}(E;F)$ of Fredholm operators is given the norm topology, then T_0 and T_1 belong to the same connected component of $\mathrm{Fred}(E;F)$ if and only if they have the same index. But, quite generally, if $\{T_t\}$ is a family of Fredholm operators, then $\chi(T_t)$ is a continuous function of t providing one can find a family $\{P_t\}$ of parametrices such that the operators $P_t \circ T_t - 1$ and $T_t \circ P_t - 1$ are compact, and vary continuously with t in the uniform topology. This is always the case if $\{T_t\}$ is a family of elliptic differential operators on a compact manifold, and T_t depends smoothly on t.

2.3 Localizing the index

Like the signature, the $\hat{A}$-genus of a manifold appears to be of an altogether global nature, and when a closed manifold X is a union $X = X_1 \cup X_2$ of two manifolds with boundary whose intersection is their common boundary Y, there seems at first no reason why $\hat{A}(X)$ should be a sum of contributions from X_1 and X_2. The Atiyah-Singer index theorem, however, tells us that $\hat{A}(X)$ is in some sense a sum of local contributions, for it asserts

$$\begin{aligned} \hat{A}(X) &= \int_X \alpha_X \\ &= \int_{X_1} \alpha_X + \int_{X_2} \alpha_X \\ &= \hat{A}(X_1) + \hat{A}(X_2), \end{aligned} \tag{2.3.1}$$

say, where α_X is a differential form constructed locally from the geometry of X.

The formula (2.3.1) splits $\hat{A}(X)$ into contributions from X_1 and X_2; but the contributions are not integers, and —more importantly— they depend on the Riemannian structure of X_1 and X_2, while $\hat{A}(X)$ does not. Nevertheless, the image of $\hat{A}(X_1)$ or $\hat{A}(X_2)$ in $\mathbb{R}/\mathbb{Z}$ depends only on the structure of X in an arbitrarily small neighbourhood of the interface Y, for if either X_1 or X_2 is replaced by another manifold which is indistinguishable in the neighbourhood of Y then $\hat{A}(X)$ can only change by an integer. Let us write Z_Y for the set of real numbers congruent to $\hat{A}(X_1)$ modulo $\mathbb{Z}$: it is a set with a free transitive action of the additive group $\mathbb{Z}$, i.e. an "affine space" for $\mathbb{Z}$, or $\mathbb{Z}$-*torsor*. (The word "torsor" seems rebarbative, but at least it is short).

That there should be a $\mathbb{Z}$-torsor Z_Y, depending only on Y, in which X_1 and X_2 define elements $\hat{A}(X_1)$ and $-\hat{A}(X_2)$ whose "difference" is $\hat{A}(X)$, is quite easy to see directly, without using the index theorem. Let us write K, K_1 and K_2 for the harmonic spinor fields on X, X_1, and X_2. Because the Dirac operator is of first order and elliptic, an element $s \in K$ is the same thing as a pair $s_1 \in K_1$, $s_2 \in K_2$ such that $s_1|Y = s_2|Y$. Furthermore, s_1 and s_2 are completely determined by their boundary values $s_1|Y$ and $s_2|Y$, and so one can regard K_1

and K_2 as subspaces of the space Γ_Y of all smooth spinor fields on Y. Thus we have $K = K_1 \cap K_2$. In fact rather more is true.

Proposition 2.3.2 *(i) The subspaces K_1 and K_2 of Γ_Y are closed, and there is an exact sequence*

$$0 \to \ker(D_X) \to K_1 \oplus K_2 \to \Gamma_Y \to \operatorname{coker}(D_X) \to 0 \tag{2.3.3}$$

where the middle map is the sum of the inclusions.

(ii) There is an orthogonal decomposition $\Gamma_Y = K_2 \bigoplus K_2^{\perp}$, and hence an exact sequence

$$0 \to \ker(D_X) \to K_1 \to K_2^{\perp} \to \operatorname{coker}(D_X) \to 0 \tag{2.3.4}$$

where the middle map is the orthogonal projection.

Before giving the proof let us notice how the result relates to the factorization of the index of D_X. If the spaces K_1 and $K_2^{\perp}$ were finite-dimensional, (2.3.4) would tell us that $\chi(D_X)$ was the difference between their dimensions. Of course they are infinite-dimensional, but we shall see that they belong to a special class of closed subspaces of Γ_Y which are sufficiently close to each other for any two of them to have a well-defined relative dimension, say $\dim(K_1 : K_2^{\perp})$. Thus (2.3.4) implies

$$\chi(D_X) = \dim(K_1 : K_2^{\perp}).$$

The subspaces of Γ_Y in question form its *restricted Grassmannian* Gr_Y. This is a space whose set of connected components $\pi_0 \mathrm{Gr}_Y$ forms a $\mathbb{Z}$-torsor Z_Y, the components being distinguished by their relative dimension.

2.4 Polarized vector spaces and the restricted Grassmannian

The concept of a polarized topological vector space will play a prominent role throughout these lectures. The vector spaces we shall consider will always be assumed to be locally convex and complete.

A polarization of a vector space E is a *class* of allowable decompositions $E = E^+ \oplus E^-$ which are fairly close to each other. The meaning of "fairly close" is somewhat elastic, depending on our precise purposes. At the least, we want to permit any *finite dimensional* changes to E^+ and E^-, but for the present a very loose definition will suffice. To state it, it is useful to identify decompositions $E = E^+ \oplus E^-$ with the corresponding operators $J : E \to E$ such that $J|E^{\pm} = \pm 1$.

Definition 2.4.1 *A* coarse polarization *of a vector space E is a class $\mathcal{J}$ of operators $J : E \to E$ such that* *(i)* $J^2 = 1$ *modulo compact operators*

(ii) any two operators in $\mathcal{J}$ *differ by a compact operator, and*

(iii) $\mathcal{J}$ *does not contain* ± 1.

Example 2.4.2 If E is the space of smooth functions on the circle S^1 then we have a decomposition $E = E^+ \oplus E^-$, where E^+ is spanned by the functions $e^{in\theta}$ for $n < 0$, and E^- by $e^{in\theta}$ for $n \geqslant 0$. The polarization so defined does not depend on the parametrization of the circle, for the operators J corresponding to two different choices differ by an integral operator with a smooth kernel. (See [PS] page 91). We could also transfer any finite number of the functions $e^{in\theta}$ from E^+ to E^-, or vice versa, without changing the polarization.

The functions $e^{in\theta}$ are characterized as the eigenfunctions of the operator $i\frac{d}{d\theta}$ on S^1, which in fact is the Dirac operator on S^1. For any odd-dimensional compact Riemannian spin manifold Y, the space Γ_Y of smooth spinor fields on Y is correspondingly polarized by $\Gamma_Y = \Gamma_Y^+ \oplus \Gamma_Y^-$ where Γ_Y^+ is spanned by the eigenfunctions of the Dirac operator with eigenvalues $\geqslant 0$, and Γ_Y^- by those with eigenvalues < 0.

When we have a polarized vector space E we can define its *restricted Grassmannian* $\mathrm{Gr}(E)$ as the set of all subspaces which can occur as the "negative energy" part E^- in one of the allowable decompositions $E = E^+ \oplus E^-$. If $E = \widetilde{E}^+ \oplus \widetilde{E}^-$ is another allowable decomposition then the projection of $\tilde{E}^-$ on to E^- along E^+ is automatically Fredholm, and its index is called the *relative dimension* dim $(\tilde{E}^- : E^-)$. The set $\mathrm{Gr}(E)$ is naturally an infinite dimensional manifold, for any $W \in \mathrm{Gr}(E)$ which is near E^- is the graph of a compact operator $E^- \to E^+$, and so a neighbourhood of E^- can be identified with the vector space $\mathrm{Hom}_{\mathrm{cpt}}(E^-; E^+)$. It is easy to see that two points of $\mathrm{Gr}(E)$ are in the same connected component if and only if their relative dimension is zero, and so $\pi_0\mathrm{Gr}(E)$ is a $\mathbb{Z}$-torsor as desired. The component of $\mathrm{Gr}(E)$ to which a subspace E^- belongs will be called its *virtual dimension*, and written simply $\dim(E^-)$.

2.5 The polarization of spinors on the boundary

My aim in this section is to show that there is a polarization of the spinor fields Γ_Y on a compact manifold Y such that whenever Y is the boundary of X_1 the boundary values of harmonic spinor fields on X_1 form a closed subspace K_{X_1} belonging to the restricted Grassmannian $\mathrm{Gr}(\Gamma_Y)$. If we only want a coarse polarization this is quite easy. We have already remarked that the self-adjoint Dirac operator on Y splits Γ_Y as $\Gamma_Y^+ \oplus \Gamma_Y^-$ according to the sign of the eigenvalues, and it is not too hard to show that the projections $K_{X_1} \to \Gamma_Y^+$ and $K_{X_1} \to \Gamma_Y^-$

are Fredholm and compact respectively. For our future purposes, however, we need a more precise result.

Let us examine the construction of the projection operator $\Gamma_Y \to K_{X_1}$, which is called the *Calderon projector*. We must use a little of the technology of pseudo-differential operators. The main points are that a pseudo-differential operator is determined up to a smoothing operator by its *symbol*, and that the symbol of the Calderon projector can be calculated explicitly from the symbol of the Dirac operator D_{X_1} by a local formula. Thus only the jets of the symbol of D_{X_1} at points of Y are relevant. The following argument is taken from Hörmander [].

There is no loss of generality in taking X_1 to be part of a closed manifold $X = X_1 \cup X_2$. Let P be a parametrix for D_X on X, i.e. an inverse modulo smoothing operators. We can choose it so that $D_X \circ P$ is exactly the identity on distributions with support in X_1. (Pseudo-differential operators extend automatically to act on distributions.) Let s_1 be a harmonic spinor field on X_1, and extend it by zero to X. Then $D_X s_1$ is a δ-function distribution along Y. To be precise, we can write $s_1 = \chi \tilde{s_1}$, where χ is the characteristic function of X_1, and $\tilde{s_1}$ is a smooth extension of s_1 to X. We have

$$\begin{aligned} D_X s_1 &= D_X(\chi \tilde{s_1}) \\ &= \chi D_X \tilde{s_1} + c_{d\chi} \tilde{s_1} \\ &= c_{d\chi} \tilde{s_1}, \end{aligned}$$

where $c_{d\chi}$ denotes Clifford multiplication by the distributional 1-form $d\chi$. Now $c_{d\chi}$ is supported on Y: it can be written $\gamma_Y \ \delta_Y$, where δ_Y is the δ-function along Y and γ_Y is the unit conormal vector field to Y. So

$$D_X s_1 = (\gamma_Y \cdot (s_1|Y))\delta_Y.$$

Let us define an operator $C : \Gamma_Y \to \Gamma_{X_1}$ by

$$C(s) = \{P((\gamma_Y s) \cdot \delta_Y)\}|X_1.$$

This has its image in K_{X_1}. (One needs to check that $C(s)$, which is automatically smooth in the interior of X_1, extends smoothly to the boundary. For this, see [].) Furthermore, if s is the boundary value of $s_1 \in K_{X_1}$ then

$$C(s) = PDs_1 = s_1 + Ss_1,$$

where S is a smoothing operator on X.

Thus C differs from the identity on $K_{X_1} \subset \Gamma_Y$ only by a smoothing operator, and we can easily correct it by adding a smoothing operator to make it exactly the identity on K_{X_1}. The resulting operator can be regarded in two ways:

(i) an operator $C : \Gamma_Y \to \Gamma_{X_1}$, in which guise it is the integral formula mentioned earlier which expresses a harmonic spinor field in terms of its boundary values, or

(ii) or as a projection operator $C : \Gamma_Y \to \Gamma_Y$.

Hörmander shows that $C : \Gamma_Y \to \Gamma_Y$ is a pseudo-differential operator of order zero, and he gives a formula ([]) for its symbol in terms of that of P.

To prove that K_{X_1} is close to Γ_Y^- we must compare the projection C with the orthogonal projection on to Γ_Y^-. The latter, however, is just the Calderon projection C_0 corresponding to regarding Y as the boundary of $Y \times \mathbb{R}_+$ with the product metric. For we can take as a parametrix for $D_{Y\times\mathbb{R}}$ the Feynman propagator

$$P_0 : C^\infty_{cpt}(\mathbb{R}; \Gamma_Y) \to \mathcal{S}(\mathbb{R}; \Gamma_Y)$$

defined by

$$(P_0 f)(t) = \sum_{\lambda<0} \int_{\mathbb{R}_+} e^{s\lambda} f_\lambda(t+s)\, ds \;\; + \;\; \sum_{\lambda>0} \int_{\mathbb{R}_-} e^{s\lambda} f_\lambda(t+s)\, ds,$$

where $f = \sum f_\lambda$ is the decomposition of an element $f \in \Gamma_Y$ into eigenfunctions of D_Y, and $\mathcal{S}$ denotes the rapidly decreasing smooth functions. The calculus of pseudo-differential operators, however, gives us a rival parametrix

$$\widetilde{P}_0 : C^\infty_{cpt}(\mathbb{R}; \Gamma_Y) \to C^\infty_{cpt}(\mathbb{R}; \Gamma_Y).$$

It is easy to see that P_0 and $\widetilde{P}_0$ must differ by a smoothing operator.

Thus, finally, we need to know how the Calderon projection C depends on the metric of the manifold X. If we are content to assume that the metric of X agrees with that of $Y \times \mathbb{R}$ to infinite order along Y then there is no more to say than that the symbol of C can be calculated locally from that of D_X. If the metric is not a product, but we are interested only in a coarse polarization, we need only check that the leading term of the symbol of C depends only on the metric of Y, for a pseudo-differential operator of order -1 is compact. But the fundamental result is

Theorem 2.5.1 *The polarization of Γ_Y defined by the Calderon projection for a manifold X_1 with $\partial X_1 = Y$ depends on the first $[n/2]$ normal derivatives of the metric of Y, where $n = dim(Y)$.*

In other words, if $\dim(X_1) = 2$ then the polarization is independent of X_1, while if $\dim(X_1) = 4$ it depends on both the metric and the second fundamental form of Y. I shall return to the importance of this for quantum field theory in section 2.

Proof of 2.5.1 We first calculate the symbol of the parametrix P of D_X. We can work in local coordinates $(x_0, \cdots, x_n;\ \xi_0, \cdots, \xi_n)$ for T^*X, so that the symbol of D_X is the matrix-valued function $\gamma_\xi = \sum \gamma_i(x)\xi_i$. Then P is of order -1, and its symbol is

$$p_{-1}(x;\xi) \; + \; p_{-2}(x;\xi) + \cdots$$

where p_{-k} is homogeneous in ξ of degree $-k$: it is not defined when $\xi = 0$. Clearly $p_{-1} = \gamma_\xi^{-1}$, and we find by recurrence that

$$p_{-k} = (-1)^{k-1}\, \gamma_\xi^{-1}\, D\, \gamma_\xi^{-1}\, D\, \gamma_\xi^{-1} \cdots D\, \gamma_\xi^{-1},$$

where there are k factors γ_ξ^{-1}, and D denotes the differential operator

$$D = \sum \gamma_i(x) \frac{\partial}{\partial x_i}.$$

Because $\gamma_\xi^2 = -||\xi||^2$ we see that $p_k(x;\xi)$ is, for each x, a matrix-valued polynomial in ξ, divided by a power of $||\xi||^2$. (Notice that $||\xi||^2$ depends on x as well as ξ.)

To calculate the symbol of C we choose the coordinates so that $(x_0;\xi_0)$ are normal to Y, which is defined by $x_0 = 0$. Then Hörmander's formula ([]) for the symbol

$$c_0(y;\eta)\ +\ c_{-1}(y;\eta)\ +\ \cdots,$$

where $y = (x_1, \cdots, x_n)$ and $\eta = (\xi_1, \cdots, \xi_n)$, amounts to saying that $c_{-k}(y;\eta)$ is obtained from $p_{-k-1}(0, y; \xi_0, \eta)$ simply by taking the residue of the latter, regarded as a matrix-valued function of ξ_0 for fixed $(y;\eta)$, at its unique pole in the upper half-plane, i.e. at $\xi_0 = i||\eta||$. Thus c_{-k}, like p_{-k-1}, involves k normal derivatives of the metric along Y.

On an n-dimensional manifold a pseudo-differential operator is Hilbert-Schimdt if its order is strictly less than $-\frac{1}{2}n$, so we need to retain all terms of order $\geqslant -\frac{1}{2}n$ to define the polarization, i.e. we need $[\frac{1}{2}n]$ normal derivatives of the metric. ∎

2.6 Subdividing the boundary: the appearance of categories

To express the localizability of the index of the Dirac operator on even-dimensional closed manifolds we were led to associate algebraic objects Z_Y to closed manifolds Y of one lower dimension. If Y in turn is the union of manifolds Y_1, Y_2 which intersect in their common boundary Σ, we may ask whether Z_Y can be constructed from objects Z_{Y_1} and Z_{Y_2} associated to the pieces. This is indeed the case. But just as, when $X = X_1 \cup_Y X_2$, the contributions of X_1 and X_2 to the index of D_X were not themselves integers, but were elements of the $\mathbb{Z}$-torsor Z_Y associated to Y, so the objects Z_{Y_1} and Z_{Y_2} will not be $\mathbb{Z}$-torsors, but instead will be objects of a new category $\mathcal{Z}_\Sigma$ associated to Σ. The category $\mathcal{Z}_\Sigma$ is a *groupoid*, i.e. all of its morphisms are isomorphisms. The group of automorphisms of each object of $\mathcal{Z}_\Sigma$ is $\mathbb{Z}$, and the morphisms from any object to any other form a $\mathbb{Z}$-torsor. In particular Z_Y is the set of all morphisms from Z_{Y_1} to Z_{Y_2}.

Before explaining this any further we should take some thought of the slipperiness of the slope we have stepped upon. It would be easy to ask what happens if $\Sigma = \Sigma_1 \cup \Sigma_2$, and to slither into a wilderness of 2-categories. Personally, I think it is worth going as far as categories. The justification for that must be mainly aesthetic, but there is one objective fact that is relevant, arising from Morse theory.

By choosing a generic Morse function $f : X \to \mathbb{R}$ on a closed manifold X we can slice X up as a union

$$X = X_0 \cup X_1 \cup \dots \cup X_m,$$

where each slice $X_k = f^{-1}([t_k, t_{k+1}])$ contains only one critical point of f, and is a cobordism $Y_k \rightsquigarrow Y_{k+1}$ between two smooth level sets of f. We cannot assume that the slices X_k have any simple standard form, though X_k differs from the cylinder $Y_k \times [0, 1]$ by attaching a single "handle". But by cutting the manifolds Y_k into two we can describe the situation much more explicitly. We write

$$Y_k = Y_k' \cup (S^{p-1} \times D^q),$$

where p is the index of the handle to be attached, and $p + q = n$. The two parts in this splitting intersect in $S^{p-1} \times S^{q-1}$. Then

$$Y_{k+1} = Y_k' \cup (D^p \times S^{q-1}),$$

and the cobordism from Y_k to Y_{k+1} is simply the union of the trivial cobordism $Y_k' \times [0, 1]$ with the standard cobordism $D^p \times D^q$ from $S^{p-1} \times D^q$ to $D^p \times S^{q-1}$. Thus in the end everything is reduced to understanding the category $\mathcal{Z}_{S^{p-1} \times S^{q-1}}$, the objects $Z_{S^{p-1} \times D^q}$ and $Z_{D^p \times S^{q-1}}$ which belong to it, and the morphism

$$Z_{S^{p-1} \times D^q} \to Z_{D^p \times S^{q-1}}$$

defined by the standard cobordism. In principle, at least, this is all very explicit.

We shall return to the preceding considerations in §?. Meanwhile, let us describe the category $\mathcal{Z}_\Sigma$ and its properties. The best known example of a groupoid is the *fundamental groupoid* $\pi_1(B)$ of a path-connected space B. This is the category whose objects are the points of B, and whose morphisms from b_0 to b_1 are the homotopy classes of path in B from b_0 to b_1. Thus the group of automorphisms of b_0 is the fundamental group $\pi_1(B, b_0)$, and the morphisms from b_0 to b_1 form a torsor for this group. Up to equivalence of categories, one can also say that $\pi_1(B)$ is the category whose objects are the universal covering spaces $\tilde{B}$ of B, and whose morphisms $\tilde{B}_0 \to \tilde{B}_1$ are the covering maps, i.e. the maps which cover the identity map of B. (The usual construction of the universal covering space of B as the space of homotopy classes of paths in B with a chosen starting-point defines a functor from the first definition of $\pi_1(B)$ to the second.)

The object Z_Y associated to an odd-dimensional manifold Y was defined as $\pi_0(\mathrm{Gr}_Y)$, where Gr_Y was a Grassmannian of subspaces of the space Γ_Y of spinor fields on Y. The groupoid $\mathcal{Z}_\Sigma$ is the fundamental groupoid of an analogous Grassmannian $\mathcal{J}_\Sigma$ formed by a certain class of subspaces of Γ_Σ. The space $\mathcal{J}_\Sigma$ is connected, and its fundamental group is $\mathbb{Z}$, so the sets of morphisms in the category $\pi_1(\mathcal{J}_\Sigma)$ are $\mathbb{Z}$-torsors, as we want. Both points of view on the fundamental groupoid are relevant. If Y is a manifold with boundary Σ then the boundary values of harmonic spinor fields on Y form a space belonging to $\mathcal{J}_\Sigma$, and hence define an object of $\pi_1(\mathcal{J}_\Sigma) = \mathcal{Z}_\Sigma$. But a point of $\mathcal{J}_\Sigma$ can also, as we shall see, be regarded as a self-adjoint boundary condition for the Dirac operator on Y. So each point σ of $\mathcal{J}_\Sigma$ defines a polarisation of Γ_Y, and hence a restricted Grassmannian $\mathrm{Gr}_{Y,\sigma}$. As σ varies the sets $Z_{Y,\sigma} = \pi_0(\mathrm{Gr}_{Y,\sigma})$ from a covering space Z_Y of $\mathcal{J}_\Sigma$, and hence an object of the category $\pi_1(\mathcal{J}_\Sigma)$.

To define $\mathcal{J}_\Sigma$ we begin with the formula which expresses the self-adjointness of the Dirac operator D_Y on an arbitrary manifold Y.

$$(2.6.1) \qquad -\langle D_Y\varphi_1, \varphi_2\rangle + \langle \varphi_1, D_Y\varphi_2\rangle = \mathrm{div}\langle \varphi_1, \gamma\varphi_2\rangle.$$

Here ψ_1 and ψ_2 are spinor fields, and $\langle\ ,\ \rangle$ denotes their pointwise inner product. The expression $\langle \psi_1, \gamma\psi_2\rangle$ denotes the 1-form (or vector field) whose components with respect to a local orthonormal framing ξ_i of the tangent bundle are, in the notation of (2.1.1), the functions $\langle \psi_1, c_{\xi_i}\psi_2\rangle$. Integrating (2.6.1) over Y with $\partial Y = \Sigma$ gives

$$(2.6.2) \qquad \begin{aligned} -\int \langle D_Y\psi_1, \psi_2\rangle dy + \int_Y \langle \psi_1, D_Y\psi_2\rangle dy &= \int_\Sigma \langle \psi_1, \gamma\psi_2\rangle d\sigma, \\ &= B(\psi_1, \psi_2), \end{aligned}$$

say, where dy and $d\sigma$ are the Riemannian volume elements. The right-hand side of (2.6.2) is a hermitian form on the space Γ_Σ of spinor fields. If Y is odd-dimensional, then Clifford multiplication by the unit normal vector to the boundary Σ splits the spin bundle Δ_Σ as $\Delta_\Sigma^0 \oplus \Delta_\Sigma^1$, and so we have

$$\langle \psi_i, \gamma\psi_2\rangle = \langle \psi_1^0, \psi_2^0\rangle - \langle \psi_1^1, \psi_2^1\rangle$$

on Σ.

2.7 Determinants

For the remainder of this lecture we turn from the index to the determinant of the Dirac operator. The next two subsections are concerned with the algebraic and analytic properties of infinite dimensional determinants, and not with the Dirac operator directly. We shall make use of this material in later lectures.

If E is a topological vector space, an operator $T : E \to E$ has a determinant in the most straightforward sense if it is of the form $T = 1 + A$, where A is of trace class. For then A has a sequence of eigenvalues $\{\lambda_k\}$ —with multiplicities— such that $\sum |\lambda_k| < \infty$, and we can define

$$\det(1 + A) = \Pi(1 + \lambda_k). \tag{2.7.1}$$

We shall say that such an operator T is of *determinant class*.

Even in finite dimensions the determinant of an operator $T : E \to F$ is not a number. If $\dim(E) = m$ and $\dim(F) = n$ we define the *determinant line* of T as the one-dimensional space

$$\mathrm{Det}(T) = (\wedge^m E)^* \otimes (\wedge^n F) = \mathrm{Hom}(\wedge^m E, \wedge^n F),$$

and then we define the *determinant* $\det(T)$ as the obvious element of the line $\mathrm{Det}(T)$ if $m = n$, and as 0 if $m \neq n$. The essential properties of $\mathrm{Det}(T)$ and $\det(T)$ are

(i) $$\det(T) \neq 0 \quad \Leftrightarrow \quad T \text{ is invertible;}$$

(ii) $$\mathrm{Det}(T_2 \circ T_1) \quad \cong \quad \mathrm{Det}(T_2) \otimes \mathrm{Det}(T_1),$$

and, in terms of this isomorphism,

$$\det(T_2 \circ T_1) \leftrightarrow \det(T_2) \otimes \det(T_1);$$

(iii) if

$$\begin{array}{ccccccccc} 0 \to & E_1 & \to & E_2 & \to & E_3 & \to & 0 \\ & \downarrow T_1 & & \downarrow T_2 & & \downarrow T_3 & & \\ 0 & \to & F_1 & \to & F_2 & \to & F_3 & \to & 0 \end{array}$$

is a commutative diagram with exact rows, then

$$\mathrm{Det}(T_2) \cong \mathrm{Det}(T_1) \otimes \mathrm{Det}(T_3)$$

canonically, and

$$\det(T_2) \leftrightarrow \det(T_3) \otimes \det(T_3).$$

Quillen was the first to point out that in this second, slightly more abstract, sense the determinant can be defined just as easily for an arbitrary Fredholm operator $T : E \to F$, and has the same three properties. I shall give the definition first in the case when T has index 0. Then a point of the line $\mathrm{Det}(T)$ is defined as an equivalence class of pairs (S, λ), where $\lambda \in \mathbb{C}$ and $S : E \to F$ is an isomorphism such that $S - T$ is of trace class, and

$$(S_1, \lambda_1) \;\sim\; (S_2, \lambda_2)$$

if and if only if

$$\lambda_2 \;=\; \det(S_2^{-1} \circ S_1)\lambda_1,$$

where $\det(S_2^{-1} \circ S_1)$ is defined in the straightforward way (2.7.1). The determinant $\det(T)$ is defined as the point of the line $\mathrm{Det}(T)$ represented by $(T, 1)$ if T is invertible, and as 0 otherwise.

The first thing to notice about this definition is that if the kernel and cokernel of T have dimension n then

$$\mathrm{Det}(T) \cong (\wedge^n \ker(T))^* \otimes \wedge^n \mathrm{coker}(T) \tag{2.7.2}$$

canonically. For if $\alpha_1, ..., \alpha_n$ is a basis of $\ker(T)^*$ and $v_1, ..., v_n \in F$ is a basis of $\mathrm{coker}(T)$ we can map the element

$$(\alpha_1 \wedge ... \wedge \alpha_n) \otimes (v_1 \wedge ... \wedge v_n)$$

of the right-hand side of (2.7.2) to the class of $(\tilde{T}, 1)$ in $\mathrm{Det}(T)$, where

$$\tilde{T} \;\; = \;\; T \, + \, \sum v_i \otimes \tilde{\alpha}_i$$

and $\tilde{\alpha}_i \in E^*$ is an extension of α_i. Having made this observation it is clear that there is a unique way to extend the definition of the $\mathrm{Det}(T)$ to Fredholm operators of any index so that the following two properties are preserved.

(a) $\mathrm{Det}(T)$ has the usual meaning if E and F are finite dimensional, and

(b) $\mathrm{Det}(T_1 \otimes T_2) \cong \mathrm{Det}(T_1) \otimes \mathrm{Det}(T_2)$.

Of course $\det(T)$ is defined as 0 if the index of T is not zero.

It is completely elementary to check that the definitions of Det and det for Fredholm operators have the basic properties $(i), (ii), (iii)$ above. One might wonder, however, what has been achieved, for to give a line with a distinguished vector conveys no information except whether the vector is non-zero. In fact the real interest of the construction appears only when we have a *family* of Fredholm operators $\{T_x\}_{x \in X}$. Then, under exactly the same very general circumstances which ensure that the index of T_x is locally constant, we find that $\{\mathrm{Det}(T_x)\}_{x \in X}$ is a complex line-bundle on X, and $\{\det(T_x)\}$ is a continuous section. Once again, the proof of this presents no difficulty.

A feature of the definition of the line $\mathrm{Det}(T)$ is that it depends on the operator T only modulo the addition of trace-class operators — and hence does not depend on T at all in finite dimensions. A situation where this can be exploited is the following.

The determinant bundle and the extension of $\mathrm{GL}_{\mathrm{res}}$

Whenever we have a polarized topological vector space E we can define not only the restricted Grassmannian $\mathrm{Gr}(E)$ as in §2.4, but also the restricted general linear group $\mathrm{GL}_{res}(E)$, which consists of all isomorphisms $E \to E$ which preserve the class of allowed splittings $E = E' \oplus E''$. Clearly $\mathrm{GL}_{res}(E)$ acts on $\mathrm{GR}(E)$.

If W_0 and W_1 both belong to $\mathrm{Gr}(E)$, then we have a preferred class of Fredholm operators $T : W_0 \to W_1$ singled out by the polarization : we compose the inclusion $W_0 \to E$ with any allowable projection $E \to W_1$. For any such T the line $\mathrm{Det}(T)$ depends only on W_0 and W_1 (and, of course, on the polarization). I shall denote it by $\mathrm{Det}(W_0 : W_1)$. Clearly we have

$$\mathrm{Det}(W_0 : W_1) \otimes \mathrm{Det}(W_1 : W_2) \cong \mathrm{Det}(W_0 : W_2) \tag{2.7.3}$$

and

$$\mathrm{Det}(W_0 : W_1) \cong \mathrm{Det}(gW_0 : gW_1)$$

for any g in $\mathrm{GL}_{res}(E)$. Putting these two facts together we see that for $g \in \mathrm{GL}_{res}(E)$ the line

$$L_g = \mathrm{Det}(W : gW)$$

is independent of the choice of $W \in \mathrm{Gr}(E)$, and satisfies

$$L_f \otimes L_g \cong L_{fg}.$$

This permits us to define the fundamental central extension $\mathrm{GL}^{\sim}_{res}(E)$ of $\mathrm{GL}_{res}(E)$ as the group of all pairs (g, λ) with $g \in \mathrm{GL}_{res}(E)$ and λ a non-zero element of L_g. It is an extension by $\mathbb{C}^\times$:

$$1 \to \mathbb{C}^\times \to \mathrm{GL}^{sim}_{res}(E) \to \mathrm{GL}_{res}(E) \to 1.$$

2.8 The ζ-function determinant

If $T : E \to E$ is a self-adjoint Fredholm operator in a vector space with an inner product then the complex line $\mathrm{Det}(T)$ is "real" i.e. there is an operation of complex conjugation in $\mathrm{Det}(T)$ which picks out a real line inside it. The element $\det(T)$ belongs to this real line.

If T is a self-adjoint first order elliptic differential operator, such as the Dirac operator, we can say a great deal more. Then T has a discrete spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$ lying on the real axis, with $\lambda_k \to \pm\infty$ as $k \to \pm\infty$, and (assuming for the moment that no λ_k is zero) we can define the ζ-function

$$\zeta_T(s) = \sum \lambda_k^{-s}.$$

Here λ_k^{-s} is defined as $|\lambda_k|^{-s} e^{-i\pi s}$ if λ_k is negative. The ζ-function is initially defined as a holomorphic function of s in a half-plane $\mathrm{Re}(s) > a$ where the series converges, but it is known that it can be analytically continued to a meromorphic function in the entire complex plane, and that ζ_T is regular at $s = 0$. Motivated by the formula

$$\zeta_T'(s) = -\sum(\log\lambda_k)\lambda_k^{-s},$$

we can now define the ζ-function determinant $\det_\zeta(T)$ as the complex number

$$\det_\zeta(T) = e^{-\zeta_T'(0)}.$$

We assumed here that 0 was not an eigenvalue of T. In fact what we have really found is an isomorphism

$$\det_\zeta : \mathrm{Det}(T) \to \mathbb{C}, \tag{2.8.1}$$

for we have

Proposition 2.8.2 *If T is a Dirac operator, and S is an invertible operator such that $S - 1$ is of trace class, then $\det_\zeta(S)$ is defined, and*

$$\det_\zeta(PS) = \det(P)\det_\zeta(S)$$

if $P \in 1+$ (trace class).

The isomorphism (2.8.1) has no reason to respect the real structure of the line $\mathrm{Det}(T)$, so $\det_\zeta$ gives us a real line contained in $\mathbb{C}$, even when T is not invertible and $\det(T) = 0$. This real line can be written $e^{i\pi\eta(T)/2}\mathbb{R}$, where $\eta(T) \in \mathbb{R}/2\mathbb{Z}$ is called the η-invariant of T. The Quillen determinant $\det(T) \in \mathrm{Det}(T)$ lies in the real sub-line, so $\eta(T)$ is essentially the *phase* of $\det_\zeta(T)$.

The fact that the phase of the determinant can be defined even when the determinant vanishes is interesting topologically. It is well known [AS] that the space of self-adjoint Fredholm operators in Hilbert space has the homotopy type of the infinite unitary group $U_\infty = \bigcup U_n$. If they are given an appropriate topology the same is true of the unbounded self-adjoint operators we are considering here. As the fundamental group $\pi_1(U_\infty)$ is $\mathbb{Z}$, there is a continuous map, defined up to homotopy, from self-adjoint Fredholm operators to the circle $\mathbb{R}/\mathbb{Z}$ which induces an isomorphism of π_1. On the subspace of Dirac operators the η-invariant is a definite choice of this map.

There is yet another way to look at the η-invariant. For each way of splitting the spectrum of T into two subsets Λ_+ and Λ_- such that almost all the positive eigenvalues belong to Λ_+ and almost all the negative ones to Λ_- we can define a holomorphic function of s for $\mathrm{Re}(s) \gg 0$ by

$$\eta(T; s) = \sum_{\lambda\in\Lambda_+} |\lambda|^{-s} - \sum_{\lambda\in\Lambda_-} |\Lambda|^{-s}.$$

It is known that this function can be continued analytically to $s = 0$, where its value is the real number $\eta(T; 0)$. It jumps by ± 2 when an eigenvalue is reassigned to the other half of the spectrum, so $\eta(T;0) \in \mathbb{R}/2\mathbb{Z}$ is independent of the chosen splitting, and is precisely the number $\eta(T)$ defined above, as we shall verify in a moment. But we can say more, for the choice of a splitting of the spectrum picks out for us the subspace of E spanned by the eigenfunctions with eigenvalues in Λ_-, and this is a point of the restricted Grassmannian $\mathrm{Gr}(E)$, and defines a point of the $\mathbb{Z}$-torsor $\pi_0\mathrm{Gr}(E)$. In other words, the η-invariant can be regarded as an embedding

$$\tfrac{1}{2}\eta \;:\; \pi_0\mathrm{Gr}(E) \to \mathbb{R}.$$

2.9 Fock spaces

To understand how the determinant of the Dirac operator behaves when manifolds are sewn together we need the concept of a Fock space. It is one of the main ideas in quantum field theory.

If E is a polarized topological vector space the Fock space $\mathcal{F}(E)$ is a "renormalized" version of the exterior algebra $\wedge(E)$, where the renormalization is defined in terms of the polarization. There is a range of possible definitions. The rough definition used by physicists — due to Dirac — makes sense when the polarization of E is defined by a self-adjoint operator $D : E \to E$, and one has an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ of E consisting of eigenvectors of D whose eigenvalues λ_k tend to $\pm\infty$ as $k \to \pm\infty$. Then $\mathcal{F}(E)$ has an orthonormal basis given by the formal expressions

$$e_{\mathbf{k}} = e_{k_0} \wedge e_{k_1} \wedge e_{k_2} \wedge \cdots ,$$

where the sequence $\mathbf{k} = \{k_0, k_1, k_2, \cdots\}$ satisfies $k_0 > k_1 > k_2 > \cdots$, and differs from $\{0, -1, -2, -3, \cdots\}$ only by including a finite number of positive integers and omitting a finite number of negative ones. The defect of this as a definition is that it seems to depend on the operator D. A good feature, however, is that it makes clear why — even when D is given — it is only the projective space of rays in $\mathcal{F}(E)$, and not the vector space itself, which can be defined canonically. For if we choose another basis $\{\tilde{e}_k\}$ of eigenvectors of D — say $\tilde{e}_k = u_k e_k$, where $|u_k| = 1$ — then

$$\widetilde{\Omega} = \tilde{e}_0 \wedge \tilde{e}_{-1} \wedge \tilde{e}_{-2} \wedge \cdots$$

should certainly define the same ray as

$$\Omega = e_0 \wedge e_{-1} \wedge e_{-2} \wedge \cdots ,$$

but there is no way of fixing a scalar κ such that $\widetilde{\Omega} = \kappa\Omega$. On the other hand, once the single number κ is prescribed there is no further indeterminacy, in the sense that we must have $\tilde{e}_{\mathbf{k}} = \kappa u_{\mathbf{k}} e_{\mathbf{k}}$, where

$$u_{\mathbf{k}} = \prod_{r\geqslant 0} u_{k_r} \Big/ \prod_{r\geqslant 0} u_r$$

is a well-defined number.

To give a mathematically more satisfactory definition of $\mathcal{F}(E)$ we observe that we should be able to multiply elements of $\mathcal{F}(E)$ by vectors in E, and so $\mathcal{F}(E)$ should be a module for the exterior algebra $\wedge(E)$. There should also be an adjoint action of the exterior algebra $\wedge(E^*)$ by the inner product: if $\alpha \in E^*$ then

$$\alpha e_{\mathbf{k}} = \sum (-1)^i \langle \alpha, e_{k_i} \rangle e_{\mathbf{k}}^{(i)},$$

where $e_{\mathbf{k}}^{(i)}$ is $e_{\mathbf{k}}$ with its i^{th} factor omitted. The action of an element of E on $\mathcal{F}(E)$ is traditionally called a *creation operator*, and that of an element of E^* an *annihilation operator*. The two actions fit together to form an action of the Clifford algebra $C(E \oplus E^*)$, where $E \oplus E^*$, has its natural hyperbolic quadratic form, i.e. if $\xi \in E$ and $\alpha \in E^*$ then

$$\alpha\xi + \xi\alpha \;=\; \langle \alpha, \xi \rangle \tag{2.9.1}$$

in $C(E \oplus E^*)$. From this point of view, the Fock space $\mathcal{F}(E)$ is characterized as an irreducible $C(E \oplus E^*)$-module which, for each splitting $E = E^+ \oplus E^-$ allowed by the polarization, contains a vector Ω_{E^-} which is annihilated by both $E^- \subset E$ and $(E^-)^\circ \subset E^*$. A finite-dimensional Clifford algebra has a unique irreducible representation, up to isomorphism, but in infinite dimensions the irreducible representations of $C(E \oplus E^*)$ are parametrized by the polarizations of $E \oplus E^*$. For a given choice of $E^- \in \mathrm{Gr}(E)$ we have a definite Fock space

$$\mathcal{F}_{E^-}(E) = \wedge((E^-)^*) \otimes \wedge(E/E^-) \tag{2.9.2}$$

with a definite vacuum vector Ω_{E^-}, but for different allowable choices E_1^-, E_2^- the isomorphism

$$\mathcal{F}_{E_1^-}(E) \to \mathcal{F}_{E_2^-}(E) \tag{2.9.3}$$

is canonical only up to a scalar: Schur's lemma replaces the physicists' renormalization constant.

The description just given was vague about the topology of $\mathcal{F}(E)$. If E is a Hilbert space then one can clearly construct $\mathcal{F}_{E^-}(E)$ as a Hilbert space, prescribing that the creation and annihilation operators are each others' adjoints. But it is worth mentioning a more general and abstract approach. We want $\mathcal{F}_{E^-}(E)$ to contain a ray L_W for each $W \in \mathrm{Gr}(E)$: roughly,

$$L_W = \mathbb{C} w_0 \wedge w_1 \wedge w_2 \wedge \cdots ,$$

where $\{w_i\}$ is a basis of W. Now we have seen that for a given $E^- \in \mathrm{Gr}(E)$ there is a holomorphic line bundle Det_{E^-} on $\mathrm{Gr}(E)$ whose fibre at W is $\mathrm{Det}(E^- : W)$. We can characterize $\mathcal{F}_{E^-}(E)$ by saying that it is a topological vector space with a holomorphic map

$$\mathrm{Det}_{E^-} \to \mathcal{F}_{E^-}(E) \tag{2.9.4}$$

which is linear on each fibre of Det_{E^-}, and that it is universal among topological vector spaces with such maps. Then the vacuum vector Ω_{E^-} is the image of $1 \in \mathrm{Det}(E^- : E^-)$ in $\mathcal{F}_{E^-}(E)$, and the isomorphism (2.9.3) arises from a *canonical* isomorphism

$$\mathcal{F}_{E_2^-}(E) \quad \cong \quad \mathcal{F}_{E_1^-} \otimes \mathrm{Det}(E_1^- : E_2^-) \tag{2.9.5}$$

(cf. (2.7.3)).

The existence of a vector space with the universal property (2.9.4) is clear: we take the dual of the space of holomorphic sections of the line bundle $\mathrm{Det}^*_{(E^-)}$. Of course we shall not get a Hilbert space, but a pre-Hilbert structure in E induces one in $\mathcal{F}_{E^-}(E)$. (See [PS] Chapter 10).

One feature of the Fock space which is clear from any of the definitions is that $\mathcal{F}(E)$ is naturally graded by the $\mathbb{Z}$-torsor $\pi_0\mathrm{Gr}(E)$: the degree of the vacuum vector Ω_{E^-} is the virtual dimension of E^-.

Finally, we need to know that reversing the polarization of the space E^-, i.e. changing J to $-J$, or interchanging E^+ and E^-, essentially changes the Fock space $\mathcal{F}(E)$ to its dual. To be precise, if $\widetilde{E}$ denotes E with the reversed polarization, and we choose $E^- \in \mathrm{Gr}(E)$ and E^+ in $\mathrm{Gr}(\widetilde{E})$, not necessarily complementary, then there is a canonical pairing

$$\mathcal{F}_{E^+}(\widetilde{E}) \times \mathcal{F}_{E^-}(E) \to L_{E^+,E^-}, \tag{2.9.6}$$

where L_{E^+,E^-} is the determinant line of the Fredholm operator $E^+ \oplus E^- \to E$ defined by adding the inclusions. Restricted to the rays

$$\mathrm{Det}(E^+ : \widetilde{W}) \times \mathrm{Det}(E^- : W)$$

the pairing (2.9.6) is

$$(\widetilde{S}, S) \;\mapsto\; \mathrm{Det}(\widetilde{S} + S : E^+ \oplus E^- \to E).$$

2.10 Patching the determinant

We shall now return to the Dirac operator on a closed even-dimensional manifold X which is a union of two pieces $X = X_1 \amalg_Y X_2$. We saw how the index of D_X can be calculated from contributions associated to X_1 and X_2, and now we should like to do the same for the determinant of D_X. There are two aspects to this. If we forget for a moment that the determinant of D_X is not quite a number, then we expect the operator D_{X_1} on a manifold with boundary to have a determinant only when we equip it with a boundary condition. An appropriate boundary condition is defined by a point W of the restricted Grassmannian $\mathrm{Gr}_{\overline{Y}}$.

A boundary condition for D_{X_2} corresponds to a point of the opposite Grassmanian Gr_Y. Thus, roughly speaking, both $\det(D_{X_1})$ and $\det(D_{X_2})$ are *functions* on Gr_Y. We are aiming for a formula of the type

$$\det(D_X) = \langle \det(D_{X_1}), \det(D_{X_2})\rangle, \tag{2.10.1}$$

expressing the result as an L^2 inner product, i.e. some kind of integral over the infinite-dimensional space of all boundary conditions. This fits in well with the point of view of quantum field theory, where $\det(D_X)$ is regarded as a path-integral over the space of all spinor fields on X, but it cannot be interpreted too literally, for the necessary integration theory is quite out of reach. Instead, we shall simply *prescribe* the Fock space $\mathcal{F}_Y$ formed from the space Γ_Y of spinor fields on Y as our candidate for the Hilbert space of functions on Gr_Y, and we shall define elements of $\mathcal{F}_Y$ so that (2.10.1) is true.

Before doing so, however, we must return to the second aspect of the problem, namely the fact that $\det(D_X)$ is not a number but actually an element of the abstractly defined line $\mathrm{Det}(D_X)$. This fits into the formalism very attractively. We saw that the object canonically associated to Y is not a Hilbert space $\mathcal{F}_Y$ but a *projective* space $\mathbb{P}\mathcal{F}_Y$. Whereas a closed manifold X gives us a line $\mathrm{Det}(D_X)$, the corresponding object for a manifold X_1 with boundary Y is a vector space $\mathcal{F}_{X_1}$ together with an isomorphism

$$\mathbb{P}(\mathcal{F}_{X_1}) \cong \mathbb{P}\mathcal{F}_Y.$$

This makes good sense, for if $\mathbb{P}$ is a complex projective space then the category of vector spaces V with isomorphisms $\mathbb{P}(V) \cong \mathbb{P}$ is equivalent to the category of complex lines. Indeed if V is one such space then any other is of the form $V \otimes L$ for some line L, for an isomorphism $\mathbb{P}(V_0) \cong \mathbb{P}(V_1)$ gives an isomorphism $V_0 \otimes L \cong V_1$, where L is the line of homomorphisms $V_0 \to V_1$ which induce the given map of projective spaces.

In the present situation we define $\mathcal{F}_{X_1}$ as the Fock space $\mathcal{F}_{K_1}(\Gamma_Y)$ formed from Γ_Y and the space K_1 of boundary values of harmonic spinor fields on X_1. For the other half, we form $\mathcal{F}_{X_2}$ similarly from $\widetilde{\Gamma}_Y = \Gamma_{\bar{Y}}$. The projective spaces $\mathbb{P}\mathcal{F}_Y$ and $\mathbb{P}\mathcal{F}_{\bar{Y}}$ are dual, and so $\mathcal{F}_{X_2}$ is in duality with $\mathcal{F}_{X_1} \otimes L$ for some line L which can be denoted by

$$\langle \mathcal{F}_{X_1}\, ,\ \mathcal{F}_{X_2}\rangle.$$

Proposition 2.10.2 *We have*

(a) $\langle \mathcal{F}_{X_1}\, ,\ , \mathcal{F}_{X_2}\rangle = \mathrm{Det}(D_X)$, *and*

(b) $\langle \det(D_{X_1})\, , \det(D_{X_2})\rangle \ = \ \det(D_X)$,

where $\det(X_i)$ *denotes the vacuum vector in* $\mathcal{F}_{X_i}$.

Proof. According to (2.9.6) the line $\langle \mathcal{F}_{X_1}, \mathcal{F}_{X_2}\rangle$ and the point $\langle \det(D_{X_1}), \det(D_{X_2})\rangle$ in it can be identified with the pointed determinant line of

$$K_1 \oplus K_2 \to \Gamma_Y.$$

But the diagram in the proof of (2.3.2) showed that this Fredholm operator was equivalent, in a sense which preserves the determinant as well as the index, to the Dirac operator

$$D_X : \Gamma_X^{even} \;\to\; \Gamma_X^{odd}.$$

■

2.11 Patching the ζ-function determinant

We now turn to the ζ-function determinant of the self-adjoint Dirac operator on an odd-dimensional manifold. Essentially the same discussion applies in even dimensions to the determinant of the self-adjoint operator $D_X = D_X^{even} \oplus D_X^{odd}$, in contrast to the "chiral" operator D_X^{even} which was treated in the previous section. We consider a closed manifold $X = X_1 \amalg_Y X_2$, as usual, but now $\dim(Y)$ is even. We shall obtain a formula

$$\det{}_\zeta(D_X) = \langle \det{}_\zeta(D_{X_1}), \det{}_\zeta(D_{X_2})\rangle, \tag{2.11.1}$$

where the determinants on the right are elements of dual Hilbert spaces $\mathcal{H}_Y, \mathcal{H}_{\bar{Y}}$ associated to Y, which we think of as consisting of functions of the boundary data for D_{X_1} and D_{X_2}.

We saw in §2.6 that appropriate boundary data in this case are maximal isotropic subspaces of Γ_Y belonging to a certain polarization-class, or, equivalently, certain unitary isomorphisms $u : \Gamma_Y^{even} \;\to\; \Gamma_Y^{odd}$ which form a space $\mathcal{U}_Y$ which is a principal homogeneous space for the group of unitary transformations of Γ_Y^{even} of determinant class. The Hilbert space $\mathcal{H}_Y$ is once again a kind of Fock space: it can be regarded as

$$\wedge^{middle}(\Gamma_Y) \cong \mathrm{Hom}(\wedge(\Gamma_Y^{even})\,;\,\wedge(\Gamma_Y^{odd})).$$

But the important thing is that it is a definite vector space, and not just a projective space. The most concrete definition is to say that $\mathcal{H}_Y$ contains a unit vector ε_u for each $u \in \mathcal{U}_Y$, and is obtained from the formal algebraic span of these vectors by completing with respect to the inner product defined by

$$\langle \varepsilon_{u_1}, \varepsilon_{u_2}\rangle = \det{}_{\frac{1}{2}}(1 + u_1^{-1}u_2). \tag{2.11.2}$$

To see that this does indeed define a positive inner product it is enough, by continuity, to consider the same formula applied to the unitary group U_n. But then (2.11.2) is simply the inner product induced by the natural embedding of U_n in $\mathrm{End}(\wedge\mathbb{C}^n)$.

For a connected manifold X_1 with non-empty boundary Y we can now define

$$\det_\zeta(X_1) = \varepsilon_u \in \mathcal{H}_Y,$$

where $u \in \mathcal{U}_Y$ represents the isotropic subspace of Γ_Y consisting of the boundary values of harmonic spinor fields on X_1. A self-adjoint boundary condition for D_{X_1} is an element $\beta \in \mathcal{U}_{\bar{Y}} = \mathcal{U}_Y^{-1}$, and we have

Proposition 2.11.3 *(i) The ζ-function determinant of D_{X_1} with boundary condition β is*

$$\begin{aligned} \det_\zeta(D_{X_1}, \beta) &= \langle \det_\zeta(D_{X_1}), \varepsilon_\beta \rangle \\ &= \det \tfrac{1}{2}(1 + \beta u). \end{aligned}$$

(ii) For the closed manifold $X = X_1 \cup X_2$ we have

$$\det_\zeta(D_X) = \langle \det_\zeta(D_{X_1}), \det_\zeta(D_{X_2}) \rangle.$$

Bibliography

[AS] AS