Chapter 1

Operator Algebras

We wish to consider concrete algebras of operators on a Hilbert space and to take advantage of the Hilbert space structure. We shall define several topologies, consider the density theorems of von Neumann and Kaplansky and then discuss the characterization of continuous linear functionals. We conclude the chapter with a discussion of some theory of sub-representations of C^*-algebras which is useful for the algebraic treatment of superselection sectors.

Topologies on $\mathcal{B}(\mathcal{H})$

We consider five topologies on $\mathcal{B}(\mathcal{H})$, the first three probably being the more familiar. Let $A \in \mathcal{B}(\mathcal{H})$ and let (B_α) be a net in $\mathcal{B}(\mathcal{H})$. We shall define the various topologies in terms of a neighbourhood basis at A and also in terms of the convergence of the net (B_α) to A — the latter being usually the more convenient.

Definition 1.1. The norm (or uniform) topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A;\varepsilon) = \{\, B \in \mathcal{B}(\mathcal{H}) : \|A - B\| < \varepsilon \,\}$$

for $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$.

$B_\alpha \to A$ in norm if $\|B_\alpha - A\| \to 0$.

Definition 1.2. The strong topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A;(x_i)_1^n,\varepsilon) = \{\, B \in \mathcal{B}(\mathcal{H}) : \sum_{i=1}^{n} \|(A-B)x_i\|^2 < \varepsilon \,\}$$

for $A \in \mathcal{B}(\mathcal{H})$, $\varepsilon > 0$ and any finite set of vectors $(x_i)_1^n$ in $\mathcal{H}$.

$B_\alpha \to A$ strongly if $\|(B_\alpha - A)x\| \to 0$ for each $x \in \mathcal{H}$.

Definition 1.3. The weak topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A;(x_i)_1^n,(y_i)_1^n,\varepsilon)=\{\,B\in\mathcal{B}(\mathcal{H}):\Big|\sum_{i=1}^{n}\langle\, y_i,(A-B)x_i\,\rangle\Big|<\varepsilon\,\}$$

for $A\in\mathcal{B}(\mathcal{H})$, $\varepsilon>0$ and any finite sets of vectors $(x_i)_1^n$ and $(y_i)_1^n$ in $\mathcal{H}$.

$B_\alpha\to A$ weakly if $\langle\, y,(B_\alpha-A)x\,\rangle\to 0$ for each $x,y\in\mathcal{H}$.

Definition 1.4. The ultrastrong topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A;(x_i)_1^\infty,\varepsilon)=\{\,B\in\mathcal{B}(\mathcal{H}):\sum_{i=1}^{\infty}\|(A-B)x_i\|^2<\varepsilon\,\}$$

for $A\in\mathcal{B}(\mathcal{H})$, $\varepsilon>0$ and any sequence (x_i) in $\mathcal{H}$ satisfying $\sum_{i=1}^\infty\|x_i\|^2<\infty$.

$B_\alpha\to A$ ultrastrongly if $\sum_{i=1}^\infty\|(B_\alpha-A)x_i\|^2\to 0$ for each sequence $(x_i)\in\mathcal{H}$ satisfying $\sum_{i=1}^\infty\|x_i\|^2<\infty$.

Definition 1.5. The ultraweak topology on $\mathcal{B}(\mathcal{H})$ is that given by the open neighbourhood base

$$\mathcal{N}(A;(x_i)_1^\infty,(y_i)_1^\infty,\varepsilon)=\{\,B\in\mathcal{B}(\mathcal{H}):\Big|\sum_{i=1}^{\infty}\langle\, y_i,(A-B)x_i\,\rangle\Big|<\varepsilon\,\}$$

for $A\in\mathcal{B}(\mathcal{H})$, $\varepsilon>0$ and sequences (x_i) and (y_i) in $\mathcal{H}$ with $\sum_{i=1}^\infty\|x_i\|^2<\infty$ and $\sum_{i=1}^\infty\|y_i\|^2<\infty$.

$B_\alpha\to A$ ultraweakly if $\sum_{i=1}^\infty\langle\, y_i,(B_\alpha-A)x_i\,\rangle\to 0$ for each pair of sequences (x_i) and (y_i) in $\mathcal{H}$ satisfying $\sum_{i=1}^\infty\|x_i\|^2<\infty$ and $\sum_{i=1}^\infty\|y_i\|^2<\infty$.

$\mathcal{B}(\mathcal{H})$ equipped with the norm topology is, of course, a C^*-algebra. Moreover, the norm topology is characterized by convergence of sequences. For the other four topologies, however, one has to consider nets, rather than just sequences (unless $\mathcal{H}$ is finite-dimensional).

It can be shown that these five topologies are all distinct whenever $\mathcal{H}$ is infinite-dimensional, otherwise they coincide. Evidently, the norm topology is finer (stronger) than the other four and the ultrastrong topology is finer than each of the strong, weak and ultraweak topologies. The strong topology and the ultraweak topology are both finer than the weak topology, but in general, the ultraweak and strong topologies are not comparable. It should be emphasized that the ultraweak topology is stronger than the weak topology, i.e., if $B_\alpha\to A$ ultraweakly, then $B_\alpha\to A$ weakly.

For each $i\in\mathbb{N}$, let $\mathcal{H}_i=\mathcal{H}$ and set $\mathcal{K}=\bigoplus_{i=1}^\infty\mathcal{H}_i$. For given $A\in\mathcal{B}(\mathcal{H})$, let $\widetilde{A}$ be the operator on $\mathcal{K}$ given by $\widetilde{A}(x_i)=(Ax_i)$. One readily checks that $\widetilde{A}\in\mathcal{B}(\mathcal{K})$.

Proposition 1.6. *A net (B_α) in $\mathcal{B}(\mathcal{H})$ converges ultrastrongly (respectively, ultraweakly) to A in $\mathcal{B}(\mathcal{H})$ if and only if the net $(\widetilde{B}_\alpha)$ converges strongly (respectively, weakly) to $\widetilde{A}$ in $\mathcal{B}(\mathcal{K})$.*

Proof. The sequence (x_i), with each $x_i \in \mathcal{H}_i = \mathcal{H}$, belongs to $\mathcal{K}$ if and only if $\sum_{i=1}^\infty \|x_i\|^2$ is convergent. For such $x = (x_i)$ and $y = (y_i)$, the result follows from the equalities

$$\|(\widetilde{B}_\alpha - \widetilde{A})x\|_{\mathcal{K}}^2 = \|(\widetilde{B}_\alpha - \widetilde{A})(x_i)\|_{\mathcal{K}}^2 = \sum_{i=1}^\infty \|(B_\alpha - A)x_i\|_{\mathcal{H}}^2$$

and

$$\langle\, y, (\widetilde{B}_\alpha - \widetilde{A})x \,\rangle_{\mathcal{K}} = \langle\, (y_i), (\widetilde{B}_\alpha - \widetilde{A})(x_i) \,\rangle_{\mathcal{K}} = \sum_{i=1}^\infty \langle\, y_i, (B_\alpha - A)x_i \,\rangle_{\mathcal{H}}\,.$$

■

This provides a useful device for converting "ultra-convergence" issues into those of strong or weak convergence, respectively, which are usually somewhat more tractable.

The next proposition tells us that on bounded sets in $\mathcal{B}(\mathcal{H})$ the occurrence of "ultra-convergence" is equivalent to that of the corresponding strong or weak convergence.

Proposition 1.7. *Suppose that (B_α) is a bounded net in $\mathcal{B}(\mathcal{H})$, i.e., there is some $M \ge 0$ such that $\|B_\alpha\| \le M$ for all α. Then $B_\alpha \to A$ ultrastrongly (respectively, ultraweakly) if and only if $B_\alpha \to A$ strongly (respectively, weakly).*

Proof. Ultrastrong convergence implies strong convergence and ultraweak convergence implies weak convergence (without any extra boundedness hypothesis).

For the converse, let (x_i) and (y_i) be any given sequences in $\mathcal{H}$, with $\sum_i \|x_i\|^2$ and $\sum_i \|y_i\|^2$ both convergent. Then, using the boundedness of the net (B_α), we have

$$\sum_{i=1}^\infty \|(B_\alpha - A)x_i\|^2 \le \sum_{i=1}^N \|(B_\alpha - A)x_i\|^2 + (M + \|A\|)^2 \sum_{i=N+1}^\infty \|x_i\|^2$$

which shows that strong convergence implies ultrastrong convergence. In a similar way, we estimate

$$\begin{aligned}\Big|\sum_{i=1}^{\infty}\langle\, y_i, (B_\alpha - A)x_i\,\rangle\Big| &\le \Big|\sum_{i=1}^{N}\langle\, y_i, (B_\alpha - A)x_i\,\rangle\Big| + (M + \|A\|) \sum_{i=N+1}^{\infty} \|y_i\|\,\|x_i\| \\ &\le \Big|\sum_{i=1}^{N}\langle\, y_i, (B_\alpha - A)x_i\,\rangle\Big| \\ &\qquad + (M + \|A\|)\Big(\sum_{i=N+1}^{\infty} \|y_i\|^2\Big)^{1/2}\Big(\sum_{i=N+1}^{\infty} \|x_i\|^2\Big)^{1/2}\end{aligned}$$

which shows that weak convergence implies ultraweak convergence. ■

Remark 1.8. Suppose that $A_\alpha \to A$ and $B_\alpha \to B$ in one of the above five topologies. One can then ask whether $A_\alpha^* \to A^*$ or whether $A_\alpha B_\alpha \to AB$ in the same topology. In other words, we wish to know whether the involution $A \mapsto A^*$ and the product $(A, B) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to AB$ are continuous maps. If $\mathcal{H}$ is finite-dimensional, then the topologies are equivalent and these maps are continuous, but in general they need not be. The situation is as follows.

The involution map $A^* \mapsto A^*$ is continuous in the norm, ultraweak and weak topologies, but not with respect to the ultrastrong or strong topologies. (This is can be a problem in the study of the scattering-matrix in which its unitarity is a non-trivial problem even though it may be constructed as a strong limit of unitary operators.)

The product $(A, B) \mapsto AB$ is jointly continuous with respect to the norm topology but only separately continuous with respect to the other topologies. On bounded sets, the product is jointly continuous with respect to the ultrastrong and strong topologies, but not with respect to the ultraweak and weak topologies.

Von Neumann's density theorem

Whilst the (ultra)strong and (ultra)weak topologies are different, nevertheless many sets in $\mathcal{B}(\mathcal{H})$ have the same closures with respect to these topologies.

Definition 1.9. Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators containing the unit $\mathbb{1}$. $\mathcal{R}$ is said to be a von Neumann algebra if it is weakly closed in $\mathcal{B}(\mathcal{H})$.

Remark 1.10. Von Neumann algebras are also called W^*-algebras. Evidently, any von Neumann algebra is also a (unital) C^*-algebra.

Definition 1.11. Let $\mathcal{M}$ be a set in $\mathcal{B}(\mathcal{H})$. The commutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$, denoted $\mathcal{M}'$, is the set

$$\mathcal{M}' = \{\, B \in \mathcal{B}(\mathcal{H}) : AB = BA \text{ for all } A \in \mathcal{M}.\,\}$$

Proposition 1.12. (i) *For any subsets $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, we have that $\mathcal{N}' \subseteq \mathcal{M}'$ and $\mathcal{M} \subseteq \mathcal{M}''$.*

(ii) *For any subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, the set $\{ \mathcal{M} \cup \mathcal{M}^* \}'$ is a von Neumann algebra.*

Proof. The verification here is straightforward. ∎

The following theorem is one of the basic results of the theory.

Theorem 1.13 (von Neumann's Density Theorem (Bicommutant Theorem)). *Let $\mathcal{R}$ be any self-adjoint algebra in $\mathcal{B}(\mathcal{H})$ containing $\mathbb{1}$. Then the ultrastrong, ultraweak, strong and weak closures of $\mathcal{R}$ in $\mathcal{B}(\mathcal{H})$ are all the same and are equal to $\mathcal{R}''$.*

Proof. Firstly, we note that $\mathcal{R} \subseteq \mathcal{R}'' = \overline{\mathcal{R}''}^{\,w}$, the weak closure of $\mathcal{R}''$, since $\mathcal{R}''$ is a von Neumann algebra. Secondly, we observe that from the definition of the topologies (and with obvious notation), we have that

$$\mathcal{R} \subseteq \overline{\mathcal{R}}^{\,us} \subseteq \mathcal{A} \subseteq \overline{\mathcal{R}}^{\,w} \subseteq \overline{\mathcal{R}''}^{\,w} = \mathcal{R}''$$

where $\mathcal{A}$ denotes either $\overline{\mathcal{R}}^{\,uw}$ or $\overline{\mathcal{R}}^{\,s}$. Hence the proof is complete if we can show that $\overline{\mathcal{R}}^{\,us} = \mathcal{R}''$, that is, if we can show that $\mathcal{R}$ is ultrastrongly dense in $\mathcal{R}''$. Thus, we wish to show that for any given $B \in \mathcal{R}''$, any sequence (x_i) in $\mathcal{H}$ such that $\sum_i \|x_i\|^2 < \infty$ and any $\varepsilon > 0$, there is $A \in \mathcal{R}$ such that $\sum_i \|Bx_i - Ax_i\|^2 < \varepsilon$. By regarding (x_i) as an element x of $\bigoplus_i \mathcal{H}_i$, where $\mathcal{H}_i = \mathcal{H}$ for each $i \in \mathbb{N}$, we see that it is enough to show that $\widetilde{B}x$ is in the closed linear span $\mathcal{V}$, say, of $\{ \widetilde{A}x : A \in \mathcal{R} \}$ in $\bigoplus_i \mathcal{H}_i$.

To show this, let P denote the projection of $\bigoplus_i \mathcal{H}_i$ onto this subspace $\mathcal{V}$. We shall show that $P\widetilde{B}x = \widetilde{B}x$. Note that $\widetilde{A}$ commutes with P, for any $A \in \mathcal{R}$, so $P \in (\widetilde{\mathcal{R}})'$ (where the commutant is taken in $\mathcal{B}(\bigoplus_i \mathcal{H}_i)$).

We claim that if $B \in \mathcal{R}''$ then $\widetilde{B} \in (\widetilde{\mathcal{R}})''$. To see this, let $C \in \mathcal{B}(\bigoplus_i \mathcal{H}_i)$. Then $C = \sum_{i,j} E_i C E_j$ where E_i is the projection of $\bigoplus_i \mathcal{H}_i$ onto $\mathcal{H}_i$ considered as a subspace of $\bigoplus_i \mathcal{H}_i$. Writing $C_{ij} = E_i C E_j$, we see that C_{ij} is an operator from $\mathcal{H}_j$ to $\mathcal{H}_i$ and so determines an element of $\mathcal{B}(\mathcal{H})$ (since $\mathcal{H}_i = \mathcal{H}_j = \mathcal{H}$). For any $y = (y_i) \in \bigoplus_i \mathcal{H}_i$, we have

$$(C\,y)_i = \sum_{j=1}^{\infty} C_{ij}\, y_j \,.$$

Hence $C \in (\widetilde{\mathcal{R}})'$ if and only if

$$\sum_j A\, C_{ij}\, y_j = \sum_j C_{ij}\, A\, y_j$$

for all $i \in \mathbb{N}$ and all $(y_j) \in \bigoplus_j \mathcal{H}_j$ for any $A \in \mathcal{R}$. In other words, $C \in (\widetilde{\mathcal{R}})'$ if and only if $C_{ij} \in \mathcal{R}'$ for all i, j. Now let $B \in \mathcal{R}''$ and $C \in (\widetilde{\mathcal{R}})'$. Then

$C_{ij} \in \mathcal{R}'$ so that B commutes with each C_{ij} and so $\widetilde{B}$ commutes with C, that is, $\widetilde{B} \in (\widetilde{\mathcal{R}})''$, as claimed.

To complete the proof, suppose that $B \in \mathcal{R}''$. Then for any $x \in \bigoplus_i \mathcal{H}_i$,

$$\begin{aligned} P\,\widetilde{B}\,x &= \widetilde{B}\,P\,x \quad \text{since } P \in (\widetilde{\mathcal{R}})' \text{ and } \widetilde{B} \in (\widetilde{\mathcal{R}})'', \\ &= \widetilde{B}\,x \end{aligned}$$

since $Px = x$ (because $\mathbb{1} \in \mathcal{R}$ and so $x = \widetilde{\mathbb{1}}x \in \widetilde{\mathcal{R}}x \subseteq P \bigoplus_i \mathcal{H}_i$). ∎

As a consequence of this theorem, we see that $\mathcal{R}$ is a von Neumann algebra if and only if $\mathcal{R}$ is closed with respect to each of the four topologies of the theorem, or if and only if $\mathcal{R} = \mathcal{R}''$. This gives several characterisations of a von Neumann algebra.

Furthermore, this theorem implies that von Neumann algebrascontain many projections and are, in fact, determined by their projections. This is to be compared with the situation concerning C^*-algebraswhich need not contain any non-trivial projections at all; for example, $\mathcal{C}[0,1]$, the C^*-algebra of continuous functions on the interval $[0,1]$. To see that a von Neumann algebra contains projections, we first note that any element in the von Neumann algebra $\mathcal{R}$ can be written as a linear combination of self-adjoint elements of $\mathcal{R}$. By the spectral theorem, the spectral projections of any $B = B^*$ are given by strong limits of polynomials in B. It follows that if $B \in \mathcal{R}$, then so are all its spectral projections. Hence $\mathcal{R}$ contains all the spectral projections of its self-adjoint elements. Evidently, if two von Neumann algebraspossess the same projections, then they possess the same self-adjoint elements and so are equal.

We also note that $B \in \mathcal{R}'$ if and only if B commutes with all projections in $\mathcal{R}$ and so $A \in \mathcal{R}$ if and only if A commutes with all projections which commute with $\mathcal{R}$. Now, any element of $\mathcal{R}'$, which is also a von Neumann algebra (and so a C^*-algebra), can be written as a linear combination of unitary elements of $\mathcal{R}'$. It follows that $A \in \mathcal{R}$ if and only if A commutes with all unitaries which commute with $\mathcal{R}$. Thus, $A \in \mathcal{R}$ if and only if $UAU^{-1} = A$ for all unitary elements U of $\mathcal{R}'$.

Linear functionals on an operator algebra

Proposition 1.14. *Suppose that $\varphi : \mathcal{R} \to \mathbb{C}$ is a strongly continuous linear functional on a von Neumann algebra $\mathcal{R}$ acting in a Hilbert space $\mathcal{H}$. Then there exists $n \in \mathbb{N}$ and vectors x_i, y_i, $1 \le i \le n$ in $\mathcal{H}$ such that*

$$\varphi(A) = \sum_{i=1}^{n} \langle\, y_i, A\,x_i \,\rangle\,.$$

In particular, φ is weakly continuous. In other words, φ is strongly continuous if and only if φ is weakly continuous and if and only if φ has the above form.

Proof. The only non-trivial part is to show that if φ is strongly continuous then it has the stated form. To show this, suppose that $\varphi : \mathcal{R} \to \mathbb{C}$ is a strongly continuous linear functional. Then the set $\varphi^{-1}(\{ z : |z| < 1 \})$ is open in $\mathcal{R}$ in the strong topology and contains 0 and so there is a strong neighbourhood $\mathcal{N}(0; (x_i)_{i=1}^n, \varepsilon)$ of 0 contained in $\varphi^{-1}(\{ z : |z| < 1 \})$. In other words, the set of elements $T \in \mathcal{R}$ satisfying $\sum_{i=1}^n \|Tx_i\|^2 < \varepsilon$ also obey $|\varphi(T)| < 1$.

Let $\mathcal{K} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n terms) and consider the set V in $\mathcal{K}$ given by

$$V = \{ (y_i)_{i=1}^n : y_i = Tx_i,\ 1 \le i \le n,\ T \in \mathcal{R} \}.$$

Evidently, V is a linear set in $\mathcal{K}$. Define the linear map $f : V \to \mathbb{C}$ by $f((y_i)) = \varphi(T)$, where $(y_i) = (Tx_i)$. By the construction of the x_is, we see that if $\|(y_i)\|_{\mathcal{K}}^2 < \delta^2 \varepsilon$ then $|\varphi(T)| < \delta$ and so $|f((y_i))| < \delta$. This means that f is well-defined on V (if $(Tx_i) = (Sx_i)$ with $S, T \in \mathcal{R}$, then $\varphi(S) = \varphi(T)$) and that f is continuous on V. It follows that f extends, by continuity, to a continuous linear functional on $\overline{V}$, the closure of V in $\mathcal{K}$. Since $\overline{V}$ is a Hilbert space, by Riesz' Lemma this linear functional is given by a vector in the space $\overline{V}$, that is, there is some $v \in \overline{V}$ such that $f(y) = \langle v, y \rangle_{\mathcal{K}}$, for all $y \in \overline{V}$ (where we now use f to denote this extension to $\overline{V}$).

Now let $A \in \mathcal{R}$. Then $(Ax_i) \in V$ and so

$$f((Ax_i)) = \langle v, (Ax_i) \rangle_{\mathcal{K}} = \varphi(A)$$

by definition of f. Writing v as $v = (y_i) \in \mathcal{H} \oplus \cdots \oplus \mathcal{H}$, we get

$$\varphi(A) = \sum_{i=1}^n \langle y_i, Ax_i \rangle$$

for all $A \in \mathcal{R}$ and the proof is complete. ∎

Remark 1.15. The assumption that $\mathcal{R}$ be a von Neumann algebra is clearly unnecessary — all one needs is for $\mathcal{R}$ to be a linear set of operators.

The above result has an "ultra" version.

Proposition 1.16. *Let $\mathcal{R}$ be a linear set in $\mathcal{B}(\mathcal{H})$ and suppose that φ is a linear functional on $\mathcal{R}$. The following are equivalent:*

(i) $\varphi : \mathcal{R} \to \mathbb{C}$ *is ultrastrongly continuous;*

(ii) $\varphi : \mathcal{R} \to \mathbb{C}$ *is ultraweakly continuous;*

(iii) *there exist sequences* (x_i) *and* (y_i) *in the Hilbert space* $\mathcal{H}$ *such that both* $\sum_i \|x_i\|^2$ *and* $\sum_i \|y_i\|^2$ *are convergent and are such that*

$$\varphi(A) = \sum_{i=1}^{\infty} \langle\, y_i, Ax_i \,\rangle$$

for all $A \in \mathcal{R}$.

Proof. The proof that (i) implies (iii) is as in the previous proposition but with the finite direct sum $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ replaced by the infinite direct sum $\bigoplus_i \mathcal{H}$. The proofs of the other implications are straightforward. ■

Proposition 1.17. *Let* K *be any convex subset of* $\mathcal{B}(\mathcal{H})$. *Then the strong and weak closures of* K *in* $\mathcal{B}(\mathcal{H})$ *are the same.*

Proof. We know that $K \subseteq \overline{K}^{\,s} \subseteq \overline{K}^{\,w}$ and $\overline{\overline{K}^{\,s}}^{\,w} = \overline{K}^{\,w}$. If we show that $\overline{K}^{\,s}$ is weakly closed, then the equality $\overline{K}^{\,s} = \overline{K}^{\,w}$ will follow. Replacing K by $\overline{K}^{\,s}$, which is also convex, we may assume that K is strongly closed. Suppose that K is a proper subset of $\overline{K}^{\,w}$ and let $A \in \overline{K}^{\,w}$ with $A \notin K$. By the Hahn-Banach Theorem, A may be separated from the convex, strongly closed set K by a continuous linear functional, that is, there is a strongly continuous linear functional φ on $\mathcal{B}(\mathcal{H})$ such that

$$\operatorname{Re}\varphi(A) > \sup\{\, \operatorname{Re}\varphi(B) : B \in K \,\}\,. \qquad (*)$$

Since φ is strongly continuous, it is also weakly continuous, by Proposition 1.14. Since $A \in \overline{K}^{\,w}$, there is a net (B_α) in K such that $B_\alpha \to A$ weakly and so $\varphi(B_\alpha) \to \varphi(A)$. This contradicts the separation inequality $(*)$. We conclude that $K = \overline{K}^{\,w}$, that is K is weakly closed and the result follows. ■

Kaplansky's Density Theorem

Consider a *-algebra of operators $\mathcal{R}$ and $\overline{\mathcal{R}}^{\,s}$ its strong closure in $\mathcal{B}(\mathcal{H})$. Then for any $A \in \overline{\mathcal{R}}^{\,s}$ there is a net A_α in $\mathcal{R}$ which converges strongly to A. It is natural to ask whether the A_α can be chosen self-adjoint whenever A is self-adjoint. Or, in any event, can the A_α be chosen such that A_α^* converges strongly to A^*, or such that A_α^2 converges to A^2? That such possibilities are available is the content of Kaplansky's Density Theorem, which we now consider.

Theorem 1.18 (Kaplansky's Density Theorem). *Let* $\mathcal{R}$ *be a self-adjoint algebra in* $\mathcal{B}(\mathcal{H})$ *with strong closure* $\overline{\mathcal{R}}^{\,s}$. *For any* $A \in \overline{\mathcal{R}}^{\,s}$, *there exists a net* A_α *in* $\mathcal{R}$ *such that*

(i) $\|A_\alpha\| \le \|A\|$ *for all* α;

(ii) A_α *converges strongly to* A;

(iii) A_α^* *converges strongly to* A^*.

Furthermore, if A *is self-adjoint, then each* A_α *can be chosen to be self-adjoint.*

Proof. Without loss of generality, we may assume that $\|A\| = 1$. Suppose first that A is self-adjoint. Let $f, g : [-1, 1] \to \mathbb{R}$ be the functions given by

$$f(t) = \frac{2t}{(1+t^2)} \quad \text{and} \quad g(t) = \frac{(1-\sqrt{1-t^2})}{t}$$

for $-1 \le t \le 1$ (and where $g(0) = 0$). Then both f and g are continuous, $-1 \le g(t) \le 1$ and $f(g(t)) = t$. Now, via Gelfand theory, A can be realized as a real function in $\mathcal{C}(Q)$, where Q is the spectrum of the unital C^*-algebra $\mathcal{A}$ generated by A. Moreover, $\|A\| = 1$ implies that this real function has absolute value not greater than 1 and so we can construct $B = g(A)$ which is self-adjoint, belongs to $\mathcal{A}$ and satisfies $f(B) = f(g(A)) = A$, that is, B satisfies $A = 2B(\mathbb{1} + B^2)^{-1}$. Since B belongs to $\mathcal{A}$, it certainly belongs to the (larger) algebra $\overline{\mathcal{R}}^s$. Therefore there is a net B_α in $\mathcal{R}$ which converges strongly to B.

We claim that we may choose B_α to be self-adjoint. To see this, we note that since B_α converges strongly to B it also converges weakly to B. Hence B_α^* converges weakly to B^* and so $\frac{1}{2}(B_\alpha + B_\alpha^*)$ converges weakly to B. From this, we see that B is in the weak closure of the convex set of self-adjoint elements of $\mathcal{R}$. By Proposition 1.17, this weak closure is the same as the strong closure and so B is in the strong closure of the set of self-adjoint elements of $\mathcal{R}$ and so we may choose the B_α self-adjoint, as claimed.

Let $A_\alpha = f(B_\alpha) = 2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1}$. Then each A_α is self-adjoint and $\|A_\alpha\| \le 1$. We claim that A_α converges strongly to A. To see this, we calculate

$$\begin{aligned}
A - A_\alpha &= 2B(\mathbb{1} + B^2)^{-1} - 2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1} \\
&= 2(\mathbb{1} + B_\alpha^2)^{-1}\{(\mathbb{1} + B_\alpha^2)B - B_\alpha(\mathbb{1} + B^2)\}(\mathbb{1} + B^2)^{-1} \\
&= 2(\mathbb{1} + B_\alpha^2)^{-1}\{(B - B_\alpha) + B_\alpha(B_\alpha - B)B\}(\mathbb{1} + B^2)^{-1} \\
&= 2(\mathbb{1} + B_\alpha^2)^{-1}(B - B_\alpha)(\mathbb{1} + B^2)^{-1} \\
&\qquad + 2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1}(B_\alpha - B)B(\mathbb{1} + B^2)^{-1}.
\end{aligned}$$

Now, $(B_\alpha - B)(\mathbb{1} + B^2)^{-1}$ converges strongly to zero. Moreover, we have $\|(\mathbb{1} + B_\alpha^2)^{-1}\| \le 1$ and $\|(2B_\alpha(\mathbb{1} + B_\alpha^2)^{-1})\| \le 1$ and so we see that it follows that A_α converges strongly to A, as required. This completes the proof for the case of A self-adjoint.

For the general case, we use a 2×2 matrix trick. Let $\mathcal{L}$ be the self-adjoint algebra of operators on the direct sum $\mathcal{H} \oplus \mathcal{H}$ comprising those 2×2 matrices (A_{ij}) with $A_{ij} \in \mathcal{R}$, $1 \le i, j \le 2$. One readily checks that $\overline{\mathcal{L}}^s$, the strong closure of $\mathcal{L}$ in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$, is given by those 2×2 matrices (A_{ij}) with $A_{ij}\in\overline{\mathcal{R}}^s$.

Let $A\in\overline{\mathcal{R}}^s$. Then $\left(\begin{smallmatrix}0 & A\\ A^* & 0\end{smallmatrix}\right)$ is a self-adjoint element of $\overline{\mathcal{L}}^s$ with norm equal to $\|A\|$. Hence, by the argument given above, there is a net (A^α_{ij}) of self-adjoint elements of $\mathcal{L}$, each with norm not greater than $\|A\|$, which converges strongly to $\left(\begin{smallmatrix}0 & A\\ A^* & 0\end{smallmatrix}\right)$ on $\mathcal{H}\oplus\mathcal{H}$. That is, for each $x\oplus y\in\mathcal{H}\oplus\mathcal{H}$,

$$\begin{aligned}\begin{pmatrix}A^\alpha_{11} & A^\alpha_{12}\\ A^\alpha_{21} & A^\alpha_{22}\end{pmatrix}(x\oplus y) &= (A^\alpha_{11}x+A^\alpha_{12}y)\oplus(A^\alpha_{21}x+A^\alpha_{22}y)\\ &\to \begin{pmatrix}0 & A\\ A^* & 0\end{pmatrix}(x\oplus y)\\ &= Ay\oplus A^*x\,.\end{aligned}$$

In particular (taking $x=0$), we see that A^α_{12} converges strongly to A on $\mathcal{H}$ and also (taking $y=0$) we see that A^α_{21} converges strongly to A^* on $\mathcal{H}$. Since each operator matrix (A^α_{ij}) is self-adjoint, we have $A^\alpha_{21}={A^\alpha_{12}}^*$ so that ${A^\alpha_{12}}^*$ converges strongly to A^*.

Finally, to complete the proof, we note that since each operator matrix (A^α_{ij}) has norm not greater than $\|A\|$, the same is true of each of its entries, so, in particular, $\|A^\alpha_{12}\|\le\|A\|$ for all α. ■

Remark 1.19. One can readily show that the properties (i), (ii) and (iii) imply that A^2_α converges strongly to A^2 and also that $A^*_\alpha A_\alpha$ converges strongly to A^*A. Indeed, for any $x\in\mathcal{H}$,

$$\begin{aligned}\|(A^2\alpha - A^2)x\| &\le \|A_\alpha(A_\alpha - A)x\| + \|A_\alpha - A)Ax\|\\ &\le \|A\|\|(A_\alpha - A)x\| + \|A(A-A_\alpha)x\|\\ &\to 0.\end{aligned}$$

The argument showing that $A^*_\alpha A_\alpha$ converges strongly to A^*A is similar.

In general, one can expect to require a net to lead to the desired strong convergence. However, as we will show next, if $\mathcal{H}$ is separable, the metrizability of the strong topology on bounded sets of $\mathcal{B}(\mathcal{H})$ can be used to yield a bounded sequence rather than just a net.

Proposition 1.20. *Let $\mathcal{H}$ be a separable Hilbert space and S a bounded set in $\mathcal{B}(\mathcal{H})$. Then the topology on S induced by the strong topology of $\mathcal{B}(\mathcal{H})$ is metrizable. In fact, it is given by a norm.*

Proof. We must find a norm $\|\!|\cdot|\!\|$ on $\mathcal{B}(\mathcal{H})$ such that a bounded net A_α converges strongly to A if and only if $\lim_\alpha \|\!|A_\alpha - A|\!\| = 0$. To construct such a norm, let $\{x_n\}$ be a countable dense set in $\mathcal{H}\setminus\{0\}$. For $A\in\mathcal{B}(\mathcal{H})$, set

$$\|\!|A|\!\| = \sum_{n=1}^\infty 2^{-n}\|Ay_n\|$$

where $y_n = x_n/\|x_n\|$. Then $|||A||| = 0$ implies that $Ax_n = 0$ for all $n \in \mathbb{N}$ and so $A = 0$, since $\{ x_n \}$ is dense in $\mathcal{H}$. It follows that $||| \cdot |||$ is a norm on $\mathcal{B}(\mathcal{H})$. It is also clear that $|||A||| \le \|A\|$.

Suppose that $A_\alpha \to A$ strongly, where A_α and A belong to any given bounded set S. Set $B_\alpha = A_\alpha - A$. Then there is $M > 0$ such that $\|B_\alpha\| \le M$ for all α. We have

$$\begin{aligned} |||B_\alpha||| &= \sum_{n=1}^{N} 2^{-n}\|B_\alpha y_n\| + \sum_{n=N+1}^{\infty} 2^{-n}\|B_\alpha y_n\| \\ &\le \sum_{n=1}^{N} 2^{-n}\|B_\alpha y_n\| + M\,2^{-N}. \end{aligned}$$

The second term on the right hand side is small for sufficiently large N. But then, for such suitably large but fixed N, the strong convergence of B_α to 0 implies that the first term on the right hand side is also small for all α sufficiently far enough along the index set of the net. We conclude that $|||B_\alpha||| \to 0$.

Conversely, suppose that (B_α) is a net in the bounded set S and suppose that $|||B_\alpha||| \to 0$. It follows that $\|B_\alpha x_n\| \to 0$ for each fixed $n \in \mathbb{N}$. Since there is $M > 0$ such that $\|B_\alpha\| \le M$ and $\{ x_n \}$ is dense in $\mathcal{H}$, it follows that $\|B_\alpha x\| \to 0$ for any $x \in \mathcal{H}$. That is, B_α converges strongly to 0 and the proof is complete. ∎

Theorem 1.21. *Let $\mathcal{R}$ be a self-adjoint algebra of operators on a separable Hilbert space, and suppose that A belongs to the strong closure of $\mathcal{R}$. Then there is a sequence (A_n) in $\mathcal{R}$ such that*

(i) $\|A_n\| \le \|A\|$ *for all* $n \in \mathbb{N}$;

(ii) A_n *converges strongly to* A;

(iii) A_n^* *converges strongly to* A^*.

Proof. By Kaplansky's Density Theorem, there is a net (A_α) in $\mathcal{R}$ satisfying $\|A_\alpha\| \le \|A\|$, $A_\alpha \to A$ strongly and $A_\alpha^* \to A^*$ strongly. By proposition 1.20 (and with the notation used there), we deduce that $|||A_\alpha - A||| \to 0$ and $|||A_\alpha^* - A^*||| \to 0$. In particular, for each $n \in \mathbb{N}$, there is some α_n such that $|||A_{\alpha_n} - A||| < 1/n$ and $|||A_{\alpha_n}^* - A^*||| < 1/n$. Setting $A_n = A_{\alpha_n}$, we have $\|A_n\| \le \|A\|$, $|||A_n - A||| \to 0$ and $|||A_n^* - A^*||| \to 0$, as $n \to \infty$, and therefore $A_n \to A$ and $A_n^* \to A^*$ strongly, as $n \to \infty$. ∎

Remark 1.22. The situation here is rather amusing. If we knew that an element belonging to the strong closure of $\mathcal{R}$ were the strong limit of a sequence from $\mathcal{R}$, then the Uniform Boundedness Principle would imply the boundedness of this sequence (irrespective of any separability assumption).

Here, we are using the boundedness of a strongly convergent net to construct a suitable strongly convergent sequence.

Positive continuous linear functionals

We have already discussed the structure of ultrastrongly and ultraweakly continuous linear functionals. We shall turn now to positivity issues. We will see that such positive functionals correspond precisely to the set of so-called density matrices.

Proposition 1.23. *Let $\mathcal{R}$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing $\mathbb{1}$ and suppose that φ is an ultrastrongly continuous positive linear functional on $\mathcal{R}$. Then there exists a sequence (x_i) of vectors in $\mathcal{H}$ such that $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$ and such that*

$$\varphi(A) = \sum_{i=1}^{\infty} \langle\, x_i, Ax_i \,\rangle$$

for all $A \in \mathcal{R}$. If φ is strongly continuous, the sequence of x_is may be chosen finite.

Proof. Suppose that φ is an ultrastrongly continuous positive linear functional on $\mathcal{R}$. Since the norm topology on $\mathcal{B}(\mathcal{H})$ is finer than the ultrastrong topology, φ uniquely extends (by continuity) to an ultrastrongly continuous linear functional on $\overline{\mathcal{R}}^{\,n}$, the norm closure of $\mathcal{R}$ in $\mathcal{B}(\mathcal{H})$. This extension is also positive. We may, therefore, without loss of generality, suppose that $\mathcal{R}$ is a C^*-algebra.

We know that we may write $\varphi(A) = \langle\, y, \widetilde{A}x \,\rangle$, for some $x = (x_i)$ and $y = (y_i)$ in $\bigoplus_i \mathcal{H}$ and where $\widetilde{A}x = (Ax_i)$. Consider the positive linear functional ψ on $\mathcal{R}$ given by

$$\psi(A) = \langle\, x + y, \widetilde{A}(x + y) \,\rangle\,.$$

Suppose that $A \in \mathcal{R}$ is positive. Then

$$\begin{aligned}
\psi(A) &= \langle\, x, \widetilde{A}x \,\rangle + \langle\, y, \widetilde{A}y \,\rangle + \langle\, y, \widetilde{A}x \,\rangle + \langle\, x, \widetilde{A}y \,\rangle \\
&= \langle\, x, \widetilde{A}x \,\rangle + \langle\, y, \widetilde{A}y \,\rangle + 2\varphi(A)
\end{aligned}$$

since $\langle\, x, \widetilde{A}y \,\rangle = \varphi(A) \geq 0$ and so $\langle\, y, \widetilde{A}x \,\rangle = \overline{\varphi(A)} = \varphi(A) = \langle\, x, \widetilde{A}y \,\rangle$. Hence

$$\psi(A) \geq 2\varphi(A)$$

that is, ψ majorizes φ. Consider the cyclic subspace $\mathcal{V}$ generated by $x+y$ in $\bigoplus_i \mathcal{H}$, i.e., $\mathcal{V}$ is the closure in $\bigoplus_i \mathcal{H}$ of the linear set $\{\, \widetilde{A}(x+y) : A \in \mathcal{R} \,\}$. Let π be the representation of $\mathcal{R}$ obtained by restricting $\widetilde{\mathcal{R}}$ to this subspace $\mathcal{V}$. By the uniqueness of the GNS construction, this representation π is equivalent to the GNS representation of $\mathcal{R}$ associated with the positive linear functional ψ. Since ψ majorizes φ, it follows that there is a positive operator T on the cyclic subspace $\mathcal{V}$, commuting with $\widetilde{A}$ for each $A \in \mathcal{R}$ and such that

$$\varphi(A) = \langle\, T(x+y), \widetilde{A}(x+y)\,\rangle = \langle\, T^{1/2}(x+y), \widetilde{A}\,T^{1/2}(x+y)\,\rangle\,.$$

Setting $z = (z_i) = T^{1/2}(x+y)$, we have

$$\varphi(A) = \sum_{i=1}^{\infty} \langle\, z_i, Az_i\,\rangle$$

as required.

If φ is strongly continuous, the proof is exactly the same except that $\bigoplus_{i=1}^{\infty} \mathcal{H}$ is replaced by a finite direct sum. ■

This result shows that φ looks rather like a trace. Indeed, we can now classify those functionals given by "density matrices".

Theorem 1.24. *Let $\mathcal{R}$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing $\mathbb{1}$ and let φ be an ultrastrongly continuous positive linear functional on $\mathcal{R}$. Then there is a positive linear trace class operator ρ such that*

$$\varphi(A) = \operatorname{Tr}(\rho A)$$

for all $A \in \mathcal{R}$.

Conversely, if ρ is a positive trace class operator, then $A \mapsto \operatorname{Tr}(\rho\, A)$ defines an ultrastrongly continuous positive linear functional on $\mathcal{B}(\mathcal{H})$.

Proof. Suppose that φ is an ultrastrongly continuous positive linear functional on $\mathcal{R}$. Then we have seen that we can write φ as

$$\varphi(A) = \sum_{i=1}^{\infty} \langle\, \zeta_i, A\zeta_i\,\rangle$$

for a suitable sequence (ζ_i) in $\mathcal{H}$ with $\sum_i \|\zeta_i\|^2 < \infty$ and for any $A \in \mathcal{R}$. Define the linear operator ρ on $\mathcal{H}$ by

$$\rho\, x = \sum_{i=1}^{\infty} \zeta_i \langle\, \zeta_i, x\,\rangle\,.$$

Then $\|\rho x\| \le \sum_i \|\zeta_i\|^2 \|x\| = \varphi(\mathbb{1})\|x\|$ and so we see that ρ is bounded. Furthermore, ρ is positive because

$$\langle\, x, \rho x\,\rangle = \sum_i \langle\, x, \zeta_i\,\rangle \langle\, \zeta_i, x\,\rangle = \sum_i |\langle\, \zeta_i, x\,\rangle|^2 \ge 0\,.$$

To show that ρ is trace class, let $\{ x_1, \dots, x_n \}$ be any orthonormal set in $\mathcal{H}$. Then

$$\begin{aligned}\sum_{j=1}^{n} \langle x_j, \rho x_j \rangle &= \sum_{j=1}^{n} \sum_{i} |\langle \zeta_i, x_j \rangle|^2 \\ &= \sum_{i} \sum_{j=1}^{n} |\langle \zeta_i, x_j \rangle|^2 \\ &\le \sum_{i} \|\zeta_i\|^2 = \varphi(\mathbb{1}) \,.\end{aligned}$$

This bound is independent of n and so we conclude that ρ is a trace class operator. Hence, for any complete orthonormal set (x_α) and any $A \in \mathcal{R}$, we have

$$\begin{aligned}\varphi(A) &= \sum_{i} \langle \zeta_i, A\zeta_i \rangle \\ &= \sum_{i,\alpha} \langle \zeta_i, x_\alpha \rangle \langle x_\alpha, A\zeta_i \rangle \\ &= \sum_{\alpha} \langle x_\alpha, A\rho x_\alpha \rangle \\ &= \mathrm{Tr}(A\rho)\end{aligned}$$

as required.

Conversely, suppose that ρ is a positive trace class operator. Then ρ can be written as

$$\rho = \sum_{i=1}^{\infty} \lambda_i^2 \, E_i$$

where the $\lambda_i^2 \ge 0$ are the eigenvalues of ρ and the E_i are the corresponding projections onto the normalized eigenvectors, ζ_i, say. Then we have

$$\rho x = \sum_{i} \lambda_i^2 \langle \zeta_i, x \rangle \, \zeta_i \,.$$

Set $\xi_i = \lambda_i \zeta_i$, so that $\sum_i \|\xi_i\|^2 = \sum_i \lambda_i^2 < \infty$ since ρ is trace class. Also, for any complete orthonormal set (x_α) in $\mathcal{H}$,

$$\begin{aligned}\mathrm{Tr}(\rho A) &= \sum_{\alpha} \langle x_\alpha, \rho A x_\alpha \rangle = \sum_{\alpha,i} \langle x_\alpha, \zeta_i \rangle \lambda_i^2 \langle \zeta_i, A x_\alpha \rangle \\ &= \sum_{\alpha,i} \langle x_\alpha, \xi_i \rangle \langle \xi_i, A x_\alpha \rangle = \sum_{\alpha,i} \langle \xi_i, A x_\alpha \rangle \langle x_\alpha, \xi_i \rangle \\ &= \sum_{i} \langle \xi_i, A\xi_i \rangle \,.\end{aligned}$$

Thus, the positive linear map $A \mapsto \mathrm{Tr}(\rho A)$ is ultraweakly continuous on $\mathcal{B}(\mathcal{H})$ and therefore, by Proposition 1.16, it is also ultrastrongly continuous. ■

Remark 1.25. The ξs are pairwise orthogonal, so we have a refinement of Proposition 1.23 in that the vectors x_i there can be chosen to be pairwise orthogonal. The trace class operator ρ is called a density matrix. Such objects play a fundamental rôle in the theory of quantum statistical mechanics. We note that, in general, ρ is not uniquely determined by φ. Indeed, ρ can be replaced by $U\rho U^*$ where U is any unitary element of the commutant $\mathcal{R}'$. However, if $\mathcal{R}$ is ultrastrongly dense in $\mathcal{B}(\mathcal{H})$ then ρ is uniquely determined by φ. Note that in this case $\mathcal{R}' = \{ \lambda \mathbb{1} : \lambda \in \mathbb{C} \}$.

Disjoint representations of a C^*-algebra

The purpose of this section is to present a proof of a theorem of Glimm and Kadison which will be used later on. First we need the polar decomposition theorem.

Definition 1.26. Let $\mathcal{H}$ be a Hilbert space. An operator $W \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if there are (closed) subspaces $\mathcal{K}, \mathcal{L}$ in $\mathcal{H}$ such that $W : \mathcal{K} \to \mathcal{L}$ is isometric onto $\mathcal{L}$ and $W : \mathcal{K}^\perp \to \{ 0 \}$. $\mathcal{K}$ is called the initial subspace and $\mathcal{L}$ the final subspace of W.

Evidently, W^* maps $\mathcal{L}$ isometrically onto $\mathcal{K}$ and maps $\mathcal{L}^\perp$ onto $\{ 0 \}$. It is also easy to see that $W^*W = P_{\mathcal{K}}$, the projection of $\mathcal{H}$ onto $\mathcal{K}$ and that $WW^* = P_{\mathcal{L}}$, the projection of $\mathcal{H}$ onto $\mathcal{L}$.

Conversely, if W is an operator such that $W^*W = P_{\mathcal{K}}$, the projection onto some (closed) subspace $\mathcal{K}$, then W is a partial isometry with initial space $\mathcal{K}$ and final space $W\mathcal{K}$.

Theorem 1.27. *Let $A \in \mathcal{B}(\mathcal{H})$. Then A can be written uniquely as $A = W\,|A|$ where $|A|$ is the positive square root of A^*A and W is a partial isometry with initial space equal to the closure of the range of $|A|$ and final space equal to the closure of the range of A.*

Proof. For any $x \in \mathcal{H}$, we have

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle = \langle x, A^*Ax \rangle \\ &= \langle x, |A|^2\, x \rangle = \|\, |A|\, x\|^2 \,. \end{aligned}$$

Thus there is a unique unitary operator W from the closure of the range of $|A|$ to the closure of the range of A given by $W\,|A|\,x = Ax$, for $x \in \mathcal{H}$. We extend W to a partial isometry on $\mathcal{H}$ by defining $Wy = 0$ if y is orthogonal to the range of $|A|$. ■

Remark 1.28. The expression $W\,|A|$ above is called the polar decomposition of A. Suppose that U is unitary and commutes with A. Then U also commutes with $|A|$ and we have

$$A = UAU^* = UWU^*\,U\,|A|\,U^* = UWU^*\;|A|\;.$$

This is another polar decomposition of A and so, by uniqueness, we see that $W = UWU^*$, that is, U also commutes with W.

Definition 1.29. Let $(\mathcal{H}, \pi)$ be a representation of a C^*-algebra $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$ and suppose that $\mathcal{H}_1 \subseteq \mathcal{H}$ is a subspace of $\mathcal{H}$ invariant under the action of $\pi(\mathfrak{A})$. Then $(\mathcal{H}_1, \pi_1)$, where $\pi_1(\mathfrak{A}) = \pi(\mathfrak{A}) \upharpoonright \mathcal{H}_1$, defines a representation of $\mathfrak{A}$ called a subrepresentation of $(\mathcal{H}, \pi)$.

If $\pi_1^\perp(\mathfrak{A}) = \pi(\mathfrak{A}) \upharpoonright \mathcal{H}_1^\perp$, then evidently, $(\mathcal{H}_1^\perp, \pi_1^\perp)$ is also a subrepresentation of $(\mathcal{H}, \pi)$.

Definition 1.30. Representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of a C^*-algebra are said to be disjoint if no subrepresentation of one is unitarily equivalent to any subrepresentation of the other.

Theorem 1.31. *Suppose $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are disjoint representations of a C^*-algebra $\mathfrak{A}$. Then the von Neumann algebra $(\pi_1 \oplus \pi_2)(\mathfrak{A})''$ is equal to the direct sum $\pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$.*

Proof. We first show that $(\pi_1 \oplus \pi_2)(\mathfrak{A})' = \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$, where the commutant on the left hand side is taken in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and those on the right hand side are taken in $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$, respectively. To see this, let $B \in (\pi_1 \oplus \pi_2)(\mathfrak{A})' \subseteq \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. We may write B as $\left(\begin{smallmatrix} X & S \\ T & Y \end{smallmatrix}\right)$ for suitable $X \in \mathcal{B}(\mathcal{H}_1)$, $Y \in \mathcal{B}(\mathcal{H}_2)$, and bounded linear operators $S : \mathcal{H}_2 \to \mathcal{H}_1$ and $T : \mathcal{H}_1 \to \mathcal{H}_2$. Since $B \in (\pi_1 \oplus \pi_2)(\mathfrak{A})'$, it follows that $X \in \pi_1(\mathfrak{A})'$, $Y \in \pi_2(\mathfrak{A})'$ and that $S\pi_2(A) = \pi_1(A)S$ and $T\pi_1(A) = \pi_2(A)T$ for all $A \in \mathfrak{A}$.

Let $\widetilde{S} = \left(\begin{smallmatrix} 0 & S \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and write π for $\pi_1 \oplus \pi_2$. Then, in 2×2 matrix terms, $\pi(a) = \left(\begin{smallmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{smallmatrix}\right)$ and we see that $\widetilde{S}\pi(A) = \pi(A)\widetilde{S}$ for all $A \in \mathfrak{A}$. In particular, for any unitary element $U \in \mathfrak{A}$, $\pi(U)$ is unitary and commutes with $\widetilde{S}$. It follows that $\pi(U)$ also commutes with the partial isometry W, where $\widetilde{S} = W|\widetilde{S}|$ is the polar decomposition of $\widetilde{S}$. However, every element of the C^*-algebra $\mathfrak{A}$ can be written as a linear combination of unitaries in $\mathfrak{A}$ and so we deduce that W commutes with $\pi(\mathfrak{A})$, that is, $W\pi(A) = \pi(A)W$ for all $A \in \mathfrak{A}$. Taking adjoints, it follows that $\pi(A)W^* = W^*\pi(A)$ for all $A \in \mathfrak{A}$ and therefore

$$\pi(A)W^*W = W^*\pi(A)W = W^*W\pi(A)$$

for all $A \in \mathfrak{A}$. In other words, $W^*W(\mathcal{H}_1 \oplus \mathcal{H}_2)$, the initial space of $\widetilde{S}$ is invariant under $\pi(\mathfrak{A})$ and so $\pi \upharpoonright W^*W(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a subrepresentation of $(\mathcal{H}_1 \oplus \mathcal{H}_2, \pi_1 \oplus \pi_2)$. However, the initial space of W is a subspace of $\mathcal{H}_2$ and so $\pi \upharpoonright W^*W(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a subrepresentation of $(\pi_2, \mathcal{H}_2)$.

Similarly, one can show that $\pi \upharpoonright WW^*(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a subrepresentation of $(\mathcal{H}_1, \pi_1)$. Now let π' denote $\pi \upharpoonright W^*W(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and let π'' denote $\pi \upharpoonright WW^*(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Then for any $\xi \in W^*W(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $A \in \mathfrak{A}$, we have

(using the fact that W^*W is a projection)

$$\begin{aligned}\pi'(A)\xi &= \pi(A)\xi = \pi(A)\,W^*W\xi \\ &= W^*\pi(A)W\xi = W^*\pi(A)W\,W^*W\xi \\ &= W^*\pi''(A)WW^*\,W\xi \\ &= W^*\pi''(A)W\xi\,.\end{aligned}$$

We see that W effects a unitary equivalence between these two subrepresentations which contradicts the hypothesis that $(\mathcal{H}_1, \pi)$ and $(\mathcal{H}_2, \pi_2)$ are disjoint representations of $\mathfrak{A}$ — unless $S = 0$ so that $W = 0$ and W^*W is the projection onto $\{0\}$.

In an analogous way, one shows that $T = 0$ and then we see that B belongs to $\pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$, that is, $(\pi_1 \oplus \pi_2)(\mathfrak{A})' \subseteq \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$. Since the reverse inclusion is trivial, we conclude that

$$(\pi_1 \oplus \pi_2)(\mathfrak{A})' = \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$$

as claimed.

We are now in a position to complete the proof. Let $B \in (\pi_1 \oplus \pi_2)(\mathfrak{A})''$ be given. By von Neumann's Density Theorem, B is in the weak closure of the set $(\pi_1 \oplus \pi_2)(\mathfrak{A})$. From this it follows that B has the form $B = C \oplus D$ where $C \in \mathcal{B}(\mathcal{H}_1)$ is in the weak closure of $\pi_1(\mathfrak{A})$ and $D \in \mathcal{B}(\mathcal{H}_2)$ is in the weak closure of $\pi_2(\mathfrak{A})$. But this means that $C \in \pi_1(\mathfrak{A})''$ and $D \in \pi_2(\mathfrak{A})''$ and therefore

$$(\pi_1 \oplus \pi_2)(\mathfrak{A})'' \subseteq \pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''\,.$$

Conversely, suppose that $A \oplus B \in \pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$. Since $(\pi_1 \oplus \pi_2)(\mathfrak{A})' = \pi_1(\mathfrak{A})' \oplus \pi_2(\mathfrak{A})'$, it follows that $A \oplus B$ commutes with $(\pi_1 \oplus \pi_2)(\mathfrak{A})'$, that is, $A \oplus B \in (\pi_1 \oplus \pi_2)(\mathfrak{A})''$ which means that

$$\pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})'' \subseteq (\pi_1 \oplus \pi_2)(\mathfrak{A})''$$

and the proof is complete. ∎

Theorem 1.32 (Glimm and Kadison (1960)). *Let ω_1 and ω_2 be states on a C^*-algebra $\mathfrak{A}$ and suppose that the associated GNS representations are disjoint. Then $\|\omega_1 - \omega_2\| = 2$.*

Proof. Let $(\mathcal{H}_1, \pi_1, \Omega_1)$ and $(\mathcal{H}_2, \pi_2, \Omega_2)$ denote the GNS representations associated with ω_1 and ω_2, respectively. Evidently, the map

$$(\pi_1 \oplus \pi_2)(A) \mapsto \langle\, \Omega_1, \pi_1(A)\Omega_1 \,\rangle - \langle\, \Omega_2, \pi_2(A)\Omega_2 \,\rangle = \omega_1(A) - \omega_2(A)$$

on $(\pi_1 \oplus \pi_2)(\mathfrak{A})$ is a weakly continuous linear functional and so extends to a weakly continuous linear form ψ, say, on the weak closure $\overline{(\pi_1 \oplus \pi_2)(\mathfrak{A})}^{\,w} = (\pi_1 \oplus \pi_2)(\mathfrak{A})''$.

We claim that $\|\psi\| = \|\omega_1 - \omega_2\|$. Evidently, the norm of ψ is not less than that of $\omega_1 - \omega_2$ since it is an extension of the latter. Let $A \in (\pi_1 \oplus \pi_2)(\mathfrak{A})''$ with $\|A\| = 1$. We need only show that $\|\psi(A)\| \le \|\omega_1 - \omega_2\|$. By von Neumann's Density Theorem, A is a strong limit point of $(\pi_1 \oplus \pi_2)(\mathfrak{A})$ and so, by Kaplansky's Density Theorem, there is a net (B_α) in $(\pi_1 \oplus \pi_2)(\mathfrak{A})$ with $\|B_\alpha\| \le 1$, for all α, such that B_α converges strongly to A. Hence $\psi(B_\alpha)$ converges to $\psi(A)$. But

$$\begin{aligned} |\psi(B_\alpha)| &= |(\omega_1 - \omega_2)(B_\alpha)| \\ &\le \|\omega_1 - \omega_2)\| \, \|B_\alpha\| \\ &\le \|\omega_1 - \omega_2)\| \end{aligned}$$

and so $|\psi(A)| \le \|\omega_1 - \omega_2\|$ and we have equality $|\psi(A)| = \|\omega_1 - \omega_2\|$, as claimed.

Finally, by hypothesis, the representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are disjoint and so $(\pi_1 \oplus \pi_2)(\mathfrak{A})'' = \pi_1(\mathfrak{A})'' \oplus \pi_2(\mathfrak{A})''$. This algebra contains $A = \mathbb{1} \oplus (-\mathbb{1})$ and so (by von Neumann's Density Theorem) there is a net $((\pi_1 \oplus \pi_2)(B_\alpha))$ in $(\pi_1 \oplus \pi_2)(\mathfrak{A})$ which converges weakly to A. In particular,

$$\langle \Omega_1 \oplus 0, (\pi_1 \oplus \pi_2)(B_\alpha)\Omega_1 \oplus 0 \rangle \to \langle \Omega_1 \oplus 0, A\,\Omega_1 \oplus 0 \rangle = 1$$

and

$$\langle 0 \oplus \Omega_2, (\pi_1 \oplus \pi_2)(B_\alpha)0 \oplus \Omega_2 \rangle \to \langle 0 \oplus \Omega_2, A\,0 \oplus \Omega_2 \rangle = 1.$$

But then

$$\begin{aligned} \psi(A) &= \lim_\alpha \psi((\pi_1 \oplus \pi_2)(B_\alpha)) \\ &= \lim_\alpha \big(\langle \Omega_1 \oplus 0, (\pi_1 \oplus \pi_2)(B_\alpha)\Omega_1 \oplus 0 \rangle \\ &\qquad + \langle 0 \oplus \Omega_2, (\pi_1 \oplus \pi_2)(B_\alpha)0 \oplus \Omega_2 \rangle \big) \\ &= 2. \end{aligned}$$

Hence, we have

$$2 = |\psi(A)| \le \|\psi\| \, \|A\| = \|\psi\| = \|\omega_1 - \omega_2\| \le 2$$

and we conclude that $\|\omega_1 - \omega_2\| = 2$. ∎

Chapter 2

Modular Theory

We introduce here the basic ingredients of Tomita-Takesaki modular theory.

The Tomita-Takesaki Modular operator

First, we need a definition.

Definition 2.1. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be a cyclic vector for $\mathcal{M}$ if the set $\{ x\Omega : x \in \mathcal{M} \}$ is dense in $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be separating for $\mathcal{M}$ if $x\Omega = 0$ for $x \in \mathcal{M}$ implies that $x = 0$.

One can verify that Ω is cyclic for $\mathcal{M}$ if and only if it is separating for the commutant $\mathcal{M}'$. To see this, suppose first that Ω is cyclic for $\mathcal{M}$ and suppose that $y\Omega = 0$ for some $y \in \mathcal{M}'$. Then for any $x \in \mathcal{M}$, $yx\Omega = xy\Omega = 0$. Since $\mathcal{M}\Omega$ is dense in $\mathcal{H}$, it follows that $y = 0$ and so Ω is separating for $\mathcal{M}'$.

Next, suppose that Ω is separating for $\mathcal{M}'$ and let p be the projection onto the closure of the subspace $\mathcal{M}\Omega$ in $\mathcal{H}$. For any $\xi \in \mathcal{H}$, $p\xi$ is given as $p\xi = \lim a_n\Omega$ for some sequence (a_n) in $\mathcal{M}$. But then, for any $b \in \mathcal{M}$, $bp\xi = \lim ba_n\Omega = \lim pba_n\Omega = pbp\xi$ and so $bp = pbp$. For any $x \in \mathcal{M}$, setting $b = x$ and then $b = x^*$ leads to $xp = pxp$ and $x^*p = px^*p$. Taking adjoints of this last equality, we find that $px = pxp = xp$, that is $px = xp$. It follows that $p \in \mathcal{M}'$. But, by construction, $p\Omega = \Omega$, that is, $(\mathbb{1} - p)\Omega = 0$. Since Ω is separating for $\mathcal{M}'$, we deduce that $p = \mathbb{1}$ and therefore the set $\mathcal{M}\Omega$ is dense in $\mathcal{H}$, as claimed.

It follows from the above that if Ω is both cyclic and separating for $\mathcal{M}$, then it is also cyclic and separating for $\mathcal{M}'$.

We note also that any von Neumann algebra with a separating vector is necessarily σ-finite. Indeed, suppose that Ω is separating for the von Neumann algebra $\mathcal{M}$ and that $\{P_\alpha\}$ is any family of (non-zero) pairwise orthogonal projections in $\mathcal{M}$. Then none of the $P_\alpha\Omega$s are zero and $\sum_{\alpha\in S} \|P_\alpha\Omega\|^2 \le \|\Omega\|^2$ for any finite set S. It follows that the family $\{P_\alpha\}$ is countable.

Suppose then that $\Omega \in \mathcal{H}$ is a cyclic and separating vector for the von Neumann algebra $\mathcal{M}$. We define conjugate linear operators S_0 and F_0 by

$$S_0 x\Omega = x^*\Omega \text{ for } x \in \mathcal{M}$$

and

$$F_0 x'\Omega = x'^*\Omega \text{ for } x' \in \mathcal{M}'.$$

S_0 and F_0 are both well-defined with dense domains, $D(S_0) = \mathcal{M}\Omega$ and $D(F_0) = \mathcal{M}'\Omega$.

Proposition 2.2. *S_0 and F_0 are closeable operators with $S_0 \subseteq F_0^*$ and $F_0 \subseteq S_0^*$ so that $\overline{S_0} \subseteq F_0^*$ and $\overline{F_0} \subseteq S_0^*$. In fact, $\overline{S_0} = F_0^*$ and $\overline{F_0} = S_0^*$.*

Proof. For $x \in \mathcal{M}$, $x' \in \mathcal{M}'$, we have

$$\begin{aligned} (\, S_0 x\Omega, x'\Omega \,) &= (\, x^*\Omega, x'\Omega \,) \\ &= (\, x'^*\Omega, x\Omega \,) \\ &= (\, F_0 x'\Omega, x\Omega \,). \end{aligned}$$

By the definition of the adjoint of a conjugate linear operator, we deduce that $x'\Omega \in D(S_0^*)$ and $S_0^* x'\Omega = F_0 x'\Omega$ and also that $x\Omega \in D(F_0^*)$ and $F_0^* x\Omega = S_0 x\Omega$, that is, $F_0 \subseteq S_0^*$ and $S_0 \subseteq F_0^*$.

Taking closures gives $\overline{F_0} \subseteq S_0^*$ and $\overline{S_0} \subseteq F_0^*$.

For a proof of the equalities $\overline{S_0} = F_0^*$ and $\overline{F_0} = S_0^*$, we refer, for example, to that given by Bratteli and Robinson. ∎

We will just write S for $\overline{S_0}$ and F for $\overline{F_0}$. Then S and F are densely-defined, closed, conjugate linear operators on $\mathcal{H}$.

Proposition 2.3. *We have that $S\, D(S) \subseteq D(S)$ and $S^2\xi = \xi$ for all $\xi \in D(S)$ and similarly $F\, D(F) \subseteq D(F)$ and $F^2\zeta = \zeta$ for all $\zeta \in D(F)$. Furthermore, there is a conjugation J (that is, J is antiunitary and $J^2 = \mathbb{1}$) and there is a non-singular positive self-adjoint operator Δ such that $D(S) = D(\Delta^{1/2})$, $D(F) = D(\Delta^{-\frac{1}{2}})$ and*

(i) $S = J\,\Delta^{1/2} = \Delta^{-1/2}\,J$;

(ii) $F = J\,\Delta^{-1/2} = \Delta^{1/2}\,J$;

(iii) $\Delta = F\,S\,,\ \Delta^{-1} = S\,F$;

(iv) $J\,\Omega = \Omega$ *and* $\Delta\,\Omega = \Omega$;

(v) $J\,\Delta^{it} = \Delta^{it}\,J$ *for all* $t \in \mathbb{R}$.

Proof. Let $\xi \in D(S)$ and let $x' \in \mathcal{M}'$. Then

$$\begin{aligned}(F_0 x'\Omega, S\xi) &= (x'^*\Omega, S\xi) \\ &= (\xi, F_0 x'^*\Omega), \quad \text{since } F_0 \subseteq S_0^* = S^*, \\ &= (\xi, x'\Omega).\end{aligned}$$

It follows that $S\xi \in D(F_0^*) = D(S)$ and that $F_0^* S\xi = SS\xi = \xi$. A similar argument applied to F shows that $F^2\zeta = \zeta$ for all $\zeta \in D(F)$.

The closed operator S has the polar decomposition $S = J\Delta^{1/2}$ where Δ is a non-negative self-adjoint operator. Since $S^2 D(S) = D(S)$, we deduce that $\operatorname{ran} S$ is dense in $\mathcal{H}$ and so J is antiunitary. (We have $SD(S) \subseteq D(S)$ so that $S^2 D(S) \subseteq SD(S)$ and therefore $D(S) = S^2 D(S) \subseteq SD(S) \subseteq D(S)$ giving equality $SD(S) = D(S)$. Alternatively, we could note that $SD(S)$ contains the elements $x^*\Omega$, for $x \in \mathcal{M}$. These are dense in $\mathcal{H}$ because Ω is cyclic for $\mathcal{M}$.)

Also, if $S\xi = 0$ then $\xi = S^2\xi = 0$ and so $\ker S = \ker \Delta = \{0\}$, that is, the operator Δ is non-singular.

Now, $\Delta = S^*S = FS$ and also $SF\Delta = SFFS = 1\!\!1 = FSSF = \Delta SF$ and therefore $SF = \Delta^{-1}$ and (iii) is proved.

Furthermore,

$$F = SSF = J\Delta^{1/2} SF = J\Delta^{-1/2}.$$

By uniqueness of the polar decomposition, the equality

$$F = S^* = \Delta^{1/2} J^* = J^* J \Delta^{1/2} J^*$$

implies that $J = J^*$ and that $J\Delta^{1/2}J^* = \Delta^{-1/2}$. Therefore $J^2 = J^*J = 1\!\!1$ and so (i) and (ii) follow.

To prove (iv), we note that $FS\Omega = F\Omega = \Omega$ and so $\Delta\Omega = \Omega$. Hence $\Delta^{1/2}\Omega = \Omega$ and therefore

$$\Omega = S\,\Omega = J\Delta^{1/2}\,\Omega = J\,\Omega.$$

Finally, to prove (v), we observe that the equality $J\Delta^{1/2}J = \Delta^{-1/2}$ implies that $J\Delta J = \Delta^{-1}$ and therefore $P_\lambda = JQ_\lambda J$, where $\Delta = \int \lambda\, dP_\lambda$ and where $\Delta^{-1} = \int \lambda\, dQ_\lambda$.

Hence, for any measurable function f on $(0, \infty)$,

$$\begin{aligned}J\,f(\Delta)\,J &= J \int_0^\infty f(\lambda)\,dP_\lambda\ J \\ &= \int_0^\infty \overline{f(\lambda)}\,dQ_\lambda\,, \quad \text{since } J \text{ is conjugate linear}, \\ &= \overline{f}(\Delta^{-1}).\end{aligned}$$

Taking $f(\lambda) = \lambda^{it}$, $t \in \mathbb{R}$, we get

$$J\,\Delta^{it}\,J = \Delta^{it}$$

and the proof is complete. ∎

Definition 2.4. The self-adjoint operator Δ is called the modular operator and J is called the modular conjugation (associated with $\mathcal{M}$ and Ω).

Example 2.5. (First version) Let ω be a faithful, positive functional on $\mathrm{M}_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. Then there is some element $\rho \in \mathrm{M}_n(\mathbb{C})$, with $\rho > 0$, such that

$$\omega(A) = \mathrm{Tr}(\rho\, A)$$

for all $A \in \mathrm{M}_n(\mathbb{C})$. To say that ω is faithful is equivalent to the equality $\ker \rho = \{\,0\,\}$.

Let $\Psi_1, \dots, \Psi_n \in \mathbb{C}^n$ be orthonormal vectors such that $\rho\Psi_i = \lambda_i \Psi_i$ with $\lambda_i > 0$. (The set $\{\,\Psi_i\,\}$ is a collection of eigenvectors of ρ.) We define a representation π of $\mathrm{M}_n(\mathbb{C})$ on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$ by the assignment

$$\pi(A) = A \otimes \mathbb{1}\,,$$

for $A \in \mathrm{M}_n(\mathbb{C})$. Set

$$\Omega = \sum_{i=1}^{n} \lambda_i^{1/2}\, \Psi_i \otimes e_i$$

where $\{\,e_i\,\}$ is any orthonormal basis of $\mathbb{C}^n$. Then

$$\begin{aligned}
(\,\pi(A)\,\Omega, \Omega\,) &= (\,\sum_i \lambda_i^{1/2}\, A\,\Psi_i \otimes e_i,\ \sum_j \lambda_j^{1/2}\, A\,\Psi_j \otimes e_j\,) \\
&= \sum_i \lambda_i\,(\,A\,\Psi_i, \Psi_i\,), \quad \text{since } (\,e_i, e_j\,) = \delta_{ij}, \\
&= \sum_i (\,A\,\Psi_i, \rho\,\Psi_i\,) \\
&= \mathrm{Tr}(\rho\,A) \\
&= \omega(A).
\end{aligned}$$

Moreover, vectors of the form $\sum_i \lambda_i^{1/2}\, A\,\Psi_i \otimes e_i$ with $A \in \mathrm{M}_n(\mathbb{C})$ span $\mathcal{H}$ and so we see that $(\mathcal{H}, \pi, \Omega)$ is the GNS representation of $\mathrm{M}_n(\mathbb{C})$ given by ω.

Since ω is faithful, the representation π is also faithful and Ω is separating for $\pi(\mathrm{M}_n(\mathbb{C}))$. Set

$$S\,\pi(A)\,\Omega = \pi(A)^*\,\Omega$$

for $A \in \mathrm{M}_n(\mathbb{C})$, and let $S = J\,\Delta^{1/2}$ be the polar decomposition of S. Since $S^2 = \mathbb{1}$, it follows that J is antiunitary. To compute the action of Δ, let

$e_i = \Psi_i$, for $1 \le i \le n$, and consider the partial isometry $E_{ij} \in \mathrm{M}_n(\mathbb{C})$ which maps Ψ_i to Ψ_j and annihilates $\{ \Psi_i \}^\perp$. We have

$$\begin{aligned}
(S\pi(E_{ij})\,\Omega, S\pi(E_{k\ell})\,\Omega) &= (J\Delta^{1/2}\pi(E_{ij})\,\Omega, J\Delta^{1/2}\pi(E_{k\ell})\,\Omega) \\
&= (\Delta^{1/2}\pi(E_{k\ell})\,\Omega, \Delta^{1/2}\pi(E_{ij})\,\Omega) \\
&= (\Delta\pi(E_{k\ell})\,\Omega, \Delta\pi(E_{ij})\,\Omega) \\
&= (\Delta\lambda_k^{1/2}\Psi_\ell \otimes \Psi_k, \lambda_i^{1/2}\Psi_j \otimes \Psi_i).
\end{aligned}$$

However, the left hand side is

$$\begin{aligned}
(\pi(E_{ij}^*)\,\Omega, \pi(E_{k\ell}^*)\,\Omega) &= (\pi(E_{ji})\,\Omega, \pi(E_{\ell k})\,\Omega) \\
&= (\lambda_j^{1/2}\Psi_i \otimes \Psi_j, \lambda_\ell^{1/2}\Psi_k \otimes \Psi_\ell) \\
&= \lambda_j^{1/2}\,\lambda_\ell^{1/2}\,\delta_{ik}\,\delta_{j\ell}.
\end{aligned}$$

It follows that

$$\lambda_k^{1/2}\,\lambda_i^{1/2}\,(\Delta\,\Psi_\ell \otimes \Psi_k, \Psi_j \otimes \Psi_i) = \lambda_j^{1/2}\lambda_\ell^{1/2}\,\delta_{ik}\,\delta_{j\ell}.$$

We see that Δ is diagonal with respect to the basis $\{ \Psi_i \otimes \Psi_j \}$ of $\mathcal{H}$. Putting $\ell = j$ and $k = i$, we find that

$$\lambda_i\,(\Delta\,\Psi_j \otimes \Psi_i, \Psi_j \otimes \Psi_i) = \lambda_j,$$

that is, $\Delta\,\Psi_j \otimes \Psi_i = \lambda_j/\lambda_i$ giving

$$\Delta = \rho \otimes \rho^{-1}\,.$$

To determine the action of J, we note that $S = J\Delta^{1/2}$ so that $J = S\,\Delta^{-1/2}$. Therefore

$$\begin{aligned}
J\;\Psi_i \otimes \Psi_j &= S\;\Delta^{-1/2}\;\Psi_i \otimes \Psi_j \\
&= \lambda_i^{-1/2}\,\lambda_j^{1/2}\,S\;\Psi_i \otimes \Psi_j \\
&= \lambda_i^{-1/2}\,\lambda_j^{1/2}\,S\;\lambda_j^{-1/2}\pi(E_{ji})\,\Omega \\
&= \lambda_i^{-1/2}\pi(E_{ji}^*)\,\Omega \\
&= \lambda_i^{-1/2}\pi(E_{ij})\,\Omega \\
&= \lambda_i^{-1/2}\,\lambda_i^{1/2}\,\Psi_j \otimes \Psi_i \\
&= \Psi_j \otimes \Psi_i\,.
\end{aligned}$$

Hence

$$J\,\sum_{i,j} c_{ij}\;\Psi_i \otimes \Psi_j = \sum_{i,j} \overline{c_{ij}}\;\Psi_j \otimes \Psi_i$$

since J is conjugate-linear.

The various relations between S, J and Δ can be checked explicitly. For example, $\Delta\Omega = \Omega$ and $J\Omega = \Omega$. Furthermore,

$$\begin{aligned}
J\,\Delta\,J\;\Psi_i\otimes\Psi_j &= J\,\Delta\;\Psi_j\otimes\Psi_i \\
&= J\,\rho\,\Psi_j\otimes\rho^{-1}\,\Psi_i \\
&= \lambda_j\,\lambda_i^{-1}\,J\;\Psi_j\otimes\Psi_i \\
&= \lambda_j\,\lambda_i^{-1}\;\Psi_i\otimes\Psi_j \\
&= \rho^{-1}\,\Psi_i\otimes\rho\,\Psi_j \\
&= \Delta^{-1}\;\Psi_i\otimes\Psi_j\,,
\end{aligned}$$

that is, $J\,\Delta\,J = \Delta^{-1}$.

For any $A\in \mathrm{M}_n(\mathbb{C})$, let A^c denote the complex conjugate matrix (with respect to the basis $\{\,\Psi_i\,\}$, so that if $A\Psi_j = \sum_{k=1}^n \alpha_k\Psi_k$, then $A^c\Psi_j = \sum_{k=1}^n \overline{\alpha}_k\,\Psi_k$). Then

$$\begin{aligned}
J\,\pi(A)\,J\;\Psi_i\otimes\Psi_j &= J\,(A\Psi_j\otimes\Psi_i) \\
&= \Psi_i\otimes A^c\Psi_j \\
&= (\mathbb{1}\otimes A^c)(\Psi_i\otimes\Psi_j)
\end{aligned}$$

and we see that $J\,\pi(A)\,J = \mathbb{1}\otimes A^c\in\pi(\mathrm{M}_n)'$. Now, $\pi(\mathrm{M}_n) = \mathrm{M}_n\otimes\mathbb{1}$ and so $\pi(\mathrm{M}_n)' = \mathbb{1}\otimes\mathrm{M}_n$ and therefore

$$J\,\pi(\mathrm{M}_n)\,J = \pi(\mathrm{M}_n)'.$$

Furthermore, for any $t\in\mathbb{R}$ and $A\in\mathrm{M}_n(\mathbb{C})$,

$$\begin{aligned}
\Delta^{it}\,\pi(A)\,\Delta^{-it} &= \rho^{it}\otimes\rho^{-it}\,(A\otimes\mathbb{1})\,\rho^{-it}\otimes\rho^{it} \\
&= \rho^{it}A\,\rho^{-it}\otimes\mathbb{1} \\
&= \pi(\rho^{it}A\,\rho^{-it})\,.
\end{aligned}$$

Therefore $t\mapsto\Delta^{it}\,\cdot\,\Delta^{-it}$ defines a group of automorphisms of $\pi(\mathrm{M}_n)$.

Since π is faithful, these induce a family of automorphisms σ_t of $\mathrm{M}_n(\mathbb{C})$ given by $A\mapsto\sigma_t(A) = \rho^{it}A\,\rho^{-it}$.

Now fix $A, B\in\mathrm{M}_n(\mathbb{C})$ and consider the complex-valued function

$$F(z) = \omega\,(\rho^{iz}A\,\rho^{-iz}B)\quad \text{for } z\in\mathbb{C}.$$

Evidently, F is entire and

$$F(t) = \omega\,(\rho^{it}A\,\rho^{-it}B) = \omega\,(\sigma_t(A)\,B)$$

for $t\in\mathbb{R}$. Furthermore,

$$F(t+i) = \omega(\rho^{-1}\rho^{it}A\,\rho^{-it}\,\rho\,B)$$

$$
\begin{aligned}
&= \mathrm{Tr}(\rho\, \rho^{-1} \rho^{it} A\, \rho^{-it}\, \rho\, B) = \mathrm{Tr}(\rho\, B\, \rho^{it} A\, \rho^{-it}) \\
&= \omega(B\, \sigma_t(A))\,.
\end{aligned}
$$

In other words, we see that $\omega(\sigma_t(A)\, B)$ and $\omega(B\, \sigma_t(A))$ are boundary values of the analytic function $F(z)$ on the strip $\{\, z : 0 \le \mathrm{Im}\, z \le 1\,\}$.

We can also derive this using the J, Δ operators, as follows. We have

$$F(z) = (\, \Delta^{iz} \pi(A)\, \Delta^{-iz} \pi(B)\, \Omega,\, \Omega\,)$$

and so

$$
\begin{aligned}
F(t+i) &= (\, \Delta^{it} \pi(A)\, \Delta^{-it} \Delta \pi(B)\, \Omega,\, \Omega\,)\,, \quad \text{since } \Delta^{-1}\Omega = \Omega, \\
&= (\, \pi(\sigma_t(A)) \Delta\, \pi(B)\, \Omega,\, \Omega\,) \\
&= (\, \Delta^{1/2} \pi(B)\, \Omega, \Delta^{1/2} \pi(\sigma_t(A))^* \Omega\,) \\
&= (\, J\, \Delta^{1/2} \pi(\sigma_t(A))^* \Omega, J\, \Delta^{1/2} \pi(B)\, \Omega\,) \\
&= (\, S \pi(\sigma_t(A))^* \Omega, S \pi(B)\ \Omega\,) \\
&= (\, \pi(\sigma_t(A)) \Omega, \pi(B)^* \Omega\,) \\
&= (\, \pi(B)\, \pi(\sigma_t(A))\, \Omega,\, \Omega\,) \\
&= \omega(\, B\, \sigma_t(A)\,)
\end{aligned}
$$

as required. Furthermore,

$$
\begin{aligned}
\omega(\, A\, B\,) &= (\, \pi(A)\, \pi(B)\, \Omega,\, \Omega\,) \\
&= (\, \pi(B)\, \Omega, \pi(A)^*\, \Omega\,) \\
&= (\, S\, \pi(B)^* \Omega, S\, \pi(A)\, \Omega\,) \\
&= (\, J\, \Delta^{1/2} \pi(B)^* \Omega, J\, \Delta^{1/2} \pi(A)\, \Omega\,) \\
&= (\, \Delta^{1/2} \pi(A)\, \Omega, \Delta^{1/2} \pi(B)^* \Omega\,) \\
&= (\, \pi(B)\, \Delta\, \pi(A)\, \Omega,\, \Omega\,) \\
&= (\, \pi(B)\, \Delta\, \pi(A) \Delta^{-1} \Omega,\, \Omega\,)
\end{aligned}
$$

that is,

$$\omega(\, A\, B\,) = \omega(\, B\, \sigma_{-i}(A)\,)\,.$$

Example 2.6. We reconsider the above example. ω is a given faithful state on $\mathrm{M}_n(\mathbb{C})$ and so is given by $\omega(\,\cdot\,) = \mathrm{Tr}(\rho\,\cdot\,)$ for positive, invertible $\rho \in \mathrm{M}_n(\mathbb{C})$ satisfying $\mathrm{Tr}(\rho) = 1$. Equip $\mathrm{M}_n(\mathbb{C})$ with the inner product

$$\langle A, B \rangle = \mathrm{Tr}(B^* A)$$

and denote the resulting Hilbert space by $\mathfrak{H}$. We define a representation π and an antirepresentation π' of $\mathrm{M}_n(\mathbb{C})$ on the Hilbert space $\mathfrak{H}$ via left and right multiplication, respectively,

$$\pi(A)\, B = A\, B \quad \text{for } B \in \mathfrak{H},$$

$$\pi'(A)\,B = B\,A^* \quad \text{for } B \in \mathfrak{H}.$$

Evidently, $\pi'(\mathrm{M}_n) \subseteq \pi(\mathrm{M}_n)'$ and $\pi(\mathrm{M}_n) \subseteq \pi'(\mathrm{M}_n)'$, where the commutants are with respect to $\mathcal{B}(\mathfrak{H})$.

Proposition 2.7. *The equalities* $\pi'(\mathrm{M}_n) = \pi(\mathrm{M}_n)'$ *and* $\pi(\mathrm{M}_n) = \pi'(\mathrm{M}_n)'$ *hold.*

Proof. Let $X \in \pi(\mathrm{M}_n)'$ and let I denote the unit matrix in $\mathrm{M}_n(\mathbb{C}) = \mathfrak{H}$. Then

$$X\,A = X\,\pi(A)\,I = \pi(A)\,XI \quad \text{for } A \in \mathfrak{H}.$$

However, $XI \in \mathfrak{H}$ and so is of the form $XI = B$ for some $B \in \mathrm{M}_n(\mathbb{C})$. Hence, for any $A \in \mathrm{M}_n(\mathbb{C})$,

$$\begin{aligned} XA &= \pi(A)XI = \pi(A)B = \pi(A)\pi'(B^*)I \\ &= \pi'(B^*)\,\pi(A)I = \pi'(B^*)A. \end{aligned}$$

Therefore $X = \pi'(B^*)$ and so $\pi(\mathrm{M}_n)' \subseteq \pi'(\mathrm{M}_n)$ and the result follows. ∎

Let Ω be the element of the Hilbert space $\mathfrak{H}$ given by the matrix $\Omega = \rho^{1/2}$. (Recall that ρ is determined by ω via $\omega(A) = \mathrm{Tr}(\rho\,A)$ for $A \in \mathrm{M}_n(\mathbb{C})$.)

Theorem 2.8. Ω *is cyclic and separating for* $\pi(\mathrm{M}_n)$ *on* $\mathfrak{H}$. *Furthermore,*

$$\omega(A) = \langle\,\pi(A)\,\Omega,\,\Omega\,\rangle$$

for all $A \in \mathrm{M}_n$.

Proof. We have

$$\pi(A\,\rho^{-1/2})\,\Omega = A\,\rho^{-1/2}\rho^{1/2} = A$$

for any $A \in \mathrm{M}_n$, which shows that Ω is cyclic for $\pi(\mathrm{M}_n)$. Similarly, for any $A \in \mathrm{M}_n$,

$$\pi'(A^*\rho^{-1/2})\,\Omega = \rho^{1/2}(A^*\rho^{-1/2})^* = A$$

and so Ω is cyclic for $\pi'(\mathrm{M}_n) = \pi(\mathrm{M}_n)'$ and therefore is separating for $\pi(\mathrm{M}_n)'' = \pi(\mathrm{M}_n)$.

We can see this directly as follows. Suppose that $\pi(A)\Omega = 0$. Now, $\pi(A)\Omega = A\,\rho^{1/2}$ and so $A\,\rho^{1/2} = 0$ in $\mathfrak{H}$ ($= \mathrm{M}_n(\mathbb{C})$). Since ρ is invertible, we deduce that $A = 0$ and therefore Ω is indeed separating for $\pi(\mathrm{M}_n)$, as claimed.

Finally, we note that ρ is positive and so is self-adjoint, and therefore the construction of the inner product in $\mathfrak{H}$ gives

$$\omega(A) = \mathrm{Tr}(\rho\,A) = \mathrm{Tr}((\rho^{1/2})^*A\,\rho^{1/2}) = \langle\,A\,\rho^{1/2},\,\rho^{1/2}\rangle = \langle\,\pi(A)\,\Omega,\,\Omega\rangle\,,$$

as required. ∎

As a consequence of this, we see that $(\mathfrak{H}, \pi, \Omega)$ is the GNS representation of $\mathrm{M}_n(\mathbb{C})$ determined by the state ω.

Define the (conjugate-linear) operator S on the Hilbert space $\mathfrak{H}$ by

$$S\,\pi(A)\,\Omega = \pi(A)^*\,\Omega\,.$$

Then, for any $B \in \mathfrak{H}$ ($= \mathrm{M}_n(\mathbb{C})$),

$$\begin{aligned} S\,B &= S\,B\rho^{-1/2}\rho^{1/2} \\ &= S\,\pi(B\rho^{-1/2})\,\Omega \\ &= \pi(B\rho^{-1/2})^*\,\Omega \\ &= \rho^{-1/2}B^*\rho^{1/2} \\ &= (\rho^{1/2}B\,\rho^{-1/2})^* \\ &= J\,\pi(\rho^{1/2})\,\pi'(\rho^{-1/2})\,B \end{aligned}$$

where J is the map on $\mathfrak{H}$ given by $J : A \mapsto A^*$ for $A \in \mathfrak{H}$.

Clearly $J^2 = \mathbb{1}$ and we calculate

$$\langle\, J\,A,\, J\,B\,\rangle = \langle\, A^*,\;\, B^*\,\rangle = \mathrm{Tr}(B\,A^*) = \mathrm{Tr}(A^*B) = \langle\, B,\, A\,\rangle$$

which implies that J is antiunitary. Set $\Delta = \pi(\rho)\,\pi'(\rho^{-1})$. For any $A \in \mathfrak{H}$,

$$\langle\,\Delta\,A,\, A\,\rangle = \mathrm{Tr}(\rho\,A\,\rho^{-1}A^*) = \mathrm{Tr}((\rho^{1/2}A\,\rho^{-1/2})(\rho^{1/2}A\,\rho^{-1/2})^*) \ge 0$$

which means that $\Delta \ge 0$. Similarly, we find that $\pi(\rho^{1/2})\,\pi'(\rho^{-1/2}) \ge 0$ and so $\Delta^{1/2} = \pi(\rho^{1/2})\,\pi'(\rho^{-1/2})$. We conclude that $S = J\,\Delta^{1/2}$ and that this is the polar decomposition of S.

Notice, further, that for $A \in \mathrm{M}_n$ and $B \in \mathfrak{H}$,

$$\begin{aligned} J\,\pi(A)\,J\,B &= J\,\pi(A)\,B^* \\ &= J\,AB^* \\ &= B\,A^* \\ &= \pi'(A)\,B \end{aligned}$$

and so $J\,\pi(\mathrm{M}_n)\,J = \pi'(\mathrm{M}_n)$.

Finally, we observe that for $A \in \mathrm{M}_n(\mathbb{C})$ and $B \in \mathfrak{H}$,

$$\begin{aligned} \Delta^{it}\pi(A)\,\Delta^{-it}B &= \rho^{it}A\,\rho^{-it}\,B\,\rho^{it}\rho^{-it} \\ &= \rho^{it}A\,\rho^{-it}\,B \\ &= \pi(\rho^{it}A\,\rho^{-it})\,B\,. \end{aligned}$$

Therefore $\Delta^{it}\pi(A)\,\Delta^{-it} = \pi(\rho^{it}A\,\rho^{-it})$ and this defines an automorphism group on $\pi(\mathrm{M}_n)$ and a corresponding automorphism group on M_n.

The general features illustrated in the above finite-dimensional case remain true in the general case.

Theorem 2.9. *Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω, and let J and Δ be the corresponding modular constructs. Then $J\,\mathcal{M}\,J = \mathcal{M}'$ and $\Delta^{it}\mathcal{M}\,\Delta^{-it} = \mathcal{M}$ for all $t \in \mathbb{R}$.*

Moreover, the modular automorphism group $\Delta^{it} \cdot \Delta^{-it}$ of $\mathcal{M}$ satisfies the modular condition (also called the KMS condition after Kubo, Martin, and Schwinger): namely, for each $x, y \in \mathcal{M}$ there is a function F continuous and bounded on the closed strip $\overline{D} = \{ z \in \mathbb{C} : 0 \le \operatorname{Im} z \le 1 \}$, analytic in the interior D and such that

$$F(t) = (\,\Delta^{it}\,x\,\Delta^{-it}y\,\Omega, \Omega\,) \quad \text{and} \quad F(t+i) = (\,y\,\Delta^{it}\,x\,\Delta^{-it}\Omega, \Omega\,)$$

for all $t \in \mathbb{R}$.

There is a non-spatial version of this.

Theorem 2.10. *Let ω be a faithful normal positive linear functional on the von Neumann algebra $\mathcal{M}$. There is a one-parameter group of automorphisms $\{\,\sigma_t : t \in \mathbb{R}\,\}$ of $\mathcal{M}$ satisfying the modular condition: for each $x, y \in \mathcal{M}$, there is a function F continuous and bounded on the closed strip $\overline{D}$, analytic in the interior D and such that*

$$F(t) = \omega(\sigma_t(x)\,y) \quad \text{and} \quad F(t+i) = \omega(y\,\sigma_t(x))$$

for all $t \in \mathbb{R}$.

Proof. Let $(\mathcal{H}, \pi, \Omega)$ be the cyclic (GNS) representation of $\mathcal{M}$ determined by ω. Since ω is faithful and normal, it can be shown that $\pi(\mathcal{M})$ is a von Neumann algebra and that π is an isomorphism from $\mathcal{M}$ onto $\pi(\mathcal{M})$. Let Δ be the modular operator associated with $\pi(\mathcal{M})$ and Ω and for $t \in \mathbb{R}$ define the automorphism σ_t on $\mathcal{M}$ by

$$\sigma_t(x) = \pi^{-1}(\Delta^{it}\,\pi(x)\,\Delta^{-it}).$$

The result now follows by applying the preceeding spatial version. ∎

Theorem 2.11. *The functional ω is invariant under each of the automorphisms σ_t it generates; i.e., $\omega(\sigma_t(x)) = \omega(x)$ for each $x \in \mathcal{M}$ and every $t \in \mathbb{R}$.*

Proof. Putting $y = \mathbb{1}$ in the modular condition implies that for each $x \in \mathcal{M}$ there is a function F, continuous and bounded on the closed strip $\overline{D}$, analytic in the interior D and such that

$$F(t) = \omega(\sigma_t(x)) = F(t+i).$$

Define the function $G(z)$ by setting $G(z) = F(z)$ for $z \in \overline{D}$ and $G(z) = G(z+i)$, $z \in \mathbb{C}$, i.e., G is obtained by extending F from $\overline{D}$ to $\mathbb{C}$ by periodicity. Note that G is well-defined because $F(t) = F(t+i)$ for each $t \in \mathbb{R}$.

The function G is continuous on $\mathbb{C}$ and is analytic in $\mathbb{C} \setminus (\mathbb{R} + i\mathbb{Z})$ and so is analytic in $\mathbb{C}$. (One can show (using the uniform continuity of G on bounded sets and Cauchy's theorem) that the contour integral of G around any triangle must vanish and so Morera's theorem tells us that G is, in fact, entire.) However, F is bounded on $\overline{D}$ and so G is bounded on $\mathbb{C}$ and therefore, by Liouville's theorem, G is constant on $\mathbb{C}$. In particular, for $t \in \mathbb{R}$, $F(t) = G(t) = G(0) = F(0)$ which gives

$$\omega(\sigma_t(x)) = \omega(x)$$

as required. ∎

Theorem 2.12. *The automorphism group $\{ \sigma_t \}$ is uniquely determined by the modular condition, that is, if $\{ \alpha_t : t \in \mathbb{R} \}$ is a one-parameter group of automorphisms of $\mathcal{M}$ satisfying the modular condition (with respect to ω), then $\alpha_t = \sigma_t$ for all $t \in \mathbb{R}$.*

Proof. As above, the modular condition implies invariance, $\omega(\alpha_t(x)) = \omega(x)$ for each $x \in \mathcal{M}$ and all $t \in \mathbb{R}$. This invariance, together with the continuity of the map $t \mapsto \omega(\alpha_t(x)\, y)$ for given $x, y \in \mathcal{M}$, implies that

$$U_t\, \pi(x)\, \Omega = \pi(\alpha_t(x))\, \Omega$$

defines a strongly continuous unitary group $\{ U_t : t \in \mathbb{R} \}$ of operators on $\mathcal{H}$, where $(\mathcal{H}, \pi, \Omega)$ is the cyclic representation of $\mathcal{M}$ determined by ω, such that $U_t \Omega = \Omega$ for all $t \in \mathbb{R}$.

U_t can be written as $U_t = e^{itK}$ where K is a (possibly unbounded) self-adjoint operator on $\mathcal{H}$. We shall show that $\Delta = e^K$ which implies that $\sigma_t = \alpha_t$ for all $t \in \mathbb{R}$.

Let $\zeta \in D(S)$, the domain of S, and let (x_n) be a sequence in $\mathcal{M}$ such that $\pi(x_n)\, \Omega \to \zeta$ and $S\, \pi(x_n)\, \Omega = \pi(x_n^*)\, \Omega \to \eta = S\zeta$. This is possible since $S = (S \restriction \pi(\mathcal{M})\, \Omega)^{**}$. Then $U_t\, \pi(x_n)\, \Omega \to U_t\, \zeta$. Also

$$\begin{aligned} U_t\, S\, \pi(x_n)\, \Omega &= U_t\, \pi(x_n^*)\, \Omega \\ &= \pi(\alpha_t(x_n)^*)\, \Omega \\ &= S\, \pi(\alpha_t(x_n))\, \Omega \\ &= S\, U_t\, \pi(x_n)\, \Omega\,. \end{aligned}$$

The left hand side converges to $U_t\, \eta = U_t\, S\, \zeta$ and we deduce that $U_t\, \zeta \in D(S)$ and that $S\, U_t\, \zeta = U_t\, S\, \zeta$. In particular, S and U_t commute. It follows that

$$S = U_t\, S\, U_t^* = U_t\, J\, U_t^*\, U_t \Delta^{1/2} U_t^*$$

and therefore $J = U_t\, J\, U_t^*$ and $\Delta = U_t\, \Delta\, U_t^*$ by the uniqueness of the polar decomposition of S.

Now let ξ and $\zeta \in D(S)$ and let (x_n) and (y_n) be sequences in $\mathcal{M}$ such that $\pi(x_n)\,\Omega \to \xi$, $S\,\pi(x_n)\,\Omega = \pi(x_n^*)\,\Omega \to S\,\xi$, $\pi(y_n)\,\Omega \to \zeta$ and $S\,\pi(y_n)\,\Omega = \pi(y_n^*)\,\Omega \to S\,\zeta$. For each n, let F_n be the function (as given by modular condition) which is analytic in the open strip D, bounded and continuous on the closed strip $\overline{D}$ and such that

$$F_n(t) = \omega(\alpha_t(x_n^*)\,y_n) = (\,\pi(y_n)\,\Omega, U_t\,\pi(x_n)\,\Omega\,)$$

and

$$F_n(t+i) = \omega(y_n\,\alpha_t(x_n^*)) = (\,U_t\,\pi(x_n^*)\,\Omega, \pi(y_n^*)\,\Omega\,).$$

Because of the convergence of the various sequences, we see that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n, m > N$, we have

$$|F_n(t) - F_m(t)| < \varepsilon \quad \text{and} \quad |F_n(t+i) - F_m(t+i)| < \varepsilon\,.$$

By the three-lines lemma, we deduce that

$$|F_n(z) - F_m(z)| < \varepsilon$$

for all $n, m > N$, uniformly on D.

Hence F_n converges uniformly on D to an analytic function F which is also bounded and continuous on $\overline{D}$. Furthermore, we find that

$$F(t) = \lim_{n\to\infty} F_n(t) = (\,\zeta, U_t\,\xi\,)$$

and

$$\begin{aligned} F(t+i) &= \lim_{n\to\infty} F_n(t+i) \\ &= (\,U_t\,S\,\xi, S\,\zeta\,) \\ &= (\,S\,U_t\,\xi, S\,\zeta\,) \\ &= (\,J\,\Delta^{1/2}U_t\,\xi, J\,\Delta^{1/2}\zeta\,) \\ &= (\,\Delta^{1/2}\zeta, \Delta^{1/2}U_t\,\xi\,). \end{aligned}$$

Now, U_t and Δ commute and so the operator $e^K + \Delta$ on $D(e^{)} \cap D(\Delta)$ is self-adjoint and positive. Let $e^K + \Delta = \int \lambda\,dp_\lambda$ be its spectral resolution. Then $\bigcup_n p_n\,\mathcal{H}$ is dense in $\mathcal{H}$ and both $p_n\,e^K p_n$ and $p_n\,\Delta\,p_n$ are bounded operators. Let $\xi, \zeta \in p_n\mathcal{H} \subseteq D(S)$ and for $z \in \mathbb{C}$, put

$$G(z) = (\,e^{-izK}\zeta, \xi\,).$$

Thus defined, G is entire and

$$G(t) = (\,e^{-itK}\zeta, \xi\,) = (\,\zeta, U_t\xi\,) = F(t)$$

for all $t \in \mathbb{R}$. It follows that $G = F$ on the strip D and therefore also $G(t+i) = F(t+i)$ for all $t \in \mathbb{R}$. (The function $G - F$ is analytic in D, continuous on $D \cup \{\operatorname{Im} z = 0\}$ and zero on $\{\operatorname{Im} z = 0\}$. In particular, $G - F$ is real on $\{\operatorname{Im} z = 0\}$ and so can be extended, by Schwarz's reflection principle, to a function analytic on the set $D \cup \{\operatorname{Im} z = 0\} \cup D^*$. But then this extension must be zero because it vanishes on $\{\operatorname{Im} z = 0\}$. Therefore $G = F$ on D.) Thus we have

$$(e^K e^{-itK} \zeta, \xi) = G(t+i) = F(t+i) = (\Delta^{1/2}\zeta, \Delta^{1/2} U_t \xi)$$

that is,

$$(e^K \zeta, U_t \xi) = (\Delta \zeta, U_t \xi).$$

In particular, setting $t = 0$ and putting $\xi = (e^K - \Delta)\zeta$ for any given $\zeta \in p_n \mathcal{H}$, we get

$$\|(e^K - \Delta)\zeta\|^2 = 0.$$

By the spectral theorem, we can write this as

$$\int_{E_n} \left|(e^K - \Delta)(q)\right|^2 \, |f(q)|^2 \, d\mu(q) = 0$$

for all n, where $p_n \equiv \chi_{E_n}$ and $\zeta \equiv \chi_{E_n} f$, with $f \in L^2(Q, d\mu)$. Since $E_n \uparrow Q$, it follows that $(e^K - \Delta)(q) = 0$ μ-a.e. and so $e^K = \Delta$ which completes the proof. ∎

Remark 2.13. From this, it follows that the group $\{\sigma_t\}$ depends only on ω and not on any particular vector realization of ω.

Example 2.14. Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and let ρ be any positive trace class operator on $\mathcal{H}$. Set $\omega(x) = \operatorname{Tr}(\rho x)$ for $x \in \mathcal{M}$. Then the modular automorphism group is given by

$$\sigma_t(x) = \rho^{it} x \rho^{-it}$$

for $x \in \mathcal{M}$ and $t \in \mathbb{R}$. This follows because σ_t, as defined above, satisfies the modular condition.

Corollary 2.15. *If $\mathcal{N} \subseteq \mathcal{M}$ is a sub-von Neumann algebra globally invariant under σ_t^ω, for $t \in \mathbb{R}$, then $\sigma_t^\omega \restriction \mathcal{N} = \sigma_t^{\omega \restriction \mathcal{N}}$. In other words, the modular automorphism group of $\mathcal{N}$ determined by ω is just the restriction to $\mathcal{N}$ of the modular automorphism group of $\mathcal{M}$ determined by ω.*

This follows because $\sigma_t \restriction \mathcal{N}$ satisfies the modular condition for ω on $\mathcal{N}$.

Corollary 2.16. *If ω is a finite (normal) trace on $\mathcal{M}$, then $\sigma_t^\omega = id \in \operatorname{Aut}(\mathcal{M})$ for all $t \in \mathbb{R}$.*

Proof. If ω is a trace, then, for any $x, y \in \mathcal{M}$,

$$\omega(x y) = \omega(y x).$$

In other words, id satisfies the modular condition and so, by uniqueness, the modular automorphism group is $\sigma_t = \text{id}$ for all $t \in \mathbb{R}$. ∎

Theorem 2.17. *Let ω be a faithful normal positive functional on $\mathcal{M}$ and let $x \in \mathcal{M}$. Then the following are equivalent:*

(i) $\omega(xy) = \omega(yx)$ *for all* $y \in \mathcal{M}$,

(ii) $\sigma_t(x) = x$ *for all* $t \in \mathbb{R}$.

Proof. To show that (i) $\implies$ (ii), note first that for any $y \in \mathcal{M}$ and $t \in \mathbb{R}$,

$$\begin{aligned}\omega(\sigma_t^\omega(x)\, y) &= \omega(x\, \sigma_{-t}^\omega(y)), \quad \text{since } \omega \circ \sigma_s^\omega = \omega \text{ for } s \in \mathbb{R},\\ &= \omega(\sigma_{-t}(y)\, x), \quad \text{by (i)},\\ &= \omega(y\, \sigma_t^\omega(x)).\end{aligned}$$

By the modular condition, we have $F(t) = F(t+i)$ and therefore F is bounded and so is constant on $\mathbb{C}$. Hence,

$$\omega(\sigma_t^\omega(x)\, y) = F(t) = F(0) = \omega(x\, y).$$

Putting $y = (\sigma_t^\omega(x) - x)^*$, we find that $\sigma_t^\omega(x) = x$ because ω is faithful.

To show that (ii) $\implies$ (i), we note that the invariance $\sigma_t^\omega(x) = x$ implies that $F(t) = \omega(\sigma_t^\omega(x)\, y)$ is constant and so $F(t) = F(t+i)$ is constant. Hence

$$F(t) = \omega(\sigma_t^\omega(x)\, y) = \omega(y\, \sigma_t^\omega(x))$$

for all $t \in \mathbb{R}$. Setting $t = 0$ gives $\omega(x\, y) = \omega(y\, x)$. ∎

Theorem 2.18. *For any σ-finite von Neumann algebra $\mathcal{M}$, the following are equivalent.*

(i) $\mathcal{M}$ *is finite,*

(ii) *there is a faithful normal positive functional on $\mathcal{M}$ whose associated modular operator is bounded.*

Proof. (i) $\implies$ (ii). If $\mathcal{M}$ is finite and σ-finite then it possesses a faithful normal finite trace. The modular operator associated with a trace vector is the identity.

(ii) $\implies$ (i). Suppose that ω is a faithful normal positive functional with associated cyclic representation $(\mathcal{H}, \pi, \Omega)$ and such that Δ_Ω, the corresponding modular operator, is bounded. Then $S = J\, \Delta_\Omega^{1/2}$ is also bounded.

Let $(\pi(x_n))$ be a sequence in the closed unit ball of $\pi(\mathcal{M})$ such that $\pi(x_n) \to \pi(y)$ strongly. Then, in particular, $\pi(x_n)\, \Omega \to \pi(y)\, \Omega$ and so $\pi(x_n)^* \Omega = S\, \pi(x_n)\, \Omega \to S\, \pi(y)\, \Omega = \pi(y)^* \Omega$.

Hence, for any $z \in \pi(\mathcal{M})'$,

$$\pi(x_n^*)\, z\, \Omega = z\, \pi(x_n^*)\, \Omega \to z\, \pi(y^*)\, \Omega = \pi(y^*)\, z\, \Omega.$$

It follows that $\pi(x_n^*) \to \pi(y^*)$ strongly. But the strong continuity of the * map on the unit ball of $\pi(\mathcal{M})$ means that $\pi(\mathcal{M})$ is finite and so the same is true of $\mathcal{M}$. ∎

Chapter 3

The canonical commutation relations

The Heisenberg relation

Probably one of the most famous aspects of quantum mechanics is the commutation relation of Heisenberg, namely, the relation

$$[Q, P] \equiv QP - PQ = i\hbar$$

where Q is (an operator representing) the position and P the momentum of a single particle moving in one-dimension and $\hbar = h/2\pi$, where h is Planck's constant. For a single particle in, say, 3-dimensions, there are 3 position components Q_1, Q_2, Q_3 and 3 momentum components P_1, P_2, P_3 with the relations

$$[Q_k, P_\ell] = i\hbar\, \delta_{k\ell}\,, \quad [Q_k, Q_\ell] = [P_k, P_\ell] = 0\,.$$

for $1 \le k, \ell \le 3$.

One is naturally led to ask for the mathematical implications of such relations. According to their physical interpretation, it is customary to require that P and Q both be self-adjoint operators on some suitable Hilbert space. To specify an operator, one must specify both its action and also its domain of definition. For (densely-defined) bounded operators this causes no special problem since such operators always have a unique natural extension (defined by continuity) to the whole Hilbert space. In this case, the products PQ and QP are well-defined bounded operators on the whole Hilbert space. We shall see that things are necessarily not that simple. We shall show that there are no such bounded operators P and Q satisfying the relation above. (We will will not even suppose that they are self-adjoint.) (In fact, the impossibility of such a relation holds in a purely algebraic context.) Suppose the contrary, namely, P and Q are bounded operators satisfying

$$[Q, P] = i$$

(from now on we set $\hbar = 1$). By induction, we see that

$$[Q^n, P] = i\, nQ^{n+1}$$

for $n \in \mathbb{N}$. Indeed, this is true for $n = 1$, and for any $m \in \mathbb{N}$,

$$\begin{aligned}[Q^{m+1}, P] &= Q^{m+1}P - PQ^{m+1} \\ &= Q(Q^m P - PQ^m) + (QP - PQ)Q^m \\ &= Q\, im\, Q^{m-1} + [Q, P]Q^m\,, \quad \text{by induction hypothesis,} \\ &= i(m+1)Q^m\,,\end{aligned}$$

as required. But then we find that

$$\begin{aligned}\|in\, Q^{n-1}\| &= n\, \|Q^{n-1}\| \\ &= \|[Q^n, P]\| = \|Q^n P - PQ^n\| \\ &\le 2\, \|P\|\, \|Q^n\| \\ &\le 2\, \|P\|\, \|Q\|\, \|Q^{n-1}\|\end{aligned}$$

which implies that $n \le 2\,\|P\|\,\|Q\|$ for any $n \in \mathbb{N}$. This clearly cannot happen. We have proved the following theorem.

Theorem 3.1. *The relation $[Q, P] = i$ has no solution in $\mathcal{B}(\mathcal{H})$, the set of bounded operators on a Hilbert space $\mathcal{H}$.*

We are "forced" to consider unbounded operators and their very delicate domain issues. Note that this is not mere mathematical fancy — the Heisenberg relations, a pillar in the realm of quantum mechanics, are simply never valid in the world of bounded operators. One can either abandon these relations altogether, or else look for realizations somewhere other than in terms of bounded operators. To stress that the domain of definition of an (unbounded) operator is an integral part of its specification, we will include it in the notation; thus $(A, D(A))$ denotes an operator A with domain $D(A)$. In most cases, $D(A)$ will be a dense linear subset of some given Hilbert space $\mathcal{H}$.

Definition 3.2. We say that self-adjoint operators $(Q, D(Q))$ and $(P, D(P))$ satisfy the Heisenberg commutation relation if there is a dense domain D with $D \subseteq D(Q) \cap D(P)$ such that $Q\,D \subseteq D(P)$, $P\,D \subseteq D(Q)$ and such that, on D,

$$QP - PQ = i\,\mathbb{1}\,.$$

This is sometimes written as $QP - PQ \subseteq i$.

A similar looking but technically different formulation is the following.

Definition 3.3. We say that symmetric operators $(Q, D(Q))$ and $(P, D(P))$ satisfy the weak Heisenberg relation if there is a dense domain D such that $D \subseteq D(Q) \cap D(P)$ and

$$\langle\, Qf, Pg\,\rangle - \langle\, Pf, Qg\,\rangle = i\,\langle\, f, g\,\rangle$$

for all $f, g \in D$.

We have seen that we cannot satisfy these with bounded operators, so one may wonder whether there are any solutions in this apparently more general setting. That the answer is, indeed, yes, is seen by considering the so-called Schrödinger representation of position and momentum. This is where $\mathcal{H} = L^2(\mathbb{R}, dx)$ and $(Qf)(x) = x\,f(x)$ on the domain given by $D(Q) = \{\, f \in L^2(\mathbb{R}, dx) : \int |x\,f(x)|^2 \; dx < \infty \,\}$. The operator P is defined in terms of the Fourier transform, namely, by $\widehat{(Pg)}(p) = p\,\widehat{g}(p)$ on the domain $D(P) = \{\, g \in L^2(\mathbb{R}, dx) : \int |p\,\widehat{g}(p)|^2 \; dp < \infty \,\}$. In other words, P is unitarily equivalent to Q, via the Fourier transform on $L^2(\mathbb{R}, dx)$. For smooth g, say $g \in \mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$, we have that $Pg = -i\,\dfrac{dg}{dx}$, that is, P is the differential operator $P = -i\,\frac{d}{dx}$ defined on a suitable domain (via the Fourier transform).

Thus defined, $(Q, D(Q))$ and $(P, (D(P))$ are self-adjoint (unbounded) operators. The Heisenberg relations above (both formulations) hold if we take $D = \mathcal{S}(\mathbb{R})$, for example. Is this Schrödinger representation unique? The answer is no — in fact, we shall construct an uncountable number of different solutions.

Let $\mathcal{H} = L^2([0,1], dx)$ and set $(Qf)(x) = x\,f(x)$ for any $f \in \mathcal{H}$. Suppose that g is differentiable on $[0,1]$, $g(0) = 0 = g(1)$ and that $g' \in \mathcal{H}$. Set $(Pg)(x) = -i\,g'(x)$. We see that Q is bounded and it can be shown (Robinson (1971)) that P has an uncountable number of distinct self-adjoint extensions P_θ, for $0 \le \theta \le 2\pi$. These correspond to specifying boundary conditions of the form $g(1) = e^{i\theta} g(0)$. One can show that P_θ has a discrete spectrum with eigenvectors $e^{i(2\pi n+\theta)x}$. Take D to be the dense linear subset of $\mathcal{H}$ comprising those $\mathcal{C}^\infty$-functions on $[0,1]$ which vanish at the end-points. Then $P_\theta = P$ on D and one checks that the Heisenberg relation holds on D. We conclude that the Heisenberg relation does not uniquely specify the self-adjoint operators Q and P.

We remark, in passing, that from the above example we see how important it is to specify precisely the domain of an operator. In the case above, this amounted to specifying the boundary conditions. An observable, a physical entity, is to be represented in quantum mechanics by an operator whose spectrum is supposed to represent its possible values. The quantum mechanical momentum is usually thought of as the operator $-id/dx$. Consider this operator on $L^2([0,1], dx)$ without any boundary conditions. Then for any $z \in \mathbb{C}$, it has e^{izx} as an eigenvector with associated eigenvalue z. Its spectrum is the whole complex plane.

Definition 3.4. Let A be an operator and let $f \in D(A)$, the domain of A, with $\|f\| = 1$. The variance of A in f is the non-negative number

$$\mathbb{V}_f(A) \equiv \|(A - \langle\, f, Af\,\rangle)\, f\|^2.$$

If A is self-adjoint and $f \in D(A^2)$, then

$$\mathbb{V}_f(A) = \langle f, (A^2 - \langle f, Af \rangle^2) f \rangle.$$

If f is a normalised eigenvector of A, then $\mathbb{V}_f(A) = 0$ (and conversely, unless $Af = 0$).

Proposition 3.5. *Suppose that Q, P satisfy the weak Heisenberg relation on the dense set D. Then, for any $f \in D$ with $\|f\| = 1$,*

$$\mathbb{V}_f(Q)\,\mathbb{V}_f(P) \geq \tfrac{1}{4}\,.$$

Proof. Let $f \in D$ and suppose that $\|f\| = 1$. Then

$$\begin{aligned} i\,2\,\mathrm{Im}\langle Pf, Qf \rangle &= \langle Pf, Qf \rangle - \overline{\langle Pf, Qf \rangle} \\ &= \langle Pf, Qf \rangle - \langle Qf, Pf \rangle \\ &= -i\,\langle f, f \rangle \text{ by the weak Heisenberg relation.} \end{aligned}$$

Hence $-2\,\mathrm{Im}\langle Pf, Qf \rangle = \|f\|^2 = 1$ and therefore

$$\begin{aligned} 1 = 2\,|\mathrm{Im}\langle Pf, Qf \rangle| &\leq 2\,|\langle Pf, Qf \rangle| \\ &\leq 2\|Pf\|\,\|Qf\| \end{aligned}$$

i.e., $\|Pf\|^2\,\|Qf\|^2 \geq \frac{1}{4}$. Now, if P, Q satisfy the Heisenberg relation, so do $Q - \alpha$ and $P - \beta$ for any real α, β. Hence,

$$\|(Q - \alpha)f\|^2\,\|(P - \beta)\|^2 \geq \tfrac{1}{4}\,.$$

Taking $\alpha = \langle f, Qf \rangle$ and $\beta = \langle f, Pf \rangle$ gives

$$\mathbb{V}_f(Q)\,\mathbb{V}_f(P) \geq \tfrac{1}{4}$$

as required. ∎

Remark 3.6. This is the well-known uncertainly relation — the better Q is known the smaller $\mathbb{V}_f(Q)$ will be and so the larger $\mathbb{V}_f(P)$ will necessarily be and vice versa.

Returning to the example of Q and P_θ on $L^2([0,1], dx)$, let $f(x) = e^{i2\pi x}$. Then $\mathbb{V}_f(P_0) = 0$ and so $\mathbb{V}_f(Q)\,\mathbb{V}_f(P_0) = 0$ — not much uncertainty here. This appears to contradict the result above. However, $f(x) = e^{i2\pi x} \notin D$. Indeed, we cannot enlarge D to include this vector and retain the Heisenberg relation — otherwise we would obtain a contradiction to the uncertainty principle, above. It is simply false that $\langle Qf, P_0 f \rangle - \langle P_0 f, Qf \rangle = i\langle f, f \rangle$ for this particular f.

In general, if Q, P satisfy the weak Heisenberg relation on D, then D cannot contain any eigenvectors of Q or P should they possess any.

We have seen that, in general, the Schrödinger representation is not unique. However, under extra conditions, this does hold. We mention here two results, without proof.

Theorem 3.7 (Dixmier). *Let Q, P be closed symmetric operators on a separable Hilbert space. Then (Q, P) is unitarily equivalent to a direct sum of Schrödinger representations of the Heisenberg relation if and only if there is a dense domain $D \subseteq D(Q) \cap D(P)$ such that*

(i) $P\,D \subseteq D$ and $Q\,D \subseteq D$,

(ii) $(P^2 + Q^2) \restriction D$ *is essentially self-adjoint,*

(iii) $QP - PQ = i$ *on D.*

These conditions ensure that Q and P are self-adjoint and that $Q \restriction D$ and $P \restriction D$ are essentially self-adjoint. Condition (ii) amounts to the fact that the number operator for the harmonic oscillator is essentially self-adjoint.

Theorem 3.8 (Tillman). *Let Q and P be closed symmetric operators on a separable Hilbert space with $D(Q) \cap D(P)$ dense and such that*

(i) $\langle Qf, Pg \rangle - \langle Pf, Qg \rangle = i \langle f, g \rangle$ for $f, g \in D(Q) \cap D(P)$,

(ii) $(Q + iP)^* = (Q - iP)$.

Then Q, P are self-adjoint and equivalent to a direct sum of Schrödinger representations.

Condition (ii) says that the harmonic oscillator creation operator is the adjoint of the annihilation operator.

For the proofs of these two theorems, we refer to the original papers. Further discussion is given by Emch.

Von Neumann's Uniqueness Theorem

The previous section should have conveyed some sense of the subtleties involved with the Heisenberg relation and domain issues of unbounded operators. In an attempt to circumvent these, we recast the Heisenberg relation into a more convenient form involving bounded operators. To do this, we begin with a formal discussion. Ignoring domain problems, we see that the relation $[Q, P] = i$ implies that $[Q^n, P] = i\,nQ^{n-1}$ and so we are led to the relation

$$\Big[\frac{Q^n}{n!}, P\Big] = i\,\frac{Q^{n-1}}{(n-1)!}\,.$$

Summing over n gives $[\,e^{iaQ}, P\,] = -a\,e^{iaQ}$, that is, $P\,e^{iaQ} = e^{iaQ}\,(P + a)$. Hence $P^n\,e^{iaQ} = e^{iaQ}\,(P + a)^n$ and so $(ibP)^n\,e^{iaQ} = e^{iaQ}\,(ib\,(P + a))^n$. Thus

$e^{ibP}\, e^{iaQ} = e^{iaQ}\, e^{ib(P+a)} = e^{iaQ}\, e^{ibP}\, e^{iab}$ and the Heisenberg relation formally implies that

$$e^{ibP}\, e^{iaQ} = e^{iaQ}\, e^{ibP}\, e^{iab}\,.$$

On the other hand, by differentiating this latter (exponential) form with respect to a and b and then setting $a = b = 0$, we recover the original Heisenberg relation $QP - PQ = i$. The relation $e^{ibP}\, e^{iaQ} = e^{iaQ}\, e^{ibP}\, e^{iab}$ is called the Weyl relation. This is formally equivalent to the Heisenberg relation but is technically very convenient because it involves only bounded operators, in fact, unitary operators. This is formalized in the following definition.

Definition 3.9. A representation of the Weyl relation (for one degree of freedom) is a pair of maps $s \mapsto U(s)$, $t \mapsto V(t)$ from $\mathbb{R}$ into the unitary operators on a Hilbert space $\mathcal{H}$ such that

(i) $s \mapsto U(s)$ and $t \mapsto V(t)$ are strongly continuous representations of $\mathbb{R}$,

(ii) $U(s)\, V(t) = e^{-ist}\, V(t)\, U(s)$ for all $s, t \in \mathbb{R}$.

Remarks 3.10.

1. For n degrees of freedom, we would have maps $\underline{s} \mapsto U(\underline{s})$ and $\underline{t} \mapsto V(\underline{t})$ as representations of $\mathbb{R}^n$ and satisfying

$$U(\underline{s})\, V(\underline{t}) = e^{-i\, \underline{s}.\underline{t}}\, V(\underline{t})\, U(\underline{s})$$

for all $\underline{s}, \underline{t} \in \mathbb{R}^n$.

2. We have chosen $U(s)$ to correspond to e^{isQ} and $V(t)$ to e^{itP}. However, this convention is common but not universal.

3. The requirement that $U(s)$ and $V(t)$ be strongly continuous means that we can recover Q and P as their generators, by Stone's Theorem. For unitary operators, weak and strong continuity are equivalent.

Definition 3.11. A representation (U, V) of the Weyl relation is said to be irreducible if the only closed subspaces of $\mathcal{H}$ invariant under the $U(s)$'s and $V(t)$'s are the trivial subspaces $\{\,0\,\}$ and $\mathcal{H}$ itself.

Definition 3.12. The Schrödinger representation of the Weyl relation is that on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$ and for $s, t \in \mathbb{R}$ and for $f \in \mathcal{H}$ given by

$$(U(s)f)(x) = e^{isx} f(x) \text{ and } (V(t)f)(x) = f(x+t)\,.$$

Theorem 3.13 (von Neumann). *Any representation $(U, V, \mathcal{H})$ of the Weyl relation is equivalent to a direct sum of irreducible representations.*

Proof. Set $W(s,t) = e^{-\frac{1}{2}ist} V(t)\, U(s)$ for $s, t \in \mathbb{R}$. It is easy to check that

$$W(s,t)\, W(s',t') = e^{\frac{i}{2}(ts'-t's)}\, W(s+s', t+t')\,. \tag{$*$}$$

Putting $s = -s'$, $t = -t'$ in $(*)$ and using $W(0,0) = \mathbb{1}$, we find that

$$W(-s,-t) = W(s,t)^{-1} = W(s,t)^*$$

for $s, t \in \mathbb{R}$. The strong continuity of $U(s)$ and $V(t)$ implies that $W(s,t)$ is jointly strongly continuous in s and t. Thus, for any $\rho \in \mathcal{S}(\mathbb{R}^2)$, say, we can define

$$A_\rho = \int \rho(s,t)\, W(s,t)\, ds\, dt$$

as a strong Riemann integral. We choose $\rho(s,t) = e^{-\frac{1}{4}(s^2+t^2)}$ and write A for this A_ρ. Clearly $A = A^*$. We claim that A is not the zero operator. To show this, suppose the contrary, namely that $A = 0$. Then, for all $f, g \in \mathcal{H}$,

$$\begin{aligned}
0 &= \langle\, f, W(-s',-t')\, A\, W(s',t')\, g\,\rangle \\
&= \int e^{-\frac{1}{4}(s^2+t^2)}\, \langle\, f, W(-s',-t')\, W(s,t)\, W(s',t')\, g\,\rangle\, ds\, dt \\
&= \int e^{-\frac{1}{4}(s^2+t^2)}\, e^{i(ts'-t's)}\, \langle\, f, W(s,t) g\,\rangle\, ds\, dt \\
&= \int e^{-\frac{1}{4}(s^2+t^2)} e^{its'}\, e^{-it's}\, \langle\, f, W(s,t) g\,\rangle\,.
\end{aligned}$$

In other words, $F(-t', s') = 0$ for all s', t', where $F(a,b)$ is the Fourier transform of the function $e^{-\frac{1}{4}(s^2+t^2)}\, \langle\, f, W(s,t) g\,\rangle$. But the only L^2-function with zero Fourier transform is zero, so we must have that $\langle\, f, W(s,t) g\,\rangle = 0$ for all s, t. Since this holds for any $f, g \in \mathcal{H}$, we deduce that $W(s,t) = 0$, which is false. We conclude that $A \neq 0$, as claimed.

A calculation with Gaussian integrals gives

$$A\, W(s,t) A = 2\pi\, A\, e^{-\frac{1}{4}(s^2+t^2)}\,. \tag{$**$}$$

Setting $s = t = 0$, we get $A^2 = 2\pi A$. This implies that $E = \frac{1}{2\pi} A$ is a projection. Let $\mathcal{M} = \operatorname{ran} E$, the range of the projection E. Then $A \neq 0$ means that $E \neq 0$ and so $\mathcal{M} \neq \{\, 0\,\}$. Let $\{\, f_\alpha\,\}$ be an orthonormal basis of $\mathcal{M}$ and let $\mathcal{H}_\alpha$ be the closed subspace of $\mathcal{H}$ spanned by vectors of the form $W(s,t) f_\alpha$, $s, t \in \mathbb{R}$ (i.e., $\mathcal{H}_\alpha$ is the cyclic subspace of $\mathcal{H}$ generated by the set $\{\, W(s,t) : s, t \in \mathbb{R}\,\}$). For any $\alpha \neq \beta$,

$$\begin{aligned}
\langle\, W(s,t) f_\alpha, W(s',t') f_\beta\,\rangle &= \langle\, W(s,t)\, E f_\alpha, W(s',t')\, E f_\beta\,\rangle \\
&= \langle\, f_\alpha, E\, W(-s,-t)\, W(s',t')\, E f_\beta\,\rangle \\
&= c\, \langle\, f_\alpha, E\, W(s'',t'')\, E f_\beta\,\rangle \quad \text{for some constant } c,
\end{aligned}$$

$$= c' \langle f_\alpha, E f_\beta \rangle$$

for some constant c' since $E\,W\,E = (\text{constant})\,E$

$$= c' \langle f_\alpha, f_\beta \rangle$$
$$= 0.$$

It follows that $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ are orthogonal. We claim that $\bigoplus_\alpha = \mathcal{H}$. To see this, suppose not and that $\mathcal{H}'$ is the orthogonal complement of $\bigoplus_\alpha \mathcal{H}_\alpha$ in $\mathcal{H}$. Since $W : \mathcal{H}\alpha \to \mathcal{H}_\alpha$, it follows that $W : \mathcal{H}' \to \mathcal{H}'$. Let $W' = W \restriction \mathcal{H}'$. Then as above, we can define a projection E' in terms of W' to conclude that $E' \neq 0$. Let $f' \in \operatorname{ran} E'$. Since W' is a restriction of W, we see that E' is a restriction of E and so $Ef' = f'$. This contradicts the fact that $f' \in \mathcal{H}'$ which is orthogonal to the subspace $\bigoplus_\alpha \mathcal{H}_\alpha$ which contains $\mathcal{M}$. This contradiction means that $\bigoplus_\alpha = \mathcal{H}$, as claimed.

Finally, we note that since $W \restriction \mathcal{H}_\alpha$ gives rise to a representation of the Weyl relation on $\mathcal{H}_\alpha$, the proof is complete if we can show that $W \restriction \mathcal{H}_\alpha$ acts irreducibly on $\mathcal{H}_\alpha$. Let $W_\alpha = W \restriction \mathcal{H}_\alpha$. We will show that any $T \in \mathcal{B}(\mathcal{H}_\alpha)$ which commutes with every $W_\alpha(s,t)$ is a multiple of the identity operator. But if T commutes with $W_\alpha(s,t)$, for each $s,t \in \mathbb{R}$, then it also commutes with E_α, the restriction of E constructed via the W_αs. Hence Tf_α satisfies $E_\alpha T f_\alpha = T\,E f_\alpha = T f_\alpha$. But E_α has a one-dimensional range and so it follows that $T f_\alpha = \lambda f_\alpha$ for some $\lambda \in \mathbb{C}$. However, $\mathcal{H}_\alpha$ is generated by $W_\alpha(s,t) f_\alpha$ for $s,t \in \mathbb{R}$ and T commutes with each $W_\alpha(s,t)$. We conclude that $T = \lambda \mathbb{1}_\alpha$ and the proof is complete. ■

The equivalence of irreducible representations is a consequence of this result, as we see next.

Corollary 3.14. *Let $(U, V, \mathcal{H})$ and $(U', V', \mathcal{H}')$ be irreducible representations of the Weyl relation. Then they are unitarily equivalent.*

Proof. As in the proof of the theorem, we construct E and $\mathcal{M} = \operatorname{ran} E$ from the U and V on $\mathcal{H}$. By irreducibility, we conclude that $\mathcal{M}$ is one-dimensional and, if f is a normalized vector in $\mathcal{M}$, then $\mathcal{H}$ is spanned by vectors of the form $W(s,t)f$ with $s,t \in \mathbb{R}$. Similarly, one constructs $\mathcal{M}'$, E' and f' so that $\mathcal{H}'$ is spanned by the corresponding vectors $W'(s,t)f'$. For $a_1, \dots, a_N \in \mathbb{C}$ and $g = \sum_{i=1}^N a_i\, W(s_i, t_i) f \in \mathcal{H}$, set

$$Ig = \sum_{i=1}^N a_i\, W'(s_i, t_i) f'.$$

Then

$$\|Ig\|^2 = \sum_{i,j} \overline{a_i}\, a_j \langle W'(s_i, t_i) f', W'(s_j, t_j) f' \rangle$$

$$
\begin{aligned}
&= \sum_{i,j} \overline{a_i}\, a_j \,\langle\, f', E'W'(-s_i,-t_i)\, W'(s_j,t_j)\, E'f' \,\rangle \\
&= \sum_{i,j} \overline{a_i}\, a_j\, c_{ij} \,\langle\, f', f' \,\rangle
\end{aligned}
$$

where, by $(**)$ above, c_{ij} depends only on s_i, t_i, s_j, t_j,

$$
\begin{aligned}
&= \sum_{i,j} \overline{a_i}\, a_j\, c_{ij} \\
&= \|g\|^2 .
\end{aligned}
$$

Therefore $I : \mathcal{H} \to \mathcal{H}'$ is isometric with dense domain and dense range and so extends to a unitary operator from $\mathcal{H}$ onto $\mathcal{H}'$. Moreover,

$$
\begin{aligned}
I\, W(s,t) g &= W'(s,t) \sum_i a_i\, W'(s_i,t_i) f' \\
&= W'(s,t)\, I g \,,
\end{aligned}
$$

that is, $W'(s,t) = I\, W(s,t) I^{-1}$ and so $(U, V, \mathcal{H})$ and $(U', V', \mathcal{H}')$ are unitarily equivalent. ■

Theorem 3.15 (von Neumann Uniqueness Theorem). *Any representation of the Weyl relation is unitarily equivalent to a direct sum of copies of the Schrödinger representation.*

Proof. By the preceding discussion, we need only show that the Schrödinger representation is irreducible. As before, this follows if we can show that $\mathcal{M} = \operatorname{ran} E$ is one-dimensional, where E is the projection constructed as above but now via the Schrödinger operators $U(s)$ and $V(t)$.

For any $g \in L^2(\mathbb{R}, dx)$, by definition of the Schrödinger operators, we have $(W(s,t)g)(x) = e^{is(x+\frac{1}{2}t)} g(x+t)$. Thus (with notation as before),

$$
\begin{aligned}
(Ag)(x) &= \int e^{-\frac{1}{4}(s^2+t^2)}\, (W(s,t)g)(x)\, ds\, dt \\
&= \int e^{-\frac{1}{4}(s^2+t^2)}\, e^{is(x+\frac{1}{2}t)}\, g(x+t)\, ds\, dt \\
&= \sqrt{4\pi} \int e^{-\frac{1}{2}(x^2+t^2)}\, g(t)\, dt \\
&= e^{-\frac{1}{2}x^2} \sqrt{4\pi} \int e^{-\frac{1}{2}t^2}\, g(t)\, dt.
\end{aligned}
$$

That is,

$$
(Eg)(x) = \frac{e^{-\frac{1}{2}x^2}}{\pi^{1/4}} \int \frac{e^{-\frac{1}{2}t^2}}{\pi^{1/4}}\, g(t)\, dt \,.
$$

This means that E is the projection onto the vector $\pi^{-1/4}\, e^{-\frac{1}{2}x^2}$ in $L^2(\mathbb{R}, dx)$. In particular, the range of E is one-dimensional, as required. ■

Remark 3.16. The projection E above is the projection onto the vacuum or "no mode" state of the harmonic oscillator.

Since the Hilbert space $L^2(\mathbb{R}, dx)$ is separable, we have the following corollary.

Corollary 3.17. *An irreducible representation of the Weyl relation necessarily acts in a separable Hilbert space.*

Example 3.18. Let $\mathcal{H}$ be the Hilbert space built from those functions on $\mathbb{R}$ belonging to

$$\{ f(x),\ x \in \mathbb{R} : \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} |f(x)|^2\, dx < \infty \}$$

equipped with the inner product

$$\langle g, f \rangle = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} \overline{g(x)}\, f(x)\, dx\,.$$

We define unitary operators $U(s)$ and $V(t)$ on $\mathcal{H}$ by

$$U(s)f(x) = e^{isx} f(x), \quad \text{for } f \in \mathcal{H}$$
$$V(t)g(x) = g(x+t), \quad \text{for } g \in \mathcal{H}.$$

One checks that U and V obey the Weyl relation $U(s)\,V(t) = e^{-ist}V(t)\,U(s)$. What are the generators Q and P here? They do not exist. Indeed, the maps $s \mapsto U(s)$ and $t \mapsto V(t)$ are not (strongly) continuous. To show this, let $f(x) = e^{ix} \in \mathcal{H}$. Then one finds that $\langle f, U(s)f \rangle = 0$ for all $s \neq 0$, so that $\langle f, (U(s) - \mathbb{1})f \rangle = 1$ for all $s \neq 0$ and we see that $U(s)$ is not weakly (and hence not strongly) continuous. Similarly, with $g(x) = e^{ix^2}$, one finds again that $\langle g, V(t)g \rangle = 0$ for all $t \neq 0$ and therefore $V(t)$ is not weakly continuous.

This means that $U(s)$ and $V(t)$ do not possess generators and so are not of the form e^{isQ} and e^{itP} for self-adjoint operators Q and P. Notice that the collection $\{ e^{i\lambda x} : \lambda \in \mathbb{R} \}$ is an orthonormal family in the Hilbert space $\mathcal{H}$, showing that $\mathcal{H}$ is a non-separable Hilbert space.

Returning to now general considerations, we have seen that, in contrast to the Heisenberg relation, the Weyl relation has an essentially unique solution — namely, the Schrödinger representation. These results can be reformulated for any finite number of degrees of freedom. The only difference will be the appearance of n-dimensional Gaussian integrals.

For n degrees of freedom, one has operators $U(\underline{\alpha})$ and $V(\underline{\beta})$, for $\underline{\alpha}$ and $\underline{\beta} \in \mathbb{R}^n$, such that

$$U(\underline{\alpha})\,V(\underline{\beta}) = e^{-i\,\langle \underline{\alpha}, \underline{\beta} \rangle} V(\underline{\beta})\,U(\underline{\alpha})\,.$$

$U(\underline{\alpha})$ and $V(\underline{\beta})$ represent $e^{i(\alpha_1 Q_1+\cdots+\alpha_n Q_n)}$ and $e^{i(\beta_1 P_1+\cdots+\beta_n P_n)}$, respectively. Replacing $\underline{\alpha}$ by $s\underline{\alpha}$ and $\underline{\beta}$ by $t\underline{\beta}$ for $s, t \in \mathbb{R}$, we get

$$U(s\underline{\alpha})\,V(t\underline{\beta}) = e^{-ist\,\langle\,\underline{\alpha},\underline{\beta}\,\rangle}\,V(t\underline{\beta})\,U(s\underline{\alpha})\,.$$

The term $\langle\,\underline{\alpha},\underline{\beta}\,\rangle$ is the inner product (a pairing) between the vectors $\underline{\alpha}$ and $\underline{\beta}$. This suggests a generalization of the Weyl relations over any real inner product space. We will not consider this in such generality but will just consider representations over $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$, the Schwartz space of smooth, rapidly decreasing real-valued functions on $\mathbb{R}^n$.

Definition 3.19. A representation of the canonical commutation relations (CCR) in Weyl form, over $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$, is a pair of maps $f \mapsto U(f)$ and $g \mapsto V(g)$ from $\mathcal{S}_{\mathbb{R}}$ into the unitary operators on a Hilbert space $\mathcal{H}$ such that

(i) $U(f_1)\,U(f_2) = U(f_1 + f_2)$ and $V(g_1)\,V(g_2) = V(g_1 + g_2)$,

(ii) the maps $s \mapsto U(sf)$ and $t \mapsto V(tg)$ are strongly continuous for fixed $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$,

(iii) for any $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$, the canonical commutation relations

$$U(f)\,V(g) = e^{-i\langle\,f,g\,\rangle}\,V(g)\,U(f)$$

hold, where $\langle\,f,g\,\rangle = \int f(x)\,g(x)\,d^n x$.

The continuity requirement allows us to recover the generators of $U(sf)$ and $V(tg)$ which will satisfy the Heisenberg relations on a suitable domain.

Example 3.20. Let $\mathcal{F}$ be the (boson) Fock space over $L^2(\mathbb{R}^n)$ and let $\Phi(f)$ and $\Pi(g)$ be the operators

$$\Phi(f) = \frac{1}{\sqrt{2}}\,(a^*(f) + a(f)), \quad \Pi(f) = \frac{1}{\sqrt{2}}\,(a^*(g) - a(g)),$$

which are essentially self-adjoint on D_0, the set of finite-particle vectors in the Fock space $\mathcal{F}$. Then $U(f) = \exp\, i\,\overline{\Phi(f)}$ and $V(g) = \exp i\,\overline{\Pi(g)}$ define a representation of the CCR over $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$. This is called the Fock representation. It can be shown that this representation is irreducible.

It turns out that von Neumann's uniqueness theorem does not extend to the case of infinitely-many degrees of freedom. We shall exhibit this by explicitly constructing an uncountable number of inequivalent (irreducible) representations.

Example 3.21. Let $\Phi(f)$ and $\Pi(g)$ be the Fock space operators, as above. For any $\mu \in \mathbb{R}$ and $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$, define

$$\Phi_\mu(f) = \Phi(f) + \mu \int f(x)\,dx$$

$$\Pi_\mu(g) = \Pi(g)\,.$$

Then $U_\mu(f) = \exp i\,\overline{\Phi_\mu(f)} = U(f)\,e^{i\mu\int f(x)\,dx}$ and $V_\mu(g) = V(g)$ define a representation of the CCR.

We claim that the representations $(U_\mu, V_\mu, \mathcal{F})$ and $(U_\lambda, V_\lambda, \mathcal{F})$ are unitarily equivalent if and only if $\mu = \lambda$. Indeed, suppose that there is unitary equivalence, i.e., there is a unitary $T : \mathcal{F} \to \mathcal{F}$ such that $TU_\mu(f)T^* = U_\lambda(f)$ and $TV_\mu(g)T^* = V_\lambda(g)$ for all $f, g \in \mathcal{S}_\mathbb{R}(\mathbb{R}^n)$. Then

$$T\,U(f)\,T^*\,e^{i\mu\int f(x)\,dx} = U(f)\,e^{i\lambda\int f(x)\,dx},$$

i.e., $T\,U(f)\,T^* = U(f)\,e^{i(\lambda-\mu)\int f(x)\,dx}$.

For $\mu \neq \lambda$, let (f_n) be a sequence in $\mathcal{S}_\mathbb{R}(\mathbb{R}^n)$ such that $\|f_n\|_{L^2} \to 0$ and $\int f_n(x)\,dx \to \pi/(\lambda-\mu))$, as $n \to \infty$. It follows that $\Phi(f_n) \to 0$ strongly on D_0 and so $\exp i\Phi(f_n)$ converges strongly to $\mathbb{1}$ on D_0 (since D_0 is a domain of entire vectors for $\Phi(f_n)$). This means that $U(f_n) \to \mathbb{1}$ strongly on $\mathcal{F}$. Hence $T\,U(f_n)\,T^* \to T\,T^* = \mathbb{1}$.

On the other hand, $U(f_n)\,e^{i(\lambda-\mu)\int f_n(x)\,dx}$ converges strongly on $\mathcal{F}$ to $e^{i\pi}\mathbb{1} = -\mathbb{1}$. This contradiction shows that T cannot exist for $\mu \neq \lambda$ and so these representations are not unitarily equivalent.

We mention here that the relativistic time-zero free field of mass m, $\phi(f)$, and its conjugate momentum, $\pi(g)$, define a representation of the Weyl CCR which is inequivalent to the Fock representation. This can be seen by showing that $\phi(f)+i\pi(f)$ annihilates no vector whereas $\Phi(f)+i\Pi(f)$ does (namely, the Fock space vacuum, or no-particle vector).

There are further generalizations of the Weyl relations. We mention only the formulation of Mackey in which the Us and Vs are defined over an abelian group and its dual, in which case there is a uniqueness theorem, and that of Segal in which one considers a symplectic form over a vector space and which yields a beautiful procedure for the quantization of free fields. For further results and details we refer to Emch and the bibliography therein.

Chapter 4

The algebraic approach to quantum theory

Rather than discuss a theory in terms of fields and commutation (or anticommutation) relations, one may wish to consider directly a theory of observables. Of course, quantum fields may well be used to construct observables but the point here is to consider observables without specific reference to any fields. In the conventional treatment developed by von Neumann, observables are represented by the self-adjoint operators on a Hilbert space. It is this that is to be generalized. It should be emphasized that our observables are mathematical or "ideal" observables. We do not discuss the act of observation nor the actual measurement within a laboratory. Indeed, this is a somewhat controversial subject. We refer the interested reader to the Varenna lectures of 1970 (D'Espagnat).

Segal's postulates

A "system" is supposed to consist of a collection of "observables" and is supposed to be capable of being in certain "states". We consider the observables as given and we can then define the "state of the system" as the knowledge of the expected values of the observables. That is to say, a state is an assignment of an expected value to each observable.

If A is an observable, then for any $a \in \mathbb{R}$, we suppose aA to also be an observable — it has an expected value in any state equal to a times that value which A has in the same state. In the same way, we assume that $A+b$ is an observable whenever A and B are. A^2 is supposed to be that observable whose possible values are equal to the squares of those of A. Furthermore, it is simpler if we suppose our observables to be bounded – that is, each can only assume values from a bounded set of real numbers (depending on the observable in question). This is not much of a restriction in that an unbounded observable could be considered as a limit or as a collection of bounded ones. This preamble leads us to Segal's postulates.

Phenomenological postulate: algebraic part

A physical system is a collection of objects called (bounded) observables, for which the operations of multiplication by a real number, squaring and addition are defined and satisfy the usual assumptions for a linear space. As remarked above, a state of the system assigns to each observable a real number, called the "expectation of the observable in the state". A state is defined to *be* this assignment. The following properties of any state E are intuitively reasonable.

(i) Linearity: $E(A+B) = E(A) + E(B)$

$$E(aA) = a\,E(A)$$

for bounded observables A and B and any $a \in \mathbb{R}$.

(ii) Positivity: $E(A^2) \geq 0$, all observables A.

(iii) Boundedness: $|E(A) \leq c_A|$, where c_A is the maximum value that A can have.

This notion of maximum value of an observable is considered further in the following postulate.

Phenomenological postulate: analytical part

To each observable A is assigned a "bound", written $\|A\|$, in such a way that

(i) $\|A\| \geq 0$ for all A and $\|A\| = 0$ if and only if $A = 0$.

(ii)
$$\|aA\| \leq |a|\ \|A\|$$
$$\|A+B\| \leq \|A\| + \|B\|$$
for any $a \in \mathbb{R}$ and observables A and B.

(iii) $\|A^2\| = \|A\|^2$ for all observables A.

The interpretation of $\|A\|$ is c_A, the maximum value that A can have. With this in mind, the requirements above seem quite reasonable. Segal makes further postulates and is able to recover many physical notions. However, the following postulate, stronger than Segal's, is the one that is currently adopted (and allows a very workable mathematical framework).

C^*-algebra Postulate

A physical system corresponds to the self-adjoint elements of a C^*-algebra with identity and where the bound $\|A\|$ of an observable A is given by the norm of the C^*-algebra.

A few remarks are in order here.

Remarks 4.1.

1. It is mathematically convenient to allow an operation of multiplication by complex numbers.

2. The property $\|A^2\| = \|A\|^2$ is just the C^*-property for a self-adjoint element A.

3. It is convenient to assume that the observables are complete with respect to the norm. Otherwise, one could just consider the appropriate completion.

4. A C^*-algebra has a product. There seems to be no justification for this assumption of our observables. Moreover, the product of self-adjoint elements need not be self-adjoint, so the product is not even defined directly on observables. (One alternative is to consider a Jordan product and Jordan algebras.)

5. The scheme here generalizes the von Neumann scheme in as much as a C^*-algebra generalizes the notion of the set of all operators on a Hilbert space. The point is that we do not need to specify any particular Hilbert space, but even if we do, we do not have to consider all the bounded operators there. To put this another way, we can consider different representations of our C^*-algebra of observables. In fact, it turns out that by simultaneously considering various such representations, one can develop a theory of superselection sectors.

6. For Segal's original postulates, we refer to Segal and Emch.

Exact values of observables

Definition 4.2. A state of a system $\mathfrak{A}$ is a state on the C^*-algebra $\mathfrak{A}$. That is, a state is a positive linear functional with norm one. The collection of states is denoted $\mathfrak{A}_1^{*+}$.

We recall that a state is said to be a mixture if it is a convex combination of two different states. A state is pure if it is not a mixture. As an example, consider the C^*-algebra (without unit) $\mathcal{C}(\mathcal{H})$ of all compact operators on the Hilbert space $\mathcal{H}$. Every continuous positive linear functional, ω, on $\mathcal{C}(\mathcal{H})$ has the form

$$\omega(A) = \operatorname{Tr}(DA)$$

for some $D \geq 0$, $D \in \mathcal{C}(\mathcal{H})$, $\operatorname{Tr}(D) < \infty$. That is, all states are given by density matrices. The pure states are the vector states, i.e., those of the form

$$\omega(A) = \langle \zeta, A\zeta \rangle$$

for $\zeta \in \mathcal{H}$. If $\mathfrak{A}$ is not equal to $\mathcal{C}(\mathcal{H})$, then there will be states which are *not* given by density matrices and pure states which are *not* given by vector states. (See Segal (1947) for an example.) We see that the scheme here is somewhat more general than the standard Hilbert space approach in which states are assumed to be given by density matrices.

Definition 4.3. For any observable $A = A^* \in \mathfrak{A}$ and state $\omega \in \mathfrak{A}_1^{*+}$, the variance of A in ω is the value

$$\mathbb{V}_\omega(A) = \omega(A^2) = \omega(A)^2.$$

We say that A has exact value in ω if $\mathbb{V}_\omega(A) = 0$, the exact value being $\omega(A)$. The collection of exact values of an observable is called its physical spectrum.

Theorem 4.4. *The physical spectrum of an observable A is equal to $\sigma(A)$, the spectrum of A.*

Proof. Let $\mathcal{A}$ denote the unital commutative C^*-algebra generated by the observable A. Then $\mathcal{A}$ is isomorphic to the uniform algebra $\mathcal{C}(K)$ over the compact Hausdorff space $K = \operatorname{Sp} \mathcal{A}$.

Let $\omega \in \mathfrak{A}_1^{*+}$ and suppose that $\mathbb{V}_\omega(A) = 0$. Let μ_ω denote the measure on K induced by ω via the Riesz-Markov representation theorem. Then

$$\begin{aligned} 0 = \mathbb{V}_\omega(A) &= \omega((A - \omega(A)\mathbb{1})^2) \\ &= \int_K |\widehat{A}(\kappa) - \omega(A)|^2 \, d\mu_\omega(\kappa) \end{aligned}$$

where $\widehat{A}$ denotes the Gelfand transform of A. It follows that $\widehat{A}(\kappa) = \omega(A)$, μ_ω-almost everywhere. In particular, there is $\kappa \in K$ such that $\widehat{A}(\kappa) = \omega(A)$. But $\sigma(A) = \operatorname{ran} \widehat{A}$, i.e., $\omega(A) \in \sigma(A)$.

Conversely, suppose that $\lambda \in \sigma(A)$. Then $\lambda = \widehat{A}(\kappa)$ for some $\kappa \in K$. For any $B \in \mathcal{A}$, define

$$\omega_\kappa(B) = \widehat{B}(\kappa).$$

Clearly, this defines a state on $\mathcal{A}$ and satisfies $\mathbb{V}_{\omega_\kappa}(A) = 0$. Any state on $\mathcal{A}$ can be extended to a state on $\mathfrak{A}$ and so $\lambda = \omega_\kappa(A)$ belongs to the physical spectrum. ■

Corollary 4.5. *Suppose that ω is a state on $\mathfrak{A}$ and that $\omega(A)$ is an exact value of the observable A. Then ω is a pure state on $\mathcal{A}$, the C^*-algebra generated by A.*

Proof. As in the proof of the theorem, we have that $\widehat{A} = \omega(A)$, μ_ω-almost everywhere. Suppose that there were κ_1 and κ_2 in K such that $\widehat{A}(\kappa_1) = \omega(A) = \widehat{A}(\kappa_2)$. Then, for any polynomial p, $p(\widehat{A})(\kappa_1) = p(\widehat{A})(\kappa_2)$. But polynomials in $\widehat{A}$ are dense in $\mathcal{C}(K)$ which contains functions which have different values at κ_1 and κ_2. Thus, there is only one $\kappa \in K$ with $\widehat{A}(\kappa) = \omega(A)$. In other words, the singleton set $\{\kappa\}$ has μ_ω-measure one. Hence $\omega(B) = \widehat{B}(\kappa)$ for all $B \in \mathcal{A}$. Thus ω is a character on $\mathcal{A}$ and so is a pure state on $\mathcal{A}$. ∎

Corollary 4.6. *The physical spectrum of an observable A is equal to the set $\{\omega(A) : \omega$ pure state on $\mathcal{A}\}$, where $\mathcal{A}$ is the C^*-algebra generated by A.*

Proof. This is clear from the above. ∎

Corollary 4.7. *Let $\mathcal{B}$ be a commutative C^*-algebra containing A. Then the physical spectrum of A is equal to the set $\{\omega(A) : \omega$ pure state on $\mathcal{B}\}$.*

Proof. If ω is pure on $\mathcal{B}$, then ω is a character on $\mathcal{B}$ and so is a character on $\mathcal{A}$, the C^*-algebra generated by A. Therefore $\omega(A)$ is an exact value.

Conversely, if $\omega(A)$ is an exact value of A, then from the above, ω is pure on $\mathcal{A}$. But a pure state on $\mathcal{A}$ can be extended to a pure state on $\mathcal{B}$. ∎

Simultaneous measurability

We discuss the notion of simultaneous measurability, following Segal (1947) and Emch (1972).

Definition 4.8. Let Γ be a collection of observables and let ω be a state. We say that ω is dispersion free on Γ if $\mathbb{V}_\omega(A) = 0$ for all $A \in \Gamma$.

We have seen that if $\mathbb{V}_\omega(A) = 0$, then ω is dispersion free on $\mathcal{A}$, the algebra generated by the observable A. To say that two observables are different is to say that we can find a state in which they differ. We can say that a collection of states specifies the system if it distinguishes between the observables.

Definition 4.9. Let $\mathcal{S}$ be a family of states on $\mathfrak{A}$ and let $\mathfrak{B}$ be a subset of $\mathfrak{A}$. The family $\mathcal{S}$ is said to be separating for $\mathfrak{B}$ if, for any $A, B \in \mathfrak{B}$, $\omega(A) = \omega(B)$ for all $\omega \in \mathcal{S}$ implies that $A = B$.

What does it mean to say that two observables are simultaneously measurable? It is natural to require that their exact values can be simultaneously realized, i.e., there are states ω such that $\mathbb{V}_\omega(A) = \mathbb{V}_\omega(B) = 0$. This should hold for sufficiently many states. Furthermore, if A and B are simultaneously measurable, we would expect the same to be true of say A^2 and $(A+B)^2$, for example. This idea is taken up in the following definition.

Definition 4.10. A collection Γ of observables is said to be simultaneously measurable if there exists a set $\mathcal{S}$ of states separating for and dispersion free on $\mathcal{A}_\mathbb{R}(\Gamma)$, the self-adjoint elements of $\mathcal{A}(\Gamma)$, the C^*-algebra generated by the collection Γ.

Theorem 4.11. *A collection of observables* Γ *is simultaneously measurable if and only if* $\mathcal{A}_{\mathbb{R}}(\Gamma)$ *is commutative.*

Proof. Suppose that Γ is a collection of observables such that $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is commutative. Then $\mathcal{A}(\Gamma)$ is commutative and $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is isometrically isomorphic to $\mathcal{C}_{\mathbb{R}}(K)$, via the Gelfand transform. For each $\kappa \in K$, define $\omega_\kappa : \mathcal{A}(\Gamma) \to \mathbb{C}$ by $\omega_\kappa(A) = \widehat{A}(\kappa)$, where $\widehat{A}$ is the Gelfand transform of $A \in \mathcal{A}(\Gamma)$. Clearly, each ω_κ is a state on $\mathcal{A}(\Gamma)$ and so has an extension to $\mathfrak{A}$. Evidently, ω_κ is dispersion free on $\mathcal{A}_{\mathbb{R}}(\Gamma)$. Moreover, the set $\{\, \omega_\kappa : \kappa \in K \,\}$ is separating for $\mathcal{A}_{\mathbb{R}}(\Gamma)$.

Conversely, suppose that $\mathcal{S}$ is a dispersion free and separating family for the set $\mathcal{A}_{\mathbb{R}}(\Gamma)$. For any $A, B \in \mathcal{A}_{\mathbb{R}}(\Gamma)$ define $A \circ B \in \mathcal{A}_{\mathbb{R}}(\Gamma)$ by the formula

$$A \circ B = \tfrac{1}{4}\big((A+B)^2 - (A-B)^2\big).$$

Then, for any $\omega \in \mathcal{S}$, we have

$$\begin{aligned} \omega(A \circ B) &= \tfrac{1}{4}\,\omega\big((A+B)^2 - (A-B)^2\big) \\ &= \tfrac{1}{4}\,\big(\,\omega((A+B)^2) - \omega((A-B)^2)\,\big) \\ &= \tfrac{1}{4}\,\big(\,\omega(A+B)^2 - \omega(A-B)^2\,\big) \end{aligned}$$

since ω is dispersion free on $\mathcal{A}_{\mathbb{R}}(\Gamma)$,

$$= \omega(A)\,\omega(B)\,.$$

Hence, for any $A, B, C \in \mathcal{A}_{\mathbb{R}}(\Gamma)$,

$$\begin{aligned} \omega((A \circ B) \circ C) &= \omega(A \circ B)\,\omega(C) = \omega(A)\,\omega(B)\,\omega(C) \\ &= \omega(A)\,\omega(B \circ C) = \omega(A \circ (B \circ C))\,. \end{aligned}$$

This holds for all $\omega \in \mathcal{S}$, which is separating, and so

$$(A \circ B) \circ C = A \circ (B \circ C)$$

i.e., "$\circ$" is an associative product on $\mathcal{A}_{\mathbb{R}}(\Gamma)$.

Without loss of generality, we may suppose that $\mathcal{A}(\Gamma)$ is an algebra of operators on a Hilbert space. By taking strong limits, together with Kaplansky's Density Theorem, we see that the operation "$\circ$" is also associative on the strong closure of $\mathcal{A}_{\mathbb{R}}(\Gamma)$. However, by the spectral theorem, this strong closure contains the spectral projections of the elements of $\mathcal{A}_{\mathbb{R}}(\Gamma)$ and the associativity of "$\circ$" on projections implies that they must commute with each other. Indeed, for projections E and F, the equality $E \circ (F \circ E) = (E \circ F) \circ E$ simplifies to the equality $EFE = FEF$. This, in turn, implies that $\|(EF - FE)x\| = 0$ for every vector x in the Hilbert space and so E and F commute. But then it follows that $\mathcal{A}_{\mathbb{R}}(\Gamma)$ is commutative. ∎

Remark 4.12. Even though A and B may be simultaneously observable, it need not be true that B has an exact value in some state even though A may have. For example, let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and let ω be the state $\omega(\,\cdot\,) = \frac{1}{2}\langle\, e_1, \cdot\, e_1\,\rangle + \frac{1}{2}\langle\, e_2, \cdot\, e_2\,\rangle$, where $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$. Then $\omega(A) = 1$ is an exact value of A, but $\omega(B) = \frac{3}{2}$ is not an exact value of B.

In view of the above remark, the following result may be somewhat gratifying.

Theorem 4.13. *Let Γ be a set of simultaneously measurable observables and suppose that $\omega(A)$ is an exact value of some given $A \in \Gamma$ for some state ω. Then there exists a state ρ such that $\rho(A) = \omega(A)$ and such that $\rho(B)$ is an exact value of B for all $B \in \Gamma$.*

Proof. The C^*-algebra $\mathcal{A}(\Gamma)$ generated by the family Γ of observables is commutative. Furthermore, since $\omega(A)$ is an exact value of A, the restriction, $\omega \restriction \mathcal{A}$, of ω to $\mathcal{A}$, the unital C^*-algebra generated by A, is pure. This can be extended to some pure state ρ on $\mathcal{A}(\Gamma)$. By corollary 4.7, $\rho(B)$ is an exact value of B, for any B in Γ. ∎

Probabilistic description

It is now straightforward to introduce the notion of joint probability distributions for simultaneous observations. Indeed, let Γ be a set of simultaneously measurable observables and let $\mathcal{A}(\Gamma)$ be the C^*-algebra they generate. Let ω be any state of the system $\mathfrak{A}$. We know that $\mathcal{A}(\Gamma)$ is commutative and so is isomorphic, via the Gelfand transform $A \mapsto \widehat{A}$, to $\mathcal{C}(K)$, for compact K. By restriction, ω defines a state on $\mathcal{C}(K)$ which, by the Riesz-Markov theorem, can be written as

$$\omega(A) = \int_K \widehat{A}(\kappa)\, d\mu_\omega(\kappa), \qquad A \in \mathcal{A}(\Gamma),$$

for some regular probability measure μ_ω on K.

Let $A_1, \dots, A_n \in \Gamma$ and let $I_1, \dots, I_n$ be Borel sets in the real line. The joint probability distribution of the observables $A_1, \dots, A_n$ in the state ω is defined to be

$$\mathbb{P}_{A_1,\dots,A_n;\omega}(I_1, \dots, I_n) = \mu_\omega(\widehat{A}_1^{-1}(I_1) \cap \cdots \cap \widehat{A}_n^{-1}(I_n)).$$

This is the probability that A_1 has value in I_1, A_2 in I_2, etc., in the state ω. For the case of one observable A, $\mathbb{P}_{A;\omega}(I)$ is just the probability that A has value in I in the state ω. If we write $\mathbb{P}_{A;\omega}(\lambda)$ for the case $I = (-\infty, \lambda]$, one can show that the expected value of A in ω is given by

$$\mathbb{E}_\omega(A) = \int_{\mathbb{R}} \lambda\, d\mathbb{P}_{A;\omega}(\lambda) = \omega(A)$$

as we would wish.

We note that the probability distribution $\mathbb{P}_{A;\omega}$ is independent of any realization of A as a function. Indeed, the characteristic function of $\mathbb{P}_{A;\omega}$ is given by $t \mapsto \omega(e^{itA})$. In the same way, the joint distribution $\mathbb{P}_{A_1,\dots,A_n;\omega}$ is uniquely determined.

Chapter 5

Local quantum theory

We shall present an account of the formalism initiated by Haag and Kastler (1964), which is concerned with a relativistic quantum theory of local observables. The idea is to base discussion on a series of modest but plausible hypotheses (postulates, building blocks or "axioms") suggested by physical arguments.

The Haag-Kastler axioms

As already discussed, it is assumed that the collection of observables generates a C^*-algebra, $\mathfrak{A}$. The first postulate represents the idea that each region of space-time gives rise to a family of observables.

Postulate 1

> To each region $\mathcal{O}$ in Minkowski space, $\mathcal{M}$, there corresponds a sub-C^*-algebra $\mathfrak{A}(\mathcal{O})$ of $\mathfrak{A}$. Moreover, $\mathfrak{A}$ is generated by the algebras $\mathfrak{A}(\mathcal{O})$ as $\mathcal{O}$ runs over regions of $\mathcal{M}$.

By definition, a region is a bounded open set in Minkowski space $\mathcal{M}$ (identified with $\mathbb{R}^4$). On physical grounds, one could argue that such regions are too general. For this reason, one often restricts attention to double-cones, i.e., regions which are formed from the intersection of a backward cone with a forward cone. This correspondence between algebras and regions should be thought of as essentially fixing the theory. Geometrical relationships between regions will lead to relationships between the various algebras.

Strictly speaking, only the self-adjoint elements of $\mathfrak{A}$ are to be considered observable, but, for terminological convenience, we use the word for any element of $\mathfrak{A}$. Thus, $\mathfrak{A}(\mathcal{O})$ is the algebra of observables associated with the region $\mathcal{O}$. The elements of the union $\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ constitute the local observables, whilst $\mathfrak{A}$ is called the algebra of quasilocal observables.

The next axiom expresses the notion that the bigger the region the more observables there may be.

Postulate 2 (Isotony)

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are regions in Minkowski space with $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then

$$\mathfrak{A}(\mathcal{O}_1) \subseteq \mathfrak{A}(\mathcal{O}_2).$$

Einstein's principle of causality is that no physical influence can propagate faster than the speed of light. Consequently, observables associated with space-like separated regions should not affect each other and so ought to be simultaneously measurable. This means that, as elements of a C^*-algebra, they must commute. This is the idea behind the next postulate.

Postulate 3 (Einstein causality, locality)

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated regions, then the associated local algebras of observables $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ commute; i.e., for any $A \in \mathfrak{A}(\mathcal{O}_1)$ and $B \in \mathfrak{A}(\mathcal{O}_2)$, we have $AB = BA$.

Poincaré covariance of the theory is expressed in the following axiom.

Postulate 4 (Poincaré covariance)

There is a representation α of $\mathcal{P}_+^\uparrow$, the restricted Poincaré group, in $\operatorname{Aut}\mathfrak{A}$, the automorphism group of $\mathfrak{A}$, such that

$$\alpha(L)\big(\mathfrak{A}(\mathcal{O})\big) = \mathfrak{A}(\Lambda\mathcal{O} + a)$$

for any region $\mathcal{O}$ and $L = (a, \Lambda) \in \mathcal{P}_+^\uparrow$.

It has been shown (Haag and Kastler (1964), Emch (1972)) that $\alpha(L)$ can never be an inner automorphism of $\mathfrak{A}$. This can be taken as an indication of the global nature of Poincaré transformations and the essentially local nature of $\mathfrak{A}$.

The next postulate is technical and excludes classical field theory (for which $\mathfrak{A}$ would be commutative).

Postulate 5

The quasilocal algebra $\mathfrak{A}$ is primitive, that is, $\mathfrak{A}$ possesses a faithful, irreducible representation.

Suppose one has a (Wightman) quantum field theory. Then one might construct a local algebra $\mathfrak{A}(\mathcal{O})$ from bounded functions of fields smeared with test-functions with support in the region $\mathcal{O}$. In this way, one can see that these axioms can be realized in terms of such C^*-algebras generated by free quantum fields. Note that for fermion theories (in which the fields anticommute at space-like separation), Postulate 3, Einstein causality, would require consideration of even functions of the fields. The algebraic approach emphasizes the observable structure and relegates fields as secondary objects or constructs.

A further specialization is to assume that the local algebras are von Neumann algebras acting on a given Hilbert space. This allows a natural introduction of the strong topology on these algebras and consequently a discussion of the energy-momentum spectrum (via the unbounded generators of the respective unitary groups).

Superselection rules

Consider a field theory describing fields with spin 0 and fields with spin $\frac{1}{2}$, say, and let ξ_0 and $\xi_{1/2}$ be vector states with spin 0 and $\frac{1}{2}$, respectively. Let η be the vector state given by the superposition of these,

$$\eta = \tfrac{1}{\sqrt{2}}\,(\xi_0 + \xi_{1/2}).$$

Under a rotation of 2π, the physics should be unchanged. However, η is transformed into

$$\eta \mapsto \eta' = \tfrac{1}{\sqrt{2}}\,(\xi_0 - \xi_{1/2}).$$

To say that a rotation of 2π has no observable effect is to say that η and η' describe the same state, that is,

$$\langle\, \eta, A\eta \,\rangle = \langle\, \eta', A\eta' \,\rangle$$

for all observables A. This is clearly a restriction on the operators which are supposed to be observable. Wick, Wightman and Wigner proposed that the Hilbert space of states, $\mathcal{H}$, could be decomposed into a suitable direct sum, $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$, such that each $\mathcal{H}_\alpha$ is mapped into itself by the algebra of observables. In the example here, we would consider $\mathcal{H}$ as decomposed into $\mathcal{H} = \mathcal{H}_o \oplus \mathcal{H}_e$, where $\mathcal{H}_o$ is the subspace of states with odd half-integer spin and $\mathcal{H}_e$ that with even half-integer spin. One then supposes that

$$\langle\, \zeta_o, A\zeta_e \,\rangle = 0, \text{ for any states } \zeta_o \in \mathcal{H}_o \text{ and } \zeta_e \in \mathcal{H}_e.$$

The subspaces $\mathcal{H}_\alpha$ are called superselection sectors and the statement that the observables have such a direct sum structure is called a superselection rule. A discussion of superselection rules has been offered by Mirman and also by Wick, Wightman and Wigner.

It has been shown by Strocchi and Wightman that the usual laws of quantum electrodynamics imply the existence of charge superselection sectors, in which case the Hilbert spaces $\mathcal{H}_\alpha$ correspond to the subspaces of different electric charge. We will consider charge sectors in some detail for the case of the free charged bose field.

The occurrence of superselection rules can be understood in this algebraic framework — the sectors correspond to inequivalent representations of the algebra of observables, $\mathfrak{A}$ and so we are thus led to a study of the representations of $\mathfrak{A}$.

Physical equivalence

Consider the measurement of the state ω on $\mathfrak{A}$. This might correspond to the measurement of a finite number of observables $A_1, \dots, A_n$, with resulting experimental values $a_1, \dots, a_n$ and with some experimental error ε, say. Then we may say that

$$|\omega(A_i) - a_i| < \varepsilon$$

for $i = 1, \dots, n$. Evidently, we cannot determine ω uniquely from this data. Indeed, as far as this experiment is concerned, we can only conclude that the system is in some state ω' with

$$\big|\omega'(A_i) - a_i\big| < \varepsilon, \quad 1 \le i \le n.$$

Thus ω' obeys

$$\big|\omega'(A_i) - \omega(A_i)\big| < 2\varepsilon, \quad 1 \le i \le n.$$

We see that each experiment corresponds to a w^*-neighbourhood of the state ω. Now, associated naturally with any representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ is the set of states given by convex combinations of vector states. We could say that two representations are physically equivalent if we cannot distinguish between them experimentally. This leads to the following definition.

Definition 5.1 (Haag and Kastler (1964)). Representations $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ of the quasilocal algebra of observables $\mathfrak{A}$ are said to be physically equivalent if every w^*-neighbourhood of any state given as a convex combination of vector states in one representation contains some state which is given as a convex combination of vector states in the other representation.

It turns out (J. Fell (1960)) that any two faithful representations are physically equivalent. This lends support to the emphasis on the algebra as an abstract object. However, if such representations correspond to different superselection sectors but are physically equivalent, is it really necessary to consider them at all? An answer to this is to agree that, in principle, one might only need to investigate the vacuum sector. However, by making various idealizations, one may well need to discuss other sectors. For example, one might make the idealization that "outside the laboratory" there is only the vacuum. Inside, one might wish to consider, say, an overall "charge" of +3. (The charge here could be the label corresponding to any of the superselection rules of the theory under discussion.) Presumably, in reality, there would be an overall corresponding charge of −3 outside the laboratory, but for our own (mathematical) convenience we imagine this to be so far away from our apparatus as not to have any experimental effect within the laboratory. In other words, we elect to work in the charge 3 superselection sector. The basic working hypothesis is that outside the laboratory there is nothing but the vacuum.

Energy and momentum as observables

So far, nothing has been said about energy and momentum, nor the notion of a vacuum state. This can be introduced as follows.

Postulate 6 (Spectrum condition)

There exists a state ω on $\mathfrak{A}$, the quasilocal algebra of observables, which is invariant under the space-time translation automorphism group $\alpha(a)$, $a \in \mathcal{M}$, and is such that the map $a \mapsto \omega(A\alpha(a)(B))$ is continuous for all $A, B \in \mathfrak{A}$.

Furthermore, if $P = (P_0, P_1, P_2, P_3)$ denotes the self-adjoint generator of the strongly continuous unitary group $U_\omega(a) = e^{i(a_0 P_0 - a_1 P_1 - a_2 P_2 - a_3 P_3)}$ implementing the group of automorphisms $\alpha(a)$ in the cyclic representation $(\mathcal{H}_\omega, \pi_\omega)$ of $\mathfrak{A}$ given by ω (the GNS representation), then the joint spectrum of (P_0, P_1, P_2, P_3) should lie in the closed forward light-cone

$$\overline{V}_+ = \{\, p \in \mathcal{M} : p_0^2 - p_1^2 - p_2^2 - p_3^2 \geq 0,\ p_0 \geq 0 \,\}.$$

Such a state ω is called a vacuum state.

Remark 5.2. The invariance of ω guarantees the existence of the unitaries $U_\omega(a)$ and the continuity hypothesis guarantees the existence of the generators $P = (P_0, P_1, P_2, P_3)$. P_0 is the energy and (P_1, P_2, P_3) the momentum.

Evidently, if Ω denotes a GNS cyclic vector for ω, then $U_\omega(a)\Omega = \Omega$, so that $P\Omega = 0$. This explains why ω is called a vacuum state — it has zero energy and momentum.

Note that the automorphisms $\alpha(a)$ are not assumed to be norm continuous in a. Indeed, this is not true for free fields. Furthermore, one can expect the energy and momentum operators to be unbounded operators. How can this fit into a C^*-algebra theory of bounded observables? This query is addressed by the following theorem (Araki (1964b), Borchers (1966)).

Theorem 5.3. *Suppose that ω is a vacuum state on $\mathfrak{A}$ with associated GNS constructs $(\mathcal{H}, \pi, U, \Omega)$. Then $U(a) \in \pi(\mathfrak{A})''$ for all $a \in \mathcal{M}$, that is, the spectral projections of the energy and momentum operators associated with ω belong to the strong closure of $\mathfrak{A}$, the quasilocal algebra of observables in the GNS representation $(\mathcal{H}, \pi, \Omega)$ induced by ω.*

Proof. Fix $A \in \mathfrak{A}$ and $X \in \pi(\mathfrak{A})'$. Then

$$\begin{aligned}\langle\, \Omega, \pi(A)U(a)\, X\Omega\,\rangle &= \langle\, \Omega, U(-a)\pi(A)U(a)\, X\Omega\,\rangle \\ &= \langle\, \Omega, \pi(\alpha(-a)(A))\, X\Omega\,\rangle \\ &= \langle\, \Omega, X\pi(\alpha(-a)(A))\Omega\,\rangle \\ &= \langle\, \Omega, X\, U(-a)\pi(A)\Omega\,\rangle.\end{aligned}$$

Hence, for any $\rho \in \mathcal{S}(\mathbb{R}^4)$,

$$\int \rho(a)\,\langle\, \Omega, \pi(A)U(a)\, X\Omega\,\rangle\, da = \int \rho(a)\,\langle\, \Omega, X\, U(-a)\pi(A)\Omega\,\rangle\, da\,.$$

Let $\widehat{\rho}(p) = (2\pi)^{-2}\int e^{ipa}\,\rho(a)\,da$ denote the Fourier transform of ρ (where $pa = p_0a_0 - p_1a_1 - p_2a_2 - p_3a_3$.) By the spectrum condition, Postulate 6, the first integral is zero if $\widehat{\rho}$ has support outside $\overline{V}_+$, whereas the second integral is zero if $\widehat{\rho}$ has support outside $\overline{V}_- = -\overline{V}_+$.

It follows that $\langle\, \Omega, \pi(A)U(a)\, X\Omega\,\rangle\widehat{}$, the Fourier transform of the function $\langle\, \Omega, \pi(A)U(a)\, X\Omega\,\rangle$ considered as a tempered distribution, has support in $\overline{V}_+ \cap \overline{V}_- = \{\, 0\,\}$. Since $\langle\, \Omega, \pi(A)U(a)\, X\Omega\,\rangle$ is a bounded and continuous function of a, we conclude that it is constant. Thus, for any $S, T \in \mathfrak{A}$,

$$\begin{aligned}\langle\, \Omega, \pi(S^*T)U(a)\, X\Omega\,\rangle &= \langle\, \pi(S)\Omega, \pi(T)U(a)\, X\Omega\,\rangle \\ &= \langle\, \pi(S)\Omega, U(a)\pi(\alpha(-a)(T))\, X\Omega\,\rangle \\ &= \langle\, \pi(S)\Omega, U(a)X\, \pi(\alpha(-a)(T))\Omega\,\rangle \\ &= \langle\, \pi(S)\Omega, U(a)XU(-a)\, \pi(T)\Omega\,\rangle \\ &= \langle\, \Omega, \pi(S^*T)\, X\Omega\,\rangle, \quad \text{setting } a = 0, \\ &= \langle\, \pi(S)\Omega, X\, \pi(T)\Omega\,\rangle\,.\end{aligned}$$

That is,

$$\langle\, \pi(S)\Omega, (U(a)XU(-a) - X)\pi(T)\Omega\,\rangle = 0\,.$$

Since Ω is cyclic, we conclude that $U(a)X = XU(a)$ for all $X \in \pi(\mathfrak{A})'$ and so $U(a) \in \pi(\mathfrak{A})''$, as required. ∎

Corollary 5.4. *With notation as in the theorem, the following are equivalent.*

(i) *ω is extremal invariant under the space-time translation group $\alpha(a)$, $a \in \mathcal{M}$.*

(ii) *$U(a)\xi = \xi$ for $\xi \in \mathcal{H}$ and for all $a \in \mathcal{M}$ implies that $\xi = \lambda\Omega$ for some $\lambda \in \mathbb{C}$.*

(iii) *$\pi(\mathfrak{A})$ is irreducible.*

Proof. We have seen that $U(\mathcal{M}) \subseteq \pi(\mathfrak{A})''$ and so $\pi(\mathfrak{A})' \subseteq U(\mathcal{M})'$, that is, $\pi(\mathfrak{A})' = \pi(\mathfrak{A})' \cap U(\mathcal{M})'$. However, ω is extremal invariant if and only if $\pi(\mathfrak{A})' \cap U(\mathcal{M})' = \mathbb{C}\,\mathbb{1}$ and so we conclude that ω is extremal invariant if and only if $\pi(\mathfrak{A})' = \mathbb{C}\,\mathbb{1}$. This shows that (i) and (ii) are equivalent.

Next we shall show that (ii) implies (iii). Let $X \in \pi(\mathfrak{A})'$. Then, since $U(a) \in \pi(\mathfrak{A})''$, it follows that $U(a)X\Omega = XU(a)\Omega = X\Omega$ for all $a \in \mathcal{M}$ and so, by hypothesis (statement (ii)), $X\Omega = \lambda\Omega$ for some $\lambda \in \mathbb{C}$. Hence

$$X\,\pi(A)\Omega = \pi(A)\,X\Omega = \lambda\pi(A)\Omega$$

for all $A \in \mathfrak{A}$. By cyclicity of Ω, it follows that $X = \lambda\mathbb{1}$ and so $\pi(\mathfrak{A})$ is irreducible.

To complete the proof, we show that (iii) implies (ii). The first step is to show that $[B, \pi(\alpha(a)(A))] \to 0$ weakly as $|a| \to \infty$ with a space-like. In fact, if $X \in \mathfrak{A}(\mathcal{O}_1)$ and $Y \in \mathfrak{A}(\mathcal{O}_2)$, then by Postulate 4 (Poincaré covariance) and Postulate 5 (Einstein causality (locality)), the commutator $[Y, \alpha(a)(X)]$ vanishes for sufficiently large space-like a since $\alpha(a)(X) \in \mathfrak{A}(\mathcal{O}_1 + a)$ and $\mathcal{O}_1 + a$ and $\mathcal{O}_2$ are space-like separated for such a. Replacing X by any $A \in \mathfrak{A}$ and Y by any $B \in \pi(\mathfrak{A})''$ leads to the weak commutativity statement we seek to prove. To begin with, we note that since $\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ is norm dense in $\mathfrak{A}$ and $\alpha(a)$ is isometric, A can be approximated in norm by some $X \in \mathfrak{A}(\mathcal{O}_1)$, for some region $\mathcal{O}_1$, so that $\alpha(a)(A)$ is approximated by $\alpha(a)(X)$ in norm, uniformly in a. We can assume, then, that $A \in \mathfrak{A}(\mathcal{O}_1)$ for some $\mathcal{O}_1$. It is now enough to prove that

$$\langle\, \xi, B\pi(\alpha(a)(A))\eta\,\rangle - \langle\, \xi, \pi(\alpha(a)(A))B\eta\,\rangle \to 0 \qquad (*)$$

for any given vectors $\xi, \eta \in \mathcal{H}$, as $a \to \infty$ with a space-like.

Now, by von Neumann's double commutant theorem, $\pi(\mathfrak{A})''$ is the strong closure of $\pi(\mathfrak{A})$. However, $\mathfrak{A}$ is the norm closure of $\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ and so $\pi(\mathfrak{A})''$ is the strong closure of $\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$. By Kaplansky's density theorem, it follows that for given $\varepsilon > 0$ there is some $Y \in \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$, say $Y \in \mathfrak{A}(\mathcal{O}_2)$, such that $\|(B - \pi(Y))\eta\| < \varepsilon$ and also $\|(B^* - \pi(Y)^*)\xi\| < \varepsilon$. Hence,

$$\begin{aligned} |\,\langle\, \xi, B\pi(\alpha(a)(A))\eta\,\rangle &- \langle\, \xi, \pi(\alpha(a)(A))B\eta\,\rangle \\ &- \big(\langle\, \xi, \pi(Y)\pi(\alpha(a)(A))\eta\,\rangle - \langle\, \xi, \pi(\alpha(a)(A))\pi(Y)\eta\,\rangle\big)\,| \\ &\le 2\varepsilon\,\|A\|\,\|\eta\| \end{aligned}$$

uniformly in $a \in \mathcal{M}$. The claim now follows from the observation, as above, that $\pi(Y)\pi(\alpha(a)(X)) - \pi(\alpha(a)(X))\pi(Y) = 0$ whenever a is sufficiently large and space-like.

Now we note that the hypothesis (iii), namely that $\pi(\mathfrak{A})$ is irreducible, means that $\pi(\mathfrak{A})'' = \mathcal{B}(\mathcal{H})$. So for any unit vectors $\xi, \eta \in \mathcal{H}$ we may take

B to be the projection onto the one-dimensional space spanned by ξ to get that

$$\langle\, \xi, \pi(\alpha(a)(A))\eta \,\rangle - \langle\, \xi, \pi(\alpha(a)(A))\xi \,\rangle\langle\, \xi, \eta \,\rangle \to 0 \qquad (**)$$

as $a \to \infty$ with a space-like. Interchanging ξ and η in $(*)$ and taking the complex conjugate we find also that

$$\langle\, \xi, \pi(\alpha(a)(A))^*\eta \,\rangle - \langle\, \eta, \pi(\alpha(a)(A))^*\eta \,\rangle\langle\, \xi, \eta \,\rangle \to 0.$$

By taking A to be self-adjoint and subtracting, we get that

$$\langle\, \xi, \pi(\alpha(a)(A))\xi \,\rangle - \langle\, \eta, \pi(\alpha(a)(A))\eta \,\rangle \to 0 \qquad (***)$$

as $a \to \infty$ with a space-like, provided that ξ and η are not orthogonal. If ξ and η are orthogonal, then we simply take B to be the operator on $\mathcal{H}$ which interchanges ξ and η and which acts as the identity on the subspace orthogonal to that spanned by ξ and η. This operator is self-adjoint and we have $(***)$ directly from $(*)$. So $(***)$ holds for any unit vectors ξ and η and, by linearity, for any $A \in \mathfrak{A}$.

To show that (ii) holds, suppose that ξ is such that $U(a)\xi = \xi$ for all $a \in \mathcal{M}$. Taking $\eta = \Omega$, $(***)$ reduces to

$$\langle\, \xi, \pi(A)\xi \,\rangle = \langle\, \Omega, \pi(A)\Omega \,\rangle$$

for all $A \in \mathfrak{A}$. Thus

$$\langle\, \xi, X\xi \,\rangle = \langle\, \Omega, X\Omega \,\rangle$$

for all X in the weak closure of $\pi(\mathfrak{A})$, that is, for all $X \in \mathcal{B}(\mathcal{H})$ since, by hypothesis, $\pi(\mathfrak{A})$ is irreducible. Taking X to be the projection F, say, onto the orthogonal complement of the one-dimensional subspace spanned by Ω, we find that

$$\|F\xi\|^2 = \langle\, \xi, F\xi \,\rangle = \langle\, \Omega, F\Omega \,\rangle = 0.$$

Hence $F\xi = 0$ and it follows that $\xi = \lambda\Omega$ for some $\lambda \in \mathbb{C}$. ■

The Reeh-Schlieder Theorem

A further postulate is introduced.

Postulate 7 (Additivity)

For any family of regions $\{\, \mathcal{O}_\nu \,\}$ covering Minkowski space $\mathcal{M}$, the quasilocal algebra $\mathfrak{A}$ is generated by the collection $\{\, \mathfrak{A}(\mathcal{O}_\nu) \,\}$.

By taking families of translates of any given region, we see that the additivity assumption prevents the algebras from becoming trivial when $\mathcal{O}$ is small.

A weaker version is the requirement that $\pi(\mathfrak{A})$ be generated by the family $\{\, \pi(\mathfrak{A}(\mathcal{O}_\nu)) \,\}$ where π is the GNS representation given by a vacuum state ω on $\mathfrak{A}$. In this case, one says that additivity holds in the vacuum sector (determined by ω).

Given any vacuum state ω, it is natural to think of the set of vectors $\{\pi(\mathfrak{A}(\mathcal{O}))\Omega\}$ as being somehow "localized" in the region $\mathcal{O}$. Taking the closure, one gets a subspace $\mathcal{H}(\mathcal{O})$, say, of the GNS Hilbert space $\mathcal{H}$. By varying $\mathcal{O}$, one might expect to get a collection of subspaces giving various families of localized states. However, this is *not* the case as the following theorem of Reeh and Schlieder (1961) (Araki (1964b)) shows. No matter how small $\mathcal{O}$ is, the set of vectors $\pi(\mathfrak{A}(\mathcal{O}))\Omega$ is dense in the whole Hilbert space $\mathcal{H}$.

Theorem 5.5 (Reeh-Schlieder Theorem). *Suppose that additivity holds in the vacuum sector given by a vacuum state ω on $\mathfrak{A}$. Let $(\mathcal{H}, \pi, U, \Omega)$ be the GNS constructs associated with ω. Then for any region $\mathcal{O}$ in Minkowski space the vector Ω is cyclic and separating for the subalgebra $\pi(\mathfrak{A}(\mathcal{O}))$.*

Proof. Let $\mathcal{O}$ be any given region. To show that Ω is cyclic for $\pi(\mathfrak{A}(\mathcal{O}))$, we need only show that $\langle\,\xi, \pi(A)\Omega\,\rangle = 0$ for all $A \in \mathfrak{A}(\mathcal{O})$ only if $\xi = 0$. But since Ω is cyclic for $\mathfrak{A}$, this follows if we can show that $\langle\,\xi, \pi(A)\Omega\,\rangle = 0$ for all $A \in \mathfrak{A}(\mathcal{O})$ implies that $\langle\,\xi, \pi(A)\Omega\,\rangle = 0$ for all $A \in \mathfrak{A}$.

So suppose that $\langle\,\xi, \pi(A)\Omega\,\rangle = 0$ for all $A \in \mathfrak{A}(\mathcal{O})$ and let $\mathcal{O}_0 \subset \mathcal{O}$ be such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}$. Let $A_1, \dots, A_n \in \mathfrak{A}(\mathcal{O}_0)$ and for $a_1, \dots, a_n \in \mathcal{M} = \mathbb{R}^4$ set

$$F(a_1, a_2 - a_1, \dots, a_n - a_{n-1}) = \langle\,\xi, \pi(\alpha(a_1)(A_1)\,\dots\,\alpha(a_n)(A_n))\Omega\,\rangle.$$

Since $\overline{\mathcal{O}}_0$ is closed and contained in the open set $\mathcal{O}$, there is a neighbourhood $N_1 \times \cdots \times N_n$ in $\mathbb{R}^{4n}$ such that $\mathcal{O}_0 + a_i \subset \mathcal{O}$ for $a_i \in N_i$, $1 \le i \le n$. But then $\alpha(a_1)(A_1)\dots\alpha(a_n)(A_n) \in \mathfrak{A}(\mathcal{O})$ and so, by hypothesis, we have that $F(a_1, a_2 - a_1, \dots, a_n - a_{n-1}) = 0$ for all $a_1 \times \cdots \times a_n \in N_1 \times \cdots \times N_n$. However, using the fact that $\alpha(a)$ is implemented by $U(a) = e^{iaP}$, we find that

$$F(a_1, a_2 - a_1, \dots, a_n - a_{n-1}) \\ = \langle\,\xi, e^{ia_1P}\pi(A_1)e^{i(a_2-a_1)P}\pi(A_2)\dots e^{i(a_n-a_{n-1})P}\pi(A_n)\Omega\,\rangle.$$

Let $(z_1, \dots, z_n) \in \mathbb{C}^{4n}$ be such that $\operatorname{Im} z_1 \in V_+$, $\operatorname{Im}(z_2 - z_1) \in V_+$, $\dots$, $\operatorname{Im}(z_n - z_{n-1}) \in V_+$. Since P has its spectrum in $\overline{V}_+$, it follows that $\operatorname{Im} z_1 P \ge 0$, $\operatorname{Im}(z_2 - z_1)P \ge 0$ ... and so on, and hence $F(z_1, z_2 - z_1, \dots, z_n - z_{n-1})$ defines a function analytic for $(z_1, z_2, \dots, z_n) \in \mathbb{C}^{4n}$ satisfying $\operatorname{Im} z_1 \in V_+$, $\operatorname{Im}(z_2 - z_1) \in V_+$, $\dots$, $\operatorname{Im}(z_n - z_{n-1}) \in V_+$ and which vanishes on the boundary $\operatorname{Im} z_1 = 0$, $\operatorname{Im}(z_2 - z_1) = 0$, $\dots$, $\operatorname{Im}(z_n - z_{n-1}) = 0$, $\operatorname{Re} z_1 \in N_1$, $\operatorname{Re} z_2 \in N_2$, $\dots$, $\operatorname{Re} z_n \in N_n$.

It follows that F is identically zero, as an analytic function, in $\operatorname{Im} z_1 \in V_+$, $\operatorname{Im}(z_2 - z_1) \in V_+$, $\dots$, $\operatorname{Im}(z_n - z_{n-1}) \in V_+$ and so we see that, in particular, $F(a_1, a_2 - a_1, \dots, a_n - a_{n-1}) = 0$ for all $a_i \in \mathbb{R}^4$, $1 \le i \le n$.

Now, by additivity in the vacuum sector, $\pi(\mathfrak{A})$ is generated by the family $\pi(\mathfrak{A}(\mathcal{O}_0 + a))$ as a varies over $\mathbb{R}^4$. We conclude that $\langle\,\xi, \pi(A)\Omega\,\rangle = 0$ for all

$A \in \mathfrak{A}$ and so, by cyclicity of Ω, $\xi = 0$ and therefore $\pi(\mathfrak{A}(\mathcal{O}))\Omega$ is dense in $\mathcal{H}$, as required.

To show that Ω is separating for $\pi(\mathfrak{A}(\mathcal{O}))$, we note that, by the preceding discussion, Ω is cyclic for $\pi(\mathfrak{A}(\mathcal{O}_1))$, for any region $\mathcal{O}_1$, and so is separating for $\pi(\mathfrak{A}(\mathcal{O}_1))'$. If we choose $\mathcal{O}_1$ to be space-like with respect to $\mathcal{O}$, then $\pi(\mathfrak{A}(\mathcal{O})) \subseteq \pi(\mathfrak{A}(\mathcal{O}_1))'$, by locality. Hence Ω is separating for $\pi(\mathfrak{A}(\mathcal{O}))$. ■

Remark 5.6. This theorem has important consequences. If $A \in \mathfrak{A}(\mathcal{O})$ and $\pi(A)\Omega = 0$, then $\pi(A) = 0$ since Ω is separating. Furthermore, if π is faithful, then $A = 0$. In other words, no local observable can annihilate the vacuum. Consequently, we cannot think of, say, the charge for a region $\mathcal{O}$ as an element of $\mathfrak{A}(\mathcal{O})$ since, presumably, such an observable should give zero on the vacuum in which case it would have to be zero itself. (One might argue that the charge should be an unbounded operator and so could not belong to any $\mathfrak{A}(\mathcal{O})$ anyway. However, one could consider instead some bounded function, say, Qe^{-Q^2} of the charge Q, which one might believe still should annihilate the vacuum.)

In the same way, one has difficulty in formulating the notion of a particle detector (see Haag (1972)). Intuitively, such a detector should correspond to an observable C such that

(i) $C = C^* = C^2$, that is, C is a projection and therefore reports "yes" or "no".

(ii) $C \in \mathfrak{A}(\mathcal{O})$, for some $\mathcal{O}$.

(iii) $C\Omega = 0$, "no" on the vacuum.

We see that, by the Reeh-Schlieder theorem, the only possibility is $C = 0$.

Chapter 6

The charged Bose field and its sectors

In this chapter, we will discuss in some detail the free charged Bose field and its charge sectors. We will see that these give rise to inequivalent, irreducible but physically equivalent representations of the algebra of observables. The observables are defined as the gauge invariant elements of the field algebra.

Our first objective will be to give a precise formulation of the charged field.

Definition of the charged field

The charged field can be thought of as a pair of fields representing "particle" and "antiparticle", respectively. We choose the particle to carry a charge $+1$ and the antiparticle a charge -1. The Fock space on which the charged field acts should contain vectors of all charges.

We recall the formalism for the free neutral field. It acts on $\mathcal{F}$, the symmetric Fock space over $L^2(\mathbb{R}^3, d^3k)$. The creation and annihilation operators $a^*(f)$ and $a(g)$ are defined as usual, for example,

$$a(g)\,\Psi(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n) = \sqrt{n}\int g(\boldsymbol{k})\,\Psi(\boldsymbol{k},\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\,d\boldsymbol{k}.$$

The free neutral scalar field of mass m is the operator-valued distribution

$$\phi(\boldsymbol{x},t) = (2\pi)^{-3/2}\int \big(e^{i(k,x)}\,a^*(\boldsymbol{k}) + e^{-i(k,x)}\,a(\boldsymbol{k})\big)\,\frac{d\boldsymbol{k}}{\sqrt{2\omega(\boldsymbol{k})}}$$

where $x=(t,\boldsymbol{x})$, $k=(k^0,\boldsymbol{k})$, and the pairing (k,x) is the Minkowski pairing given by $(k,x)=k^0t-\boldsymbol{k}.\boldsymbol{x}$ with $k^0=\omega(\boldsymbol{k})=\sqrt{\boldsymbol{k}^2+m^2}$. In smeared form, this becomes, for $f\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$,

$$\phi(f) = 2^{-1/2}\big(a^*(F)+a(\overline{F})\big)$$

where $F(\boldsymbol{k}) = \sqrt{\frac{2\pi}{\omega(\boldsymbol{k})}}\,\widehat{f}(\omega(\boldsymbol{k}),\boldsymbol{k})$ and $\widehat{f}(p) = (2\pi)^{-1/2}\int e^{i(p,x)}\,f(x)\,d^4x$.

Let $\mathcal{F}^+$ and $\mathcal{F}^-$ be two distinguished Fock spaces over $L^2(\mathbb{R}^3, d^3k)$. The Fock space for the charged field is

$$\mathcal{K} = \mathcal{F}^+ \otimes \mathcal{F}^- .$$

Let $a^*_\pm(\,\cdot\,)$ and $a_\pm(\,\cdot\,)$ be the creation and annihilation operators on $\mathcal{F}^\pm$, respectively. We interpret $a^*_+(\,\cdot\,)\otimes \mathbb{1}$ as the operator in $\mathcal{K}$ creating a particle wiht charge $+1$ and $a_+(\,\cdot\,)\otimes \mathbb{1}$ as that destroying a particle with charge $+1$. Similarly, we interpret $\mathbb{1}\otimes a^*_-(\,\cdot\,)$ and $\mathbb{1}\otimes a_-(\,\cdot\,)$ as creating and destroying, respectively, charge -1.

Let $D^\pm$ denote the set of finite-particle vectors in $\mathcal{F}^\pm$. For $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, we define the charged field $\phi(f)$ on the domain $D = D^+ \otimes D^-$ to be

$$\phi(f) = 2^{-1/2}\big(a^*_+(F)\otimes \mathbb{1} + \mathbb{1}\otimes a_-(\overline{F})\big)$$

where $F(\boldsymbol{k}) = \sqrt{\frac{2\pi}{\omega(\boldsymbol{k})}}\,\widehat{f}(\omega(\boldsymbol{k}), \boldsymbol{k})$.

Its "complex conjugate" $\phi_c(f)$ is defined on $D = D^+ \otimes D^-$ as

$$\phi_c(f) = 2^{-1/2}\big(a_+(\overline{F})\otimes \mathbb{1} + \mathbb{1}\otimes a^*_-(F)\big)\,.$$

Clearly, $\phi(f)^*$, the adjoint of $\phi(f)$ is an extension of $\phi_c(f)$. We will define $\phi(f)$ and $\phi_c(f)$ on bigger domains so that they become adjoints of each other (the notation ϕ_c is temporary).

Note that $\mathcal{K}$ is spanned by vectors of the form

$$a^*_+(h_1)\ldots a^*_+(h_n)\Omega_+ \otimes a^*_-(g_1)\ldots a^*_-(g_m)\Omega_-$$

with each h_i and g_j in $L^2(\mathbb{R}^3, d^3k)$. If we write $\Omega = \Omega_+ \otimes \Omega_-$, then these vectors are of the form $a^*_+(h_1)\ldots a^*_+(h_n)\otimes a^*_-(g_1)\ldots a^*_-(g_m)\,\Omega$. The vector Ω is cyclic for the $a^*_+(\,\cdot\,)\otimes \mathbb{1}$ and $\mathbb{1}\otimes a^*_-(\,\cdot\,)$.

Definition 6.1. Let $N^\pm$ be the number operators on $\mathcal{F}^\pm$, respectively. The number operator on $\mathcal{K}$ is

$$N = N^+ \otimes \mathbb{1} + \mathbb{1}\otimes N^-\,.$$

The total charge operator on $\mathcal{K}$ is

$$Q = N^+ \otimes \mathbb{1} - \mathbb{1}\otimes N^-\,.$$

Evidently, D is a domain of analytic vectors for N and Q on which they are therefore essentially self-adjoint. N has eigenvalues $0, 1, 2, \ldots$, whereas Q has eigenvalues $0, \pm 1, \pm 2, \ldots$.

Clearly, $a^*_+(h_1)\ldots a^*_+(h_n)\otimes a^*_-(g_1)\ldots a^*_-(g_m)\,\Omega$ is an eigenvector of both N and Q with corresponding eigenvalues $n+m$ and $n-m$, respectively. We also see that $\phi(f)$ creates a charge of $+1$ and destroys a charge of -1. Accordingly, we say that $\phi(f)$ carries charge $+1$. Similarly, $\phi_c(f)$ carries charge -1.

Proposition 6.2. *Let $\Psi \in D$ and suppose that Ψ contains no more than n particles. Then, for any $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$,*

$$\|(\phi(f) \pm \phi_c(f))\,\Psi\| \le 2\sqrt{2}\,\sqrt{n+1}\,\|F\|_2\,\|\Psi\|$$

where $F(\boldsymbol{k}) = \sqrt{\frac{2\pi}{\omega(\boldsymbol{k})}}\,\widehat{f}(\omega(\boldsymbol{k}), \boldsymbol{k})$.

Proof. We show that

$$\|(a_+^*(G) \otimes \mathbb{1})\,\Psi\| \le \sqrt{n+1}\,\|G\|_2\,\|\Psi\|$$

and

$$\|(a_+(G) \otimes \mathbb{1})\,\Psi\| \le \sqrt{n+1}\,\|G\|_2\,\|\Psi\|$$

for any $G \in L^2(\mathbb{R}^3, d^3k)$. The same proof holds for $\mathbb{1} \otimes a_-^*$ and $\mathbb{1} \otimes a_-$.

For the first inequality, let $\mathcal{H}_m^\pm = \mathcal{F}_0^\pm \oplus \mathcal{F}_1^\pm \oplus \cdots \oplus \mathcal{F}_n^\pm$ be the subspace of $\mathcal{F}^\pm$ containing n or fewer particles. Then we have

$$\begin{aligned}\|a_+^*(G) \otimes \mathbb{1}\,\Psi\|^2 &= \langle\, \Psi, (a_+^*(G) \otimes \mathbb{1})^*(a_+^*(G) \otimes \mathbb{1})\,\Psi\,\rangle \\ &= \langle\, \Psi, (a_+(\overline{G})\,a_+^*(G) \otimes \mathbb{1})\,\Psi\,\rangle\,.\end{aligned}$$

By hypothesis, Ψ has at most n particles and so $\Psi \in \mathcal{H}_n^+ \otimes \mathcal{H}_n^-$. Moreover, $a_+(\overline{G})\,a_+^*(G)$ is a bounded self-adjoint operator from $\mathcal{H}_n^+$ into $\mathcal{H}_n^+$ with norm not greater than $(n+1)\|G\|_2^2$.

Similarly, $a_+^*(\overline{G})\,a_+(G)$ is bounded from $\mathcal{H}_n^+ \to \mathcal{H}_n^+$, with norm not greater than $n\,\|G\|_2^2$. The proof is therefore complete once we have proved the following lemma. ∎

Lemma 6.3. *Suppose that $A = A^*$ is a bounded operator on a Hilbert space $\mathcal{H}_1$. Then for any Hilbert space $\mathcal{H}_2$, $A \otimes \mathbb{1}$ is bounded on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\|A \otimes \mathbb{1}\| = \|A\|$.*

Proof. By the spectral theorem, there are measure spaces (X, μ) and $(Y\nu)$ such that $\mathcal{H}_1 \simeq L^2(X, \mu)$, $\mathcal{H}_2 \simeq L^2(Y, \nu)$ and $A \simeq A(x) \in L^\infty(X, \mu)$. Then $\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq L^2(X \times Y, \mu \otimes \nu)$. For any $z \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we have

$$\begin{aligned}\|A \otimes \mathbb{1}\,z\|^2 &= \iint_{X \times Y} |A(x)z(x,y)|^2\,d\mu(x)\,d\nu(y) \\ &\le \|A\|_\infty^2\,\|z\|^2.\end{aligned}$$

But $\|A\|_\infty = \|A\|$ and so $\|A \otimes \mathbb{1}\| \le \|A\|$. Taking z of the form $z(x,y) = z_1(x)z_2(y)$, we see that equality $\|A \otimes \mathbb{1}\| = \|A\|$ holds. ∎

This completes the proof of proposition 6.2.

Proposition 6.4. *For any given $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, the operators $\phi(f) + \phi_c(f)$ and $i(\phi(f) - \phi_c(f))$ are essentially self-adjoint on D.*

Proof. By Proposition 6.2, D is a domain of entire vectors for $\phi(f) \pm \phi_c(f)$ and so the result follows. ■

(For a discussion of analytic vectors, essential self-adjointness etc, see, for example, Simon (1972)).

Notation.

Let us denote by $\xi(f)$ the self-adjoint operator $\frac{1}{2}(\phi(f) + \phi_c(f))^*$ and by $\eta(f)$ the self-adjoint operator $-\frac{i}{2}(\phi(f) - \phi_c(f))^*$).

Proposition 6.5. *For any $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, the one-parameter unitary groups generated by $\xi(f)$ and $\eta(f)$ commute.*

Proof. The self-adjoint operators $\xi(f)$ and $\eta(f)$ commute on D. Taking expectation values in elements of D, we can write the unitary groups $e^{it\xi(f)}$ and $e^{it\eta(f)}$ as exponential power series, since D is a domain of entire vectors. By the first remark, the unitaries commute on D. Since D is dense in $\mathcal{K}$, the result follows. ■

Notice that on D, $\phi(f) = \xi(f) + i\eta(f)$ and $\phi_c(f) = \xi(f) - i\eta(f)$. We can now give a precise definition of $\phi(f)$ and its conjugate.

Definition 6.6. For any $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, the charged field $\phi(f)$ and its conjugate field, $\phi^*(f)$, are the operators on $\mathcal{K}$ with domain

$$D(\phi(f)) = D(\phi^*(f)) = D(\xi(f)) \cap D(\eta(f))$$

and action

$$\phi(f) = \xi(f) + i\eta(f) \quad \text{and} \quad \phi^*(f) = \xi(f) - i\eta(f)\,.$$

Theorem 6.7. *For any $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$, the operators $\phi(f)$ and $\phi^*(f)$ are normal operators and are mutual adjoints.*

Proof. By proposition 6.5 and the spectral theorem, there is a measure space (X, μ) such that $\mathcal{K} \simeq L^2(X, \mu)$ and such that $\xi(f)$ and $\eta(f)$ are equivalent to multiplication operators by real measurable functions. Let us denote these also by ξ and η. Then the operator $(D(\phi(f)), \phi(f))$ is equivalent to the operator $(D(\xi) \cap D(\eta), \xi + i\eta)$ and the operator $(D(\phi^*(f)), \phi^*(f))$ is equivalent to $(D(\xi) \cap D(\eta), \xi - i\eta)$.

But $D(\xi) \cap D(\eta)$ is the set

$$\begin{aligned} D(\xi) \cap D(\eta) &= \{\, u \in L^2(X, \mu) : (\xi + i\eta)u \in L^2(X, \mu) \,\} \\ &= \{\, u \in L^2(X, \mu) : (\xi - i\eta)u \in L^2(X, \mu) \,\}\,. \end{aligned}$$

That is, $D(\xi) \cap D(\eta)$ is the domain of the multiplication operators (by the complex functions) $\xi \pm i\eta$. These operators are normal and mutual adjoints. Such properties are preserved under unitary equivalence. ■

Remark 6.8. This justifies the notation $\phi^*(f)$.

We will need the following technical result.

Theorem 6.9. *Let* $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$ *with* $f \not\equiv 0$. *Then* $\ker \xi(f) = \{0\}$, *that is,* $\xi(f)\Psi = 0$ *if and only if* $\Psi = 0$.

Proof. Fix $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$ with $f \not\equiv 0$. Define the function G on $\mathbb{R}^3$ by $G(\boldsymbol{k}) = F(\boldsymbol{k})/2\|F\|_2^2$, where $F(\boldsymbol{k}) = \sqrt{\frac{2\pi}{\omega(\boldsymbol{k})}}\, \widehat{f}(\omega(\boldsymbol{k}), \boldsymbol{k})$.

Let $\phi_\pm$ denote the time-zero free neutral field on $\mathcal{F}^\pm$ and let $\pi_\pm$ denote the time-zero conjugate momentum fields. Set

$$B = 2^{-1/2}\big(\pi_+(G) \otimes \mathbb{1} + \mathbb{1} \otimes \pi_-(G)\big)$$

on D. Then, as earlier, we note that D is a domain of entire vectors for B, so that B is essentially self-adjoint on D. Moreover, on D, we find that

$$[\xi(f), B] = i\,.$$

Because D is a domain of entire vectors for both B and $\xi(f)$, we are able to exponentiate this relation to conclude that the unitaries $U(s) = e^{is\xi(f)}$ and $V(t) = e^{it\overline{B}}$ give a representation of the Weyl relation for one degree of freedom. By von Neumann's uniqueness theorem, it follows that we may write $\mathcal{K} \simeq \bigoplus_{j\in I} L^2(\mathbb{R}_j, dx_j)$ and $U(s) \simeq \bigoplus_{j\in I} e^{isx_j}$ for some index set I.

Now, to say that $\xi(f)\Psi = 0$, for some non-zero element Ψ in the domain of $\xi(f)$, is to say that $U(s)\Psi = \Psi$ for all $s \in \mathbb{R}$. This would mean that 1 is an eigenvalue of $U(s)$ and so 1 would be an eigenvalue of e^{isx_j}, which is false. ∎

Let $V : L^2(\mathbb{R}^3, d^3k) \to L^2(\mathbb{R}^3, d^3k)$ be a "one-particle operator". There is a natural action, $\Gamma^\pm(V)$ on $\mathcal{F}^\pm$ induced by that of V given by linear extension of

$$\Gamma^\pm(V) \restriction \mathcal{F}_n^\pm = \otimes^n V\,,$$

that is, $\Gamma^\pm(V) : \mathcal{F}^\pm \to \mathcal{F}^\pm$ is given by

$$\Gamma^\pm(V) = \mathbb{1} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

If $\|V\| \le 1$, then $\|\Gamma^\pm(V)\| \le 1$, otherwise $\Gamma^\pm(V)$ is unbounded. Moreover, if V is unitary, then so is $\Gamma^\pm(V)$. In this case, the operator $\Gamma^+(V) \otimes \Gamma^-(V)$ defined on $\mathcal{K}$ is also unitary.

For any Poincaré transformation $(a, \Lambda) \in \mathcal{P}_+^\uparrow$, we define its action on $L^2(\mathbb{R}^3, d^3k)$ by

$$u(a,\Lambda) : \Psi(\boldsymbol{k}) \mapsto e^{i(a,k)}\, \frac{\omega(\underline{\Lambda^{-1}k})^{1/2}}{\omega(\boldsymbol{k})^{1/2}}\, \Psi(\underline{\Lambda^{-1}k})\, \Big|_{k^0=\omega(\boldsymbol{k})}$$

where $(a,k) = a^0k^0 - \boldsymbol{a}.\boldsymbol{k}$ and $\underline{\Lambda^{-1}k}$ is the spatial component of the 4-vector $\Lambda^{-1}k$ with $k = (\omega(\boldsymbol{k}), \boldsymbol{k})$.

One verifies that $u(a,\Lambda)$ is a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$ in $L^2(\mathbb{R}^3, d^3k)$. In $\mathcal{K}$, the action of $\mathcal{P}_+^\uparrow$ is defined as

$$(a,\Lambda) \mapsto \Gamma^+(u(a,\Lambda)) \otimes \Gamma^-(u(a,\Lambda)) \equiv U(a,\Lambda)\,.$$

Proposition 6.10. *For any $f \in \mathcal{S}_\mathbb{R}(\mathbb{R}^4)$ and any given $(a,\Lambda) \in \mathcal{P}_+^\uparrow$, let $f_{a,\Lambda}(x)$ be the function $f_{a,\Lambda}(x) = f(\Lambda^{-1}(x-a))$. Then, on D, we have*

$$U(a,\Lambda)\,\phi^\#(f)\,U(a,\Lambda)^{-1} = \phi^\#(f_{a,\Lambda})$$

where $\phi^\#(f)$ stands for either $\phi(f)$ or $\phi^*(f)$.

Proof. The proof is straightforward by direct calculation. ∎

The field algebra

We wish to construct local algebras associated with the fields ϕ and ϕ^*. However, these are not self-adjoint, so we must use their real and imaginary parts.

Definition 6.11. For any given region $\mathcal{O}$, the local field algebra $\mathfrak{F}(\mathcal{O})$ is the von Neumann algebra generated by the unitary operators with generators $\xi(f)$ and $\eta(f)$, as f runs over $\mathcal{D}_\mathbb{R}(\mathcal{O})$ (the subset of $\mathcal{S}_\mathbb{R}(\mathbb{R}^4)$ whose elements have support in the region $\mathcal{O}$). The field algebra $\mathfrak{F}$ is the norm closure of the union of all the $\mathfrak{F}(\mathcal{O})$ as $\mathcal{O}$ runs over all regions in Minkowski space $\mathcal{M}$.

Evidently, if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathfrak{F}(\mathcal{O}_1) \subseteq \mathfrak{F}(\mathcal{O}_2)$. One can check that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated regions, then $\mathfrak{F}(\mathcal{O}_1)$ and $\mathfrak{F}(\mathcal{O}_2)$ commute. Furthermore, the map $A \mapsto \alpha(a,\Lambda)(A) = U(a,\Lambda)\,A\,U(a,\Lambda)^*$ defines an automorphism of $\mathcal{B}(\mathcal{K})$ which satisfies

$$\alpha(a,\Lambda)\,\mathfrak{F}(\mathcal{O}) = \mathfrak{F}(\Lambda\mathcal{O} + a)\,.$$

Clearly $\alpha(a,\Lambda) : \mathfrak{F} \to \mathfrak{F}$.

We should point out that there is another localization which is also Poincaré covariant but which is anti-local with respect to the one set-up here (Wilde (1971)).

Theorem 6.12. *The algebra $\mathfrak{F}$ is irreducible.*

Proof. We will only sketch the proof. Let $\dot{\phi}$ be the operator-valued distribution obtained from ϕ by taking its time-derivative. For $f \in \mathcal{S}_\mathbb{R}(\mathbb{R}^4)$, let $f_1 \in \mathcal{S}_\mathbb{R}(\mathbb{R}^4)$ satisfy

$$\widehat{f}_1(\omega(\boldsymbol{k}), \boldsymbol{k}) = \widehat{f}(\omega(\boldsymbol{k}), \boldsymbol{k})/\omega(\boldsymbol{k})$$

Using the fact that D is a domain of entire vectors, one shows that the unitary operators $\exp i(\phi(f) + \phi^*(f))^-$ and $\exp(\dot\phi(f_1) - \dot\phi^*(f_1))^-$ commute and that

$$e^{i\phi_+(f)} \otimes \mathbb{1} = e^{\frac{i}{2}(\phi(f)+\phi^*(f))^-} e^{\frac{1}{2}(\dot\phi(f_1)-\dot\phi^*(f_1))^-}.$$

Since D is a core for the self-adjoint operators involved and since $\dot\phi(f_1)$ is a limit of operators of the form $\frac{1}{t}(\phi(g_t) - \phi(g))$, the semigroup convergence theorem implies that $\mathfrak{F}''$ contains all the operators of the form $\{\, e^{i\phi_+(f)} \otimes \mathbb{1} \,\}$ and $\{\, \mathbb{1} \otimes e^{i\phi_-(f)} \,\}$ with $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$.

Let $\mathcal{A} = \{\, e^{i\phi_+(f)} : f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4) \,\}''$ and let $\mathcal{B} = \{\, e^{i\phi_-(f)} : f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4) \,\}''$ (where the commutants are taken in $\mathcal{B}(\mathcal{F}^\pm)$, respectively). Then $\mathfrak{F}'' \supseteq \mathcal{A} \otimes \mathcal{B}$. But $\phi_\pm$ act irreducibly on $\mathcal{F}^\pm$, that is, $\mathcal{A} = \mathcal{B}(\mathcal{F}^+)$ and $\mathcal{B} = \mathcal{B}(\mathcal{F}^-)$. It follows that $\mathfrak{F}' = \mathbb{C}\mathbb{1}$, that is, $\mathfrak{F}$ is irreducible. ■

Gauge transformations and the observables

For any $0 \le \theta < 2\pi$, the maps $g \mapsto e^{\pm i\theta} g$ are unitary multiplication operators on the one-particle space $L^2(\mathbb{R}^3, d^3k)$. These give rise to unitary operators $U(\theta)$ on $\mathcal{K}$ by setting

$$U(\theta) = \Gamma^+(e^{i\theta}) \otimes \Gamma^-(e^{-i\theta})\,.$$

The mapping $\theta \mapsto U(\theta)$ defines a strongly continuous representation of the torus, $\mathbb{T}$, on $\mathcal{K}$.

Proposition 6.13. *For any $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$ and $0 \le \theta < 2\pi$, we have*

$$U(\theta)\,\phi(f)\,U(\theta)^* = e^{i\theta}\,\phi(f) \quad \text{and} \quad U(\theta)\,\phi^*(f)\,U(\theta)^* = e^{-i\theta}\,\phi^*(f)$$

on the dense domain D in $\mathcal{K}$.

Proof. The proof is a straightforward verification. ■

Definition 6.14. The transformation $\phi(f) \mapsto e^{i\theta}\phi(f)$, $\phi^*(f) \mapsto e^{-i\theta}\phi^*(f)$ is called a gauge transformation of the first kind. The gauge group is the torus.

We note that the generator of $U(\theta)$ is nothing other than Q, the charge operator.

Proposition 6.15. *For any region $\mathcal{O}$ in $\mathcal{M}$, $U(\theta)\mathfrak{F}(\mathcal{O})U(\theta)^* = \mathfrak{F}(\mathcal{O})$, for all $0 \le \theta < 2\pi$.*

Proof. On the domain D, we have

$$U(\theta)\,\xi(f)\,U(\theta)^* = 2\cos\theta\,\xi(f) \quad \text{and} \quad U(\theta)\,\eta(f)\,U(\theta)^* = 2\sin\theta\,\eta(f)\,.$$

The result follows on exponentiation. ■

According to Lagrangian field theory, gauge transformations should have no physical consequences. That is to say, the observables should be invariant under a gauge transformation. This suggests the following definition.

Definition 6.16. The local observables associated with any given region $\mathcal{O}$ in Minkowski space $\mathcal{M}$ are the elements

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{F}(\mathcal{O}) \cap \{\, U(\theta) : 0 \leq \theta < 2\pi \,\}' .$$

In other words, $\mathfrak{A}(\mathcal{O})$ is the subalgebra of elements $A \in \mathfrak{F}(\mathcal{O})$ which commute with the $U(\theta)$s, $A = U(\theta) A U(\theta)^*$.

The quasilocal algebra $\mathfrak{A}$ is defined to be the C^*-algebra generated by the local algebras $\mathfrak{A}(\mathcal{O})$ as $\mathcal{O}$ runs over regions in Minkowski space $\mathcal{M}$.

Proposition 6.17. *For each region $\mathcal{O}$, $\mathfrak{A}$ is a von Neumann algebra. The algebras $\{\, \mathfrak{A}, \mathfrak{A}(\mathcal{O}) \,\}$ satisfy the Haag-Kastler axioms, Postulates 1–4, where $\alpha(a, \Lambda)$ is given by $\alpha(a,\Lambda)(A) = U(a,\Lambda)\, A\, U(a,\Lambda)^*$ for $(a,\Lambda) \in \mathcal{P}_+^\uparrow$.*

Proof. It is clear that each $\mathfrak{A}(\mathcal{O})$ is a weakly closed $*$-algebra containing $\mathbb{1}$, so it is a von Neumann algebra.

Postulates 1,2 and 3 hold because they hold for the $\mathfrak{F}(\mathcal{O})$. Postulate 4 holds because it holds for the $\mathfrak{F}(\mathcal{O})$ and the $U(\theta)$ commute with each $U(a, \Lambda)$. ∎

The charge sectors

As we have already noted, the charge operator Q is the generator of the group $U(\theta)$ and has eigenvalues $0, \pm 1, \pm 2, \ldots$. Let $\mathcal{K}_q$ be the subspace of $\mathcal{K}$ corresponding to charge q. Then

$$\mathcal{K} = \bigoplus_{q=-\infty}^{\infty} \mathcal{K}_q .$$

Since $\mathfrak{A}$ commutes with each $U(\theta)$, we see that $\mathfrak{A}$ maps each $\mathcal{K}_q$ into itself. Therefore we can define a representation $(\mathcal{K}_q, \pi_q)$ of $\mathfrak{A}$ by

$$\pi_q \mathfrak{A} = \mathfrak{A} \restriction \mathcal{K}_q .$$

Definition 6.18. The representations $(\mathcal{K}_q, \pi_q)$, for $-\infty < q < \infty$, are called the charge sectors of the charged field.

Suppose that we are given a vector belonging to $\mathcal{K}_q$. This determines a state on $\mathfrak{A}$. If we were to add some charge to this state, but in a very remote region of space, then we would not expect this to make very much difference as far as local observables are concerned. In other words, we might expect that the various charge representations are physically equivalent. This is the "particle behind the moon" argument of Haag and Kastler. Observations within our laboratory ought not to be too sensitive as to whether there is or is not an extra charged particle sitting somewhere behind the moon.

Before pursuing this, we first need two lemmas.

Lemma 6.19. *Suppose that* $A \in \mathfrak{A}(\mathcal{O})$ *and* $f \in \mathcal{D}_{\mathbb{R}}(\mathcal{O}_1)$*, where* $\mathcal{O}$ *and* $\mathcal{O}_1$ *are space-like separated. Then for any* $z, z' \in D(\phi(f)) = D(\phi^*(f))$*, we have*

$$\langle\, z, A\,\phi(f)\, z'\,\rangle = \langle\, \phi^*(f)\, z, Az'\,\rangle\,,$$

that is, $\phi(f)$ *and* A *weakly commute on* $D(\phi(f))$*.*

Proof. We know that both $e^{is\xi(f)}$ and $e^{it\eta(f)}$ commute with A for all $s, t \in \mathbb{R}$. Hence

$$\langle\, e^{-is\xi(f)z}, Az'\,\rangle = \langle\, z, A\, e^{is\xi(f)z'}\,\rangle$$

and

$$\langle\, e^{-it\eta(f)z}, Az'\,\rangle = \langle\, z, A\, e^{it\eta(f)z'}\,\rangle\,.$$

The result follows by taking derivatives with respect to s and t at $s = t = 0$ and adding the resulting equations. ∎

Lemma 6.20. *Suppose that* $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$ *and let* $f_{\boldsymbol{a}}$ *be the space-translate of* f*, that is,* $f_{\boldsymbol{a}}(t, \boldsymbol{x}) = f(t, \boldsymbol{x} - \boldsymbol{a})$*. Suppose further that* $\|F(\boldsymbol{k})\|_2^2 = 2$*, where* $F(\boldsymbol{k}) = (2\pi)^{1/2}\omega(\boldsymbol{k})^{-1/2}\widehat{f}(\omega(\boldsymbol{k}), \boldsymbol{k})$*. Then* $\phi^*(f_{\boldsymbol{a}})\phi(f_{\boldsymbol{a}})\, z$ *converges weakly to* z *as* $|\boldsymbol{a}| \to \infty$*, for any* $z \in D$*.*

Proof. Fix $z \in D$. By proposition 6.10 (unitary implementation of Poincaré transformations), it follows that the collection $\phi^*(f_{\boldsymbol{a}})\phi(f_{\boldsymbol{a}})z$ is bounded uniformly in $\boldsymbol{a}$, so it is enough to show that

$$\langle\, z', \phi^*(f_{\boldsymbol{a}})\phi(f_{\boldsymbol{a}})z\,\rangle \to \langle\, z', z\,\rangle \quad \text{as } |\boldsymbol{a}| \to \infty$$

for all z' in some dense set in $\mathcal{K}$. We choose $z' \in D$.

Writing ϕ^* and ϕ in terms of creation and annihilation operators, we find that

$$\begin{aligned} 2\,\langle\, z', \phi^*(f_{\boldsymbol{a}})\phi(f_{\boldsymbol{a}})z\,\rangle = \langle\, z', a_+(\overline{F_{\boldsymbol{a}}})\, a_+^*(F_{\boldsymbol{a}}) \otimes \mathbb{1}\, z\,\rangle \\ + \langle\, z', a_+(\overline{F_{\boldsymbol{a}}}) \otimes a_-(\overline{F_{\boldsymbol{a}}})\, z\,\rangle \\ + \langle\, z', a_+^*(F_{\boldsymbol{a}}) \otimes a_-^*(F_{\boldsymbol{a}})\, z\,\rangle \\ + \langle\, z', \mathbb{1} \otimes a_-^*(F_{\boldsymbol{a}})\, a_-(\overline{F_{\boldsymbol{a}}})\, z\,\rangle \end{aligned}$$

where $F_{\boldsymbol{a}}(\boldsymbol{k}) = e^{-i\boldsymbol{k}.\boldsymbol{a}}\, F(\boldsymbol{k})$. The second, third and fourth terms all converge to zero, as $|\boldsymbol{a}| \to \infty$, by the Riemann-Lebesgue Lemma (because they each contain a term of the form $a_\pm(F_{\boldsymbol{a}})z$ or $a_\pm(F_{\boldsymbol{a}})z'$). The first term can be written as

$$\langle\, z', a_+^*(F_{\boldsymbol{a}})\, a_+(\overline{F_{\boldsymbol{a}}}) \otimes \mathbb{1}\, z\,\rangle + \langle\, z', z\,\rangle \int F_{\boldsymbol{a}}(\boldsymbol{k})\, \overline{F_{\boldsymbol{a}}}(\boldsymbol{k})\, d^3k\,,$$

using the commutation relations. Once again, the first term converges to zero by the Riemann-Lebesgue Lemma. The last term is equal to $2\langle\, z', z\,\rangle$ because of our normalization $\|F\|_2^2 = 2$. ∎

Theorem 6.21. *Suppose that ω is a vector state on $\mathfrak{A}$ in the representation $(\mathcal{K}_q, \pi_q)$. Then any w^*-neighbourhood of ω contains a vector state ρ in the representation $(\mathcal{K}_{q'}, \pi_{q'})$, for any q, q'. In particular, the charge sectors are physically equivalent.*

Proof. By a $|q - q'|\,\varepsilon$-argument, it is enough to consider $q' = q + 1$. Let ω be a vector state on $\mathfrak{A}$ in the representation $(\mathcal{K}_q, \pi_q)$, that is, ω has the form

$$\omega(\,\cdot\,) = \langle\, z, \pi_q(\,\cdot\,)\, z\,\rangle$$

for some $z \in \mathcal{K}_q$ with $\|z\| = 1$. Let $\mathcal{N}(\omega; A_1, \dots, A_p, \varepsilon)$ be a w^*-neighbourhood of ω,

$$\mathcal{N}(\omega; A_1, \dots, A_p, \varepsilon) = \{\, \omega' \in \mathfrak{A}_1^{*+} : \big|\omega'(A_j) - \omega(A_j)\big| < \varepsilon,\ j = 1, 2, \dots, p\,\}\,.$$

We can choose $h \in D \cap \mathcal{K}_q$ with $\|h\| = 1$ such that $\omega'(\,\cdot\,) = \langle\, h, \pi_q(\,\cdot\,)\, h\,\rangle$ belongs to $\mathcal{N}(\omega; A_1, \dots, A_p, \frac{1}{2}\varepsilon)$. This is possible because $D \cap \mathcal{K}_q$ is dense in $\mathcal{K}_q$ and p is finite.

Suppose, for the moment, that $A_1, \dots, A_p \in \mathfrak{A}(\mathcal{O})$ for some local region $\mathcal{O}$. We define a positive linear functional $\rho_{\boldsymbol{a}}$ on $\mathfrak{A}$ by

$$\rho_{\boldsymbol{a}}(\,\cdot\,) = \langle\, \phi(f_{\boldsymbol{a}})h, \pi_{q+1}(\,\cdot\,)\phi(f_{\boldsymbol{a}})h\,\rangle$$

where $f \in \mathcal{D}_{\mathbb{R}}(\mathcal{O}_1)$, for some region $\mathcal{O}_1$, and where f obeys the normalization as in lemma 6.20. Now, by the weak commutativity, Lemma 6.19, we see that each $\rho_{\boldsymbol{a}}(A_j)$ can be written as

$$\rho_{\boldsymbol{a}}(A_j) = \langle\, \phi^*(f_{\boldsymbol{a}})\phi(f_{\boldsymbol{a}})\, h, \pi_q(A_j)\, h\,\rangle$$

for $j = 1, 2, \dots, p$, provided $|\boldsymbol{a}|$ is sufficiently large. By Lemma 6.20, it follows that $\rho_{\boldsymbol{a}}(X) \to \omega'(X)$, as $|\boldsymbol{a}| \to \infty$, for $X = \mathbb{1}, A_1, A_2, \dots, A_p$.

In other words, for all sufficiently large $|\boldsymbol{a}|$, the vector state $\sigma(\,\cdot\,) = \rho_{\boldsymbol{a}}(\,\cdot\,)/\rho_{\boldsymbol{a}}(\mathbb{1})$ belongs to the w^*-neighbourhood $\mathcal{N}(\omega'; A_1, \dots, A_p, \frac{1}{2}\varepsilon)$ and therefore σ belongs to $\mathcal{N}(\omega; A_1, \dots, A_p, \varepsilon)$. σ is a vector state in the representation $(\mathcal{K}_{q+1}, \pi_{q+1})$.

It remains to remove the restriction that $A_1, \dots, A_p \in \mathfrak{A}(\mathcal{O})$. This requires yet further approximation. Let $A_1, \dots, A_p \in \mathfrak{A}$ be given. By definition of $\mathfrak{A}$, there is some region $\mathcal{O}$ and elements $A_1', A_2', \dots, A_p' \in \mathfrak{A}(\mathcal{O})$ such that $\|A_j - A_j'\| < \varepsilon$ for all $1 \le j \le p$. Thus, given ω, we construct a vector state σ, as above, which belongs to the w^*-neighbourhood $\mathcal{N}(\omega; A_1', \dots, A_p', \varepsilon)$. But then, for each $1 \le j \le p$,

$$\begin{aligned}|\omega(A_j) - \sigma(A_j)| &\le \big|\omega(A_j') - \sigma(A_j')\big| + 2\,\|A_j' - A_j\| \\ &\le 3\varepsilon\,.\end{aligned}$$

That is, $\sigma \in \mathcal{N}(\omega; A_1, \dots, A_p, 3\varepsilon)$ and the result follows. ∎

Remark 6.22. It has been shown by Fell that two representations of a C^*-algebra are physically equivalent if and only if they have the same kernel. Using this result, we see that π_q and $\pi_{q'}$ have the same kernel. But then $\bigoplus_q \pi_q$ has this same kernel which is zero since it is the identity representation of $\mathfrak{A}$. We deduce that each $(\mathcal{K}_q, \pi_q)$ is faithful. We have therefore proved the following.

Corollary 6.23. *The representations $(\mathcal{K}_q, \pi_q)$ are faithful representations of the quasilocal algebra $\mathfrak{A}$.*

The faithfulness of these representations also follows from the strong local equivalence of the sectors (see the next section). We turn to a discussion of the irreducibility of the charge representations. Following Doplicher, Haag and Roberts, we use the notion of a "mean". Recall that the gauge group here is the torus, $\mathbb{T}$, which is represented on $\mathcal{K}$ by $U(\theta)$.

Definition 6.24. The mean of an operator $X \in \mathcal{B}(\mathcal{K})$ with respect to the unitary representation U of the gauge group $\mathbb{T}$ is the operator $m(X)$ given by

$$m(X) = \int_{\mathbb{T}} U(\theta)\, X\, U(\theta)^*\, d\theta$$

where the integral is a weak integral in $\mathcal{B}(\mathcal{K})$. Note that $\int_{\mathbb{T}} \cdot\, d\theta$ is the normalized integral over $\mathbb{T}$.

Lemma 6.25. *The following hold:*

(i) $U(\theta)m(X)U(\theta)^* = m(U(\theta)XU(\theta)^*) = m(X)$ *for any* $\theta \in \mathbb{T}$*;*

(ii) $m : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ *is weakly continuous on bounded sets.*

Proof. (i) is straightforward. For (ii), we offer an explicit alternative proof to that of Doplicher, Haag and Roberts (1969). Suppose that the net (X_ν) is bounded and converges weakly to X. For each ν, set $A_\nu = X - X_\nu$. We wish to show that $m(X_\nu) \to 0$ weakly. Let $z, z' \in \mathcal{K}$ be given. Then, for any fixed $\alpha \in \mathbb{T}$, $\langle\, z', U(\alpha)\, A_\nu\, U(\alpha)^*\, z\,\rangle \to 0$. Hence, for given α and any given $\varepsilon > 0$, there is some $\nu(\alpha)$ such that

$$\big|\langle\, z', U(\beta)\, A_\nu\, U(\beta)^*\, z\,\rangle\big| < \varepsilon$$

for all β in some neighbourhood $\mathcal{N}(\alpha)$ of α and all $\nu > \nu(\alpha)$. By varying α over $\mathbb{T}$, we obtain a family of neighbourhoods and $\nu(\alpha)$s. These neighbourhoods cover $\mathbb{T}$, which is compact, and so there is a finite collection $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\mathbb{T} = \bigcup_{j=1}^k \mathcal{N}(\alpha_j)$. Let $\nu' > \nu(\alpha_j)$ for all $1 \le j \le k$. Then for any $\nu > \nu'$ and any $\theta \in \mathbb{T}$,

$$\big|\langle\, z', U(\theta)\, A_\nu\, U(\theta)^*\, z\,\rangle\big| < \varepsilon$$

because $\theta \in \mathcal{N}(\alpha_j)$ for some $1 \le j \le k$ and $\nu > \nu(\alpha_j)$. Hence we find that $|\langle\, z', m(A_\nu) z\,\rangle| < \varepsilon$ for all $\nu > \nu'$ and the result follows. ■

Lemma 6.26. *Let $\mathfrak{B}$ be a C^*-algebra in $\mathcal{B}(\mathcal{K})$ such that $m(\mathfrak{B}) \subseteq \mathfrak{B}$. Then*

$$\{\, \mathfrak{B} \cap U(\mathbb{T})' \,\}^{-w} = \overline{\mathfrak{B}}^{\,w} \cap U(\mathbb{T})'.$$

Proof. It is clear that $\{\, \mathfrak{B} \cap U(\mathbb{T})' \,\}^{-w} \subseteq \overline{\mathfrak{B}}^{\,w} \cap U(\mathbb{T})'$. Let $A \in \overline{\mathfrak{B}}^{\,w} \cap U(\mathbb{T})'$. Then A belongs to the weak closure $\overline{\mathfrak{B}}^{\,w}$ and so by Kaplansky's density theorem, there is a net A_ν in $\mathfrak{B}$ with $\|A_\nu\| \le \|A\|$ such that $A_\nu \to A$, weakly. Then, by lemma 6.25, $m(A_\nu) \to m(A)$ weakly. However, $A \in U(\mathbb{T})'$ and so $m(A) = A$. Hence $m(A_\nu) \to A$ weakly. Since $m(A_\nu) \in \mathfrak{B} \cap U(\mathbb{T})'$, we conclude that $A \in \{\, \mathfrak{B} \cap U(\mathbb{T})' \,\}^{-w}$ ∎

With these preliminaries done, we can now prove irreducibility of the charge sectors.

Theorem 6.27. *The representations $(\mathcal{K}_q, \pi_g)$, $q = 0, \pm 1, \pm 2, \dots$ of $\mathfrak{A}$, the quasilocal algebra of observables, are irreducible.*

Proof. By the construction of the algebras $\mathfrak{A}(\mathcal{O})$ (together with the fact that $U(\theta)\mathfrak{F}(\mathcal{O})U(\theta)^* = \mathfrak{F}(\mathcal{O})$), we see that

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{F}(\mathcal{O}) \cap U(\mathbb{T})' = m(\mathfrak{F}(\mathcal{O}))$$

and so

$$\mathfrak{A} = m(\mathfrak{F}) = \mathfrak{F} \cap U(\mathbb{T})'.$$

By lemma 6.26, we have

$$\overline{\mathfrak{A}}^{\,w} = \overline{\mathfrak{F}}^{\,w} \cap U(\mathbb{T})'.$$

But by the irreducibility of $\mathfrak{F}$, theorem 6.12, $\overline{\mathfrak{F}}^{\,w} = \mathcal{B}(\mathcal{K})$ and so $\overline{\mathfrak{A}}^{\,w} = U(\mathbb{T})'$. Hence $\overline{\mathfrak{A}}^{\,w} : \mathcal{K}_q \to \mathcal{K}_q$ for each q. Moreover, $U(\mathbb{T})' = \bigoplus_q \mathcal{B}(\mathcal{K}_q)$ and therefore $\overline{\mathfrak{A}}^{\,w} \restriction \mathcal{K}_q = \mathcal{B}(\mathcal{K}_q)$. But $\overline{\mathfrak{A}}^{\,w} \restriction \mathcal{K}_q$ is in the weak closure of $\pi_q(\mathfrak{A})$. Hence $(\mathcal{K}_q, \pi_q)$ is irreducible. ∎

Remark 6.28. We have now shown that the algebras $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}$ obey the postulates 1–5.

Corollary 6.29. *The representations $(\mathcal{K}_q, \pi_q)$, $q = 0, \pm 1, \pm 2, \dots$ of $\mathfrak{A}$ are unitarily inequivalent.*

Proof. According to the theorem, for any $\theta \in \mathbb{T}$, $U(\theta) \in U(\mathbb{T})' = \overline{\mathfrak{A}}^{\,w}$. So suppose that $A_\nu \in \mathfrak{A}$ is some net such that $A_\nu \to U(\theta)$ weakly. Then for any $z \in \mathcal{K}_q$,

$$\langle\, z, \pi_q(A_\nu) z \,\rangle = \langle\, z, A_\nu\, z \,\rangle \to \langle\, z, U(\theta)\, z \,\rangle = e^{iq\theta} \langle\, z, z \,\rangle.$$

If $(\mathcal{K}_q, \pi_q)$ and $(\mathcal{K}_{q'}, \pi_{q'})$ were unitarily equivalent, say, $W \pi_q(\,\cdot\,) W^{-1} = \pi_{q'}(\,\cdot\,)$ for some unitary map W from $\mathcal{K}_q$ onto $\mathcal{K}_{q'}$, then we would have

$$\langle z, \pi_q(A_\nu) z \rangle = \langle Wz, \pi_{q'}(A_\nu) Wz \rangle$$
$$\to \langle Wz, U(\theta) Wz \rangle = e^{iq'\theta} \langle Wz, Wz \rangle = e^{iq'\theta} \langle z, z \rangle.$$

This is impossible unless $q = q'$. ■

Remark 6.30. One can show that the representations $(\mathcal{K}_q, \pi_q)$, $q \in \mathbb{Z}$, are mutually disjoint (take z in the appropriate subspace).

To summarize: the charge sectors are irreducible, physically equivalent, but unitarily inequivalent representations of the quasilocal algebra of observables $\mathfrak{A}$.

Strong local equivalence

A further notion of equivalence has been introduced by Borchers (1967) which is stronger than physical equivalence but weaker than unitary equivalence.

Definition 6.31. Representations $(\mathcal{H}, \pi)$, $(\mathcal{H}', \pi')$ of a quasilocal algebra $\mathfrak{A}$ are said to be locally equivalent if for each region $\mathcal{O}$ in Minkowski space there is a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that

$$U \pi(A) U^{-1} = \pi'(A) \quad \text{for all } A \in \mathfrak{A}(\mathcal{O}),$$

that is, for each region $\mathcal{O}$, the representations π and π' of $\mathfrak{A}(\mathcal{O})$ are unitarily equivalent.

For any region $\mathcal{O}$, let us denote by $\mathfrak{A}(\mathcal{O}^s)$ the C^*-algebra algebra generated by those observables associated with regions space-like separated from $\mathcal{O}$. Thus, $\mathfrak{A}(\mathcal{O}^s)$ is the C^*-algebra generated by $\mathfrak{A}(\mathcal{O}_1)$ where $\mathcal{O}_1$ runs over all regions space-like separated from $\mathcal{O}$ (the latter denoted by $\mathcal{O}_1 \between \mathcal{O}$).

Definition 6.32. The representations $(\mathcal{H}, \pi)$, $(\mathcal{H}', \pi')$ of a quasilocal algebra $\mathfrak{A}$ are said to be strongly locally equivalent if for each region $\mathcal{O}$ in Minkowski space there is a unitary operator $V : \mathcal{H} \to \mathcal{H}'$ such that

$$V \pi(A) V^{-1} = \pi'(A) \quad \text{for all } A \in \mathfrak{A}(\mathcal{O}^s),$$

that is, for each region $\mathcal{O}$, the representations π and π' of $\mathfrak{A}(\mathcal{O}^s)$ are unitarily equivalent.

We will say that representations $(\mathcal{H}, \pi)$, $(\mathcal{H}', \pi')$ are strongly locally equivalent for a region $\mathcal{O}$ if $\pi \upharpoonright \mathfrak{A}(\mathcal{O}^s)$ and $\pi' \upharpoonright \mathfrak{A}(\mathcal{O}^s)$ are unitarily equivalent.

Remark 6.33. We note that, of course, the unitary involved may depend on the region $\mathcal{O}$ selected. Also, one readily checks that these really are equivalence relations. Furthermore, if two representations are strongly locally equivalent then they are locally equivalent.

We shall see that whilst the charge sectors $(\mathcal{K}_q, \pi_q)$ are unitarily inequivalent, nevertheless, they are strongly locally equivalent. The idea of the proof of this is simple. The charged field $\phi(f)$ carries charge $+1$ in $\mathcal{K}$. However, $\phi(f)$ has polar decomposition $\phi(f) = VM$, where $M^2 = \phi^*(f)\phi(f)$. Since $\phi^*(f)\phi(f)$ is neutral, the charge must be carried by the isometry V. It turns out that V is unitary and commutes with the fields associated with regions space-like with respect to the support of f. In other words, V effects the strong local equivalence of sectors differing by unit charge.

Theorem 6.34. *The representations $(\mathcal{K}_q, \pi_q)$, $q \in \mathbb{Z}$, of $\mathfrak{A}$ are strongly locally equivalent.*

Proof. It is enough to show that for any $q \in \mathbb{Z}$ the representations $(\mathcal{K}_q, \pi_q)$ and $(\mathcal{K}_{q+1}, \pi_{q+1})$ are strongly locally equivalent. Let $\mathcal{O}$ be a given region and let $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^4)$ with support $\operatorname{supp} f \subseteq \mathcal{O}$. Let $\phi(f) = VM$, with $M^2 = \phi^*(f)\phi(f)$, be the polar decomposition of the normal operator $\phi(f)$. Since $\phi(f)$ has no kernel (theorem 6.9), it follows that V is unitary on $\mathcal{K}$. Furthermore, $\phi(f)$ maps $D(\phi(f)) \cap \mathcal{K}_q$ into $\mathcal{K}_{q+1}$. Using the fact that D is a domain of analytic (even if not entire) vectors for M^2 (by the estimate of proposition 6.2), we see that M^2 commutes with $U(\theta)$ for all $\theta \in \mathbb{T}$. Hence M maps $D(\phi(f) \cap \mathcal{K}_q$ into $\mathcal{K}_q$. Moreover, M is self-adjoint, $M > 0$ and, by theorem 6.9, has no kernel and so $\operatorname{ran} M \restriction D(\phi(f)) \cap \mathcal{K}_q$ is dense in $\mathcal{K}_q$. It follows that V maps $\mathcal{K}_q$ into $\mathcal{K}_{q+1}$.

By the same argument applied to $\phi^*(f) = V^*M$, we see that V^* maps $\mathcal{K}_{q+1}$ into $\mathcal{K}_q$ and therefore V maps $\mathcal{K}_q$ unitarily onto $\mathcal{K}_{q+1}$.

Now, by the spectral theorem, V commutes with $\mathfrak{F}(\mathcal{O})'$ which means that $V \in \mathfrak{F}(\mathcal{O})'' = \mathfrak{F}(\mathcal{O})$. In particular, V commutes with $\mathfrak{A}(\mathcal{O}^s)$ and, since $\mathfrak{A}(\mathcal{O}^s)$ leaves each $\mathcal{K}_q$ invariant, we have

$$V\,\pi_q(A) = \pi_{q+1}(A)\,V \restriction \mathcal{K}_q$$

for all $A \in \mathfrak{A}(\mathcal{O}^s)$ and all $q \in \mathbb{Z}$. ∎

Remark 6.35. It is easy to see from this result that the representations $(\mathcal{K}_q, \pi_q)$ of $\mathfrak{A}$ all have the same kernel and so are faithful. In fact, suppose that $A \in \mathfrak{A}$ is such that $\pi_q(A) = 0$. Then there is a sequence of regions $(\mathcal{O}_n)$ and elements $A_n \in \mathfrak{A}(\mathcal{O}_n)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$. In particular,

$$\|\pi_q(A_n)\| = \|\pi_q(A_n) - \pi_q(A)\| = \|\pi_q(A_n - A)\| \le \|A_n - A\|$$

so that $\|\pi(A_n)\| \to 0$. By (strong) local equivalence, for any fixed q' in $\mathbb{Z}$, there are unitaries $V_n : \mathcal{K}_q \to \mathcal{K}_{q'}$ such that $\pi_{q'}(A_n) = V_n\,\pi_q(A_n)\,V_n^{-1}$. Therefore

$$\|\pi_{q'}(A_n)\| = \|V_n\,\pi_q(A_n)\,V_n^{-1}\| \le \|\pi_q(A_n)\| \to 0.$$

Since $\|\pi_{q'}(A_n - A)\| \le \|A_n - A\| \to 0$, we deduce that $\pi_{q'}(A) = 0$, as claimed.

Remark 6.36. Let $A \in \mathfrak{A}$. Since V, the unitary operator in the polar decomposition of $\phi(f)$, belongs to $\mathfrak{F}$, we see that $V^*AV \in \mathfrak{F}$ and moreover, V^*AV maps each $\mathcal{K}_q$ into itself, that is, V^*AV commutes with the gauge group $\mathbb{T}$ (as represented by U). Hence $V^*AV \in \mathfrak{A}$. In other words, the mapping $A \mapsto \gamma(A) = V^*AV$ is an automorphism of $\mathfrak{A}$.

Furthermore, if $\operatorname{supp} f \subseteq \mathcal{O}$ and $A \in \mathfrak{A}(\mathcal{O}^s)$, then $\gamma(A) = A$, that is, $\gamma \restriction \mathfrak{A}(\mathcal{O}^s)$ is the identity automorphism of $\mathfrak{A}(\mathcal{O}^s)$. We could therefore call γ an automorphism localized in $\mathcal{O}$.

Consider now the representation π of $\mathfrak{A}$ acting on $\mathcal{K}_0$ given by the prescription $\pi(A) = \pi_0 \circ \gamma(A)$. Then, for $z \in \mathcal{K}_0$, we have

$$\pi(A)z = \pi_0 \circ \gamma(A)z = \gamma(A)z = V^* \, A \, V z = V^* \, \pi_1(A) \, V z$$

since $Vz \in \mathcal{K}_1$. This shows that π_1 is unitarily equivalent to π, that is, $\pi_1 \simeq \pi_0 \circ \gamma$. Similarly, $\pi_q \simeq \pi_0 \circ \gamma^q$.

In other words, up to unitary equivalence, *the sectors* $(\mathcal{K}_q, \pi_q)$, $q \in \mathbb{Z}$, *are given by localized automorphisms acting in the charge zero sector.*

A general discussion of this situation is the subject of the next chapter.

Chapter 7

The general structure of sectors

We have constructed the quasilocal algebra of observables $\mathfrak{A}$ for the free charged Bose field from the field algebra $\mathfrak{F}$ and have found the sectors which occur. A similar analysis has been carried out by Doplicher, Haag and Roberts (1969a) for a general field algebra and gauge group, G. They find that there is a one-one correspondence between the sectors (i.e., the unitary equivalence classes of representations of the algebra of observables) occurring and inequivalent irreducible unitary representations of G.

We would like to consider the converse problem of constructing the sectors, given the algebra of observables in the vacuum sector. As in the case of the free charged field, we think of the sectors as being obtained from the vacuum sector via charge carrying fields. These must be constructed and one can ask if they are Bose or Fermi fields or neither. This analysis was initiated by Borchers and re-examined by Doplicher, Haag and Roberts.

We will follow the treatment of Doplicher, Haag and Roberts (1969b, 1971, 1974). (See also Haag (1970)).

States of interest

The aim is to construct the sectors from the algebra of observables, $\mathfrak{A}$. This means that we must determine irreducible representations of $\mathfrak{A}$. To make the problem somewhat more tractable, one appeals to physical arguments to single out some of these representations. Now, to any representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$, we can associate a family of states — namely, the vector states (which are pure if $(\mathcal{H}, \pi)$ is irreducible), or, more generally, the density matrices (or, equivalently, the ultraweakly continuous states) of $(\mathcal{H}, \pi)$. Evidently, unitarily equivalent representations give rise to the same set of such states.

Conversely, any pure state on $\mathfrak{A}$ defines, courtesy of the GNS construction, an irreducible representation. We see then, a certain correspondence between representations of $\mathfrak{A}$ and families of states of $\mathfrak{A}$. The idea now is to single out a family of representations by invoking physical arguments to focus attention on a particular collection of states (Haag (1995)). We shall suppose that our "laboratory" is isolated in an otherwise empty universe

— that is, we shall restrict our investigations to those states which behave like the vacuum in remote regions of space. We shall formulate this more precisely.

We shall suppose that we are given a quasilocal C^*-algebra $\mathfrak{A}$ of observables, generated by local algebras, C^*-algebras $\mathfrak{A}(\mathcal{O})$, for regions $\mathcal{O}$. We shall assume that these algebras satisfy the postulates of isotony, causality (locality) and Poincaré covariance.

Let ω_0 be a vacuum state on $\mathfrak{A}$ according to postulate 6 (spectrum condition) and let $(\mathcal{H}_0, \pi_0)$ be the associated GNS representation. We suppose that $(\mathcal{H}_0, \pi_0)$ is irreducible (or, equivalently, that ω_0 is pure) and also faithful. The representation $(\mathcal{H}_0, \pi_0)$ is called the vacuum sector. The preceding discussion leads to the following definition.

Definition 7.1. We say that a state ω on $\mathfrak{A}$ agrees asymptotically with the vacuum, ω_0, if for any sequence $\{ \mathcal{O}_n \}$ of increasing regions which cover $\mathcal{M}$ (i.e., $\bigcup_n \mathcal{O}_n = \mathcal{M}$), it is true that

$$\|(\omega - \omega_0) \upharpoonright \mathfrak{A}(\mathcal{O}_n^s)\| \to 0$$

as $n \to \infty$. (Recall that $\mathfrak{A}(\mathcal{O}_n^s)$ is the C^*-algebra generated by the $\mathfrak{A}(\mathcal{O}_1)$ as $\mathcal{O}_1$ runs over regions space-like with respect to $\mathcal{O}$.)

In other words, ω begins to look like the vacuum far away — the convergence being in norm. This presumably will not do for a study of cosmology or indeed, for electrodynamics, where Gauss' law indicates that measurements made in far off regions can nevertheless determine the situation (charge) within our laboratory. We ought not expect a state with non-zero charge to behave like the vacuum in the sense of definition 7.1.

Theorem 7.2. *Suppose that the representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ is strongly locally equivalent to the vacuum sector, $(\mathcal{H}_0, \pi_0)$, for some region $\mathcal{O}$. Let ω be any state on $\mathfrak{A}$ which is ultraweakly continuous in $(\mathcal{H}, \pi)$. Then ω asymptotically agrees with ω_0.*

Proof. Suppose that ω and ω_0 do not agree asymptotically. Then there is some sequence $\{ \mathcal{O}_n \}$ of regions with $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$ and $\bigcup_n \mathcal{O}_n = \mathcal{M}$ and some $\varepsilon > 0$ such that

$$\|(\omega - \omega_0) \upharpoonright \mathfrak{A}(\mathcal{O}_n^s)\| \geq \varepsilon$$

for all n. Then , by definition of the norm of a functional, there exists $B_n \in \mathfrak{A}(\mathcal{O}_n^s)$ with $\|B_n\| = 1$ such that

$$|(\omega - \omega_0)(B_n)| > \tfrac{1}{2}\varepsilon \qquad (*)$$

for each n.

Now, $\|\pi_0(B_n)\| = \|B_n\|$ and so $\{ \pi_0(B_n) \}$ is a sequence in the unit ball of $\mathcal{B}(\mathcal{H}_0)$, which is weakly compact. Hence there is a subnet $\{ B_\alpha \}$ of $\{ B_n \}$

such that $\pi_0(B_\alpha)$ converges weakly to some B in the unit ball of $\mathcal{B}(\mathcal{H}_0)$. Since $\mathcal{O}_n^s \supseteq \mathcal{O}_{n+1}^s$ and $B_n \in \mathfrak{A}(\mathcal{O}_n^s)$, we see that B commutes with each $\pi_0(\mathfrak{A}(\mathcal{O}_n))$, i.e.,

$$B \in \{ \bigcup_n \pi_0(\mathfrak{A}(\mathcal{O}_n)) \}' .$$

This implies that $B \in \pi_0(\mathfrak{A})' = \mathbb{C}\,\mathbb{1}$ since $(\mathcal{H}_0, \pi_0)$ is irreducible. Write $B = c\pi_0(\mathbb{1})$ for some $c \in \mathbb{C}$ with $|c| \le 1$. Then $\pi_0(B_\alpha)$ converges weakly to $\pi_0(c\mathbb{1})$.

However, by hypothesis, there is a unitary $V : \mathcal{H}_0 \to \mathcal{H}$ such that

$$\pi(A) = V\,\pi_0(A)\,V^{-1}$$

for all $A \in \mathfrak{A}(\mathcal{O}^s)$. For sufficiently large n, $\mathcal{O} \subseteq \mathcal{O}_n$, so that $\mathcal{O}_n^s \subseteq \mathcal{O}^s$ and therefore $B_\alpha \in \mathfrak{A}(\mathcal{O}^s)$ for all sufficiently large α. The above unitary equivalence implies that $\pi(B_\alpha)$ converges weakly to $c\pi(\mathbb{1})$. Since $\pi(B_\alpha)$ lies in the unit ball of $\mathcal{B}(\mathcal{H})$, the convergence is also with respect to the ultraweak topology and therefore $\omega(B_\alpha)$ converges to c.

On the other hand, ω_0 is a vector state in $(\mathcal{H}_0, \pi_0)$ and so the net $\omega_0(B_\alpha)$ converges to c. Hence $(\omega - \omega_0)(B) = 0$, which contradicts $(*)$. ∎

Borchers' Property

In order to obtain a converse to theorem 7.2, we need to make a technical assumption concerning the representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$. Following Doplicher, Haag and Roberts (1971), we call this property B.

For any region $\mathcal{O}$ and representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$, we denote the von Neumann algebra $\pi(\mathfrak{A}(\mathcal{O}^s))'$, the commutant of $\pi(\mathfrak{A}(\mathcal{O}^s))$ in $\mathcal{B}(\mathcal{H})$, by $\mathcal{R}_\pi(\mathcal{O})$.

Definition 7.3. A representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ is said to satisfy property B if, for any region $\mathcal{O}$, included in the interior of another region $\mathcal{O}_1$, and for any non-zero projection E in $\mathcal{R}_\pi(\mathcal{O})$, there is an isometry $W \in \mathcal{R}_\pi(\mathcal{O}_1)$ such that $WW^* = E$ and $W^*W = \mathbb{1}$.

It was shown by Borchers (1967b) that this property holds under the assumptions of the spectrum condition and additivity in $(\mathcal{H}, \pi)$ together with some extra regularity. (Since π_0 is faithful, we may consider $\pi(\mathfrak{A}(\mathcal{O}))$ as a representation of $\pi_0(\mathfrak{A}(\mathcal{O}))$. We need to know that π extends to a representation of $\pi_0(\mathfrak{A}(\mathcal{O}))''$. We also need irreducibility of π, which allows $W^*W = \mathbb{1}$ — otherwise we would have $W^*W = F$ for some projection F.) Property B holds in the vacuum representation $(\mathcal{H}_0, \pi_0)$.

Let us suppose then that $(\mathcal{H}, \pi)$ satisfies property B. Let $(\mathcal{H}_1, \pi_1)$ be a subrepresentation of $(\mathcal{H}, \pi \restriction \mathfrak{A}(\mathcal{O}^s))$ and consider $(\mathcal{H}_1, \pi_1 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ as a representation of $\mathfrak{A}(\mathcal{O}_1^s))$, where $\mathcal{O}_1$ is a region containing $\mathcal{O}$ within its interior.

We claim that this is unitarily equivalent to $(\mathcal{H}, \pi \restriction \mathfrak{A}(\mathcal{O}_1^s))$. To see this, we note that the projection E of $\mathcal{H}$ onto $\mathcal{H}_1$ belongs to $\pi(\mathfrak{A}(\mathcal{O}^s))' = \mathcal{R}_\pi(\mathcal{O})$. Hence, using property B, there is an isometry $W \in \mathcal{R}_\pi(\mathcal{O}_1) = \pi(\mathfrak{A}(\mathcal{O}_1^s))$ such that $WW^* = E$ and $W^*W = \mathbb{1}$. Hence

$$\begin{aligned} W\, \pi \restriction \mathfrak{A}(\mathcal{O}_1^s) &= \pi \restriction \mathfrak{A}(\mathcal{O}_1^s)\, W \\ &= \pi_1 \restriction \mathfrak{A}(\mathcal{O}_1^s)\, W \end{aligned}$$

since $\mathcal{H}_1$ is the final space of W. Evidently, W effects the claimed unitary equivalence.

Proposition 7.4. *Suppose that $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are representations of $\mathfrak{A}$ both satisfying property B and are such that for some given region $\mathcal{O}$ the representations $(\mathcal{H}_1, \pi_1 \restriction \mathfrak{A}(\mathcal{O}^s))$ and $(\mathcal{H}_2, \pi_2 \restriction \mathfrak{A}(\mathcal{O}^s))$ of $\mathfrak{A}(\mathcal{O}^s)$ are non-disjoint. Then, for any given region $\mathcal{O}_1$ containing $\mathcal{O}$ in its interior, the representations $(\mathcal{H}_1, \pi_1 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ and $(\mathcal{H}_2, \pi_2 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ of $\mathfrak{A}(\mathcal{O}_1^s)$ are unitarily equivalent.*

Proof. By hypothesis (non-disjointness), we know that there are subrepresentations $(\widehat{\mathcal{H}}_1, \widehat{\pi}_1 \restriction \mathfrak{A}(\mathcal{O}^s))$ and $(\widehat{\mathcal{H}}_2, \widehat{\pi}_2 \restriction \mathfrak{A}(\mathcal{O}^s))$ of $(\mathcal{H}_1, \pi_1 \restriction \mathfrak{A}(\mathcal{O}^s))$ and $(\mathcal{H}_2, \pi_2 \restriction \mathfrak{A}(\mathcal{O}^s))$, respectively, which are unitarily equivalent.

By the above remark, however, $(\widehat{\mathcal{H}}_1, \widehat{\pi}_1 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ is unitarily equivalent to $(\mathcal{H}_1, \pi_1 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ for any region $\mathcal{O}_1$ containing $\mathcal{O}$ in its interior. Similarly, $(\widehat{\mathcal{H}}_2, \widehat{\pi}_2 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ is unitarily equivalent to $(\mathcal{H}_2, \pi_2 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ and the result follows. ∎

Theorem 7.5. *Suppose that ω is a state on the quasilocal algebra $\mathfrak{A}$ which asymptotically agrees with the vacuum state, ω_0. If the GNS representation $(\mathcal{H}, \pi)$ associated with ω satisfies property B, then there is a region $\mathcal{O}$ such that*

$$\pi \restriction \mathfrak{A}(\mathcal{O}^s) \simeq \pi_0 \restriction \mathfrak{A}(\mathcal{O}^s)\,,$$

that is, $(\mathcal{H}, \pi)$ and $(\mathcal{H}_0, \pi_0)$ are strongly locally equivalent for $\mathcal{O}$.

Proof. Let $\{\,\mathcal{O}_n\,\}$ be a sequence of increasing regions which exhaust $\mathcal{M}$. Since ω and ω_0 asymptotically agree, it follows that

$$\|(\omega - \omega_0) \restriction \mathfrak{A}(\mathcal{O}_n^s)\| < 2$$

for all sufficiently large n. Hence, by the theorem of Glimm and Kadison, theorem 1.32, and for large n, the GNS representations $(\mathcal{H}, \pi \restriction \mathfrak{A}(\mathcal{O}^s))$ and $(\mathcal{H}_0, \pi_0 \restriction \mathfrak{A}(\mathcal{O}^s))$ of $\mathfrak{A}(\mathcal{O}_n^s)$ given by the states $\omega \restriction \mathfrak{A}(\mathcal{O}_n^s)$ and $\omega_0 \restriction \mathfrak{A}(\mathcal{O}_n^s)$ are non-disjoint.

Fix n large, and let $\mathcal{O}$ be any region containing $\mathcal{O}_n$ in its interior. By proposition 7.4, $(\mathcal{H}, \pi \restriction \mathfrak{A}(\mathcal{O}^s))$ is unitarily equivalent to $(\mathcal{H}_0, \pi_0 \restriction \mathfrak{A}(\mathcal{O}^s))$, as required. ∎

Duality

Suppose that $(\mathcal{H}, \pi)$ is a representation of $\mathfrak{A}$, the quasilocal algebra, satisfying

$$\pi \restriction \mathfrak{A}(\mathcal{O}^s) \simeq \pi_0 \restriction \mathfrak{A}(\mathcal{O}^s)$$

for some double cone $\mathcal{O}$. If $(\mathcal{H}, \pi)$ carries a unitary representation of the translation group, we can translate any given region $\mathcal{O}_1$ into the set $\mathcal{O}^s$. It follows that

$$\pi \restriction \mathfrak{A}(\mathcal{O}_1) \simeq \pi_0 \restriction \mathfrak{A}(\mathcal{O}_1)\,.$$

Since π_0 is faithful, we can think of π as a representation of $\pi_0(\mathfrak{A})$. The above unitary equivalence implies that π extends to a representation of $\pi_0(\mathfrak{A}(\mathcal{O}_1))''$ — this being unitarily equivalent to $\pi_0(\mathfrak{A}(\mathcal{O}_1))''$.

We see, then, that if we restrict our attention to those representations $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ which are strongly locally equivalent to $(\mathcal{H}_0, \pi_0)$ for some $\mathcal{O}$, then (assuming that we also have translations implemented in $(\mathcal{H}, \pi)$) π defines a representation of $\pi_0(\mathfrak{A}(\mathcal{O}_1))''$ for any region $\mathcal{O}_1$. As a consequence, we may consider local von Neumann algebras(see Haag, Kadison and Kastler (1970)).

From now on, we shall consider $\mathfrak{A}$ as being defined by its vacuum representation. More precisely, we suppose that $\mathfrak{A}$ is a C^*-algebra of operators on a Hilbert space $\mathcal{H}_0$ containing a vacuum vector Ω_0. We shall suppose that the local algebras $\mathfrak{A}(\mathcal{O})$ are weakly closed, i.e., are von Neumann algebras. The symbol π_0 is therefore redundant, but it can often be used for emphasis.

To proceed, we shall need more structure — the notion of duality. This plays a central rôle in the work of Doplicher, Haag and Roberts. Consider the statement of locality. If $\mathcal{O}$ is any region, then $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}^s)$ commute. In the vacuum representation, this can be expressed as

$$\pi_0(\mathfrak{A}(\mathcal{O})) \subseteq \pi_0(\mathfrak{A}(\mathcal{O}^s))'\,.$$

Now suppose that $X \in \pi_0(\mathfrak{A}(\mathcal{O}^s))'$. One might ask whether X belongs to $\pi_0(\mathfrak{A}(\mathcal{O}))$. In general, the answer is no — indeed, this could never be true for all $X \in \pi_0(\mathfrak{A}(\mathcal{O}^s))'$ if $\pi_0(\mathfrak{A}(\mathcal{O}))$ were not a von Neumann algebra.

Definition 7.6. We say that the representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ satisfies duality for a region $\mathcal{O}$ if

$$\pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}^s))'\,.$$

Note that, in particular, this implies that $\pi(\mathfrak{A}(\mathcal{O}))$ is a von Neumann algebra.

We shall assume that, in the vacuum sector, duality holds for all regions. This means that

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}^s)'$$

for any region $\mathcal{O}$. It is important here that we adopt the definition of region as a double cone. Indeed, for the free field, Araki (1964a) has shown that duality does hold for double cones but that it does not hold for a region given by two double cones, one on top of the other. The point here is that one cannot expect duality to hold for arbitrary regions but rather for some restricted class of regions (here taken to be double cones). We also note that duality fails even for double cones for some generalized free fields (see Landau (1974)).

As noted by Haag (1970), duality implies a certain maximality of the local algebras $\mathfrak{A}(\mathcal{O})$. Indeed, let $\mathfrak{R}(\mathcal{O}) \supset \mathfrak{A}(\mathcal{O})$. Then if $\mathfrak{R}(\mathcal{O})$ is to be interpreted as an algebra of local observables within $\mathcal{O}$, locality requires that $\mathfrak{R}(\mathcal{O})$ commutes with $\mathfrak{A}(\mathcal{O}^s)$, i.e., $\mathfrak{R}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}^s)'$. But then duality implies that $\mathfrak{R}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O})$ and we conclude that $\mathfrak{R}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})$. It is not clear whether maximality implies duality (Haag (1970)).

Localized Monomorphisms

We recall that a monomorphism of a C^*-algebra is an injective *-homomorphism. Given a representation $(\mathcal{H}, \pi)$ of a C^*-algebra, $\mathfrak{R}$, and a monomorphism ρ, we can define another representation $(\mathcal{H}, \widehat{\pi})$ on the same Hilbert space by setting $\widehat{\pi}(A) = \pi \circ \rho(A)$ for $A \in \mathfrak{R}$.

Definition 7.7. We say that a monomorphism ρ on the quasilocal algebra $\mathfrak{A}$ is localized in a region $\mathcal{O}$ if $\rho(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}^s)$.

The importance of localized monomorphisms can be seen in the following theorem.

Theorem 7.8. *Suppose that $(\mathcal{H}, \pi)$ is a faithful representation of $\mathfrak{A}$. Then $(\mathcal{H}, \pi)$ and $(\mathcal{H}_0, \pi_0)$ are strongly locally equivalent for a double cone $\mathcal{O}_1$ if and only if there is a monomorphism ρ, localized in $\mathcal{O}_1$, such that $(\mathcal{H}, \pi)$ is unitarily equivalent to $(\mathcal{H}_0, \pi_0 \circ \rho)$.*

Proof. Suppose first that $(\mathcal{H}, \pi)$ and $(\mathcal{H}_0, \pi_0)$ are strongly locally equivalent for $\mathcal{O}_1$. Then there is a unitary operator $V : \mathcal{H}_0 \to \mathcal{H}$ such that

$$\pi(A)\, V = V\, \pi_0(A)$$

for all $A \in \mathfrak{A}(\mathcal{O}_1^s)$.

For given $A \in \mathfrak{A}$, define $\rho(A) = V^*\pi(A)V$. Evidently, $\rho(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}_1^s)$. Let $B \in \mathfrak{A}(\mathcal{O}^s)$ where $\mathcal{O} \supset \mathcal{O}_1$. Then for any $A \in \mathfrak{A}(\mathcal{O})$, we have

$$\begin{aligned}
\rho(A)\, \pi_0(B) &= V^*\, \pi(A)\, V\, \pi_0(B) \\
&= V^*\, \pi(A)\, V\, \pi_0(B)\, V^*V \\
&= V^*\, \pi(A)\, \pi(B)\, V \quad \text{since } B \in \mathfrak{A}(\mathcal{O}^s) \subseteq \mathfrak{A}(\mathcal{O}_1^s) \\
&= V^*\, \pi(AB)\, V = V^*\, \pi(BA)\, V \\
&= V^*\, \pi(B)\, VV^*\, \pi(A)\, V
\end{aligned}$$

$$= \pi_0(B)\, V^* \pi(A)\, V$$
$$= \pi_0(B)\, \rho(A).$$

Thus $\rho(A)$ commutes with $\mathfrak{A}(\mathcal{O}^s)$ and so, by duality for $(\mathcal{H}_0, \pi_0)$, $\rho(A) \in \mathfrak{A}(\mathcal{O})$. In other words, $\rho : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(\mathcal{O})$. Moreover, π is faithful and so ρ is injective and therefore extends to a monomorphism of $\mathfrak{A}$, by continuity. Also, $\rho(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}_1^s)$ which is to say that ρ is localized in $\mathcal{O}_1$.

By construction, we have $\pi(\mathfrak{A}) = V\rho(\mathfrak{A})V^*$, i.e., $\pi(\mathfrak{A}) = V\pi_0 \circ \rho(\mathfrak{A})V^*$ and so $\pi \simeq \pi_0 \circ \rho$.

For the converse, suppose that ρ is a monomorphism localized in the region $\mathcal{O}_1$ and set $\widehat{\pi} = \pi_0 \circ \rho$. Then for any $A \in \mathfrak{A}(\mathcal{O}_1^s)$, we have

$$\widehat{\pi}(A) = \pi_0 \circ \rho(A) = \pi_0(A),$$

that is, $\widehat{\pi}$ and π define the same representation of $\mathfrak{A}(\mathcal{O}_1^s)$. It follows that the unitary equivalence $(\mathcal{H}, \pi) \simeq (\mathcal{H}_0, \widehat{\pi})$ clearly implies the required strong local equivalence. ∎

Theorem 7.8 says that sectors strongly locally equivalent to the vacuum sector are given by localized monomorphisms.

Localized Automorphisms

Definition 7.9. We denote by $\Gamma(\mathcal{O})$ those automorphisms of the quasilocal algebra $\mathfrak{A}$ localized in the region $\mathcal{O}$.

We can improve on theorem 7.8 under the additional hypothesis of duality in $(\mathcal{H}, \pi)$.

Theorem 7.10. *Suppose that $(\mathcal{H}, \pi)$ is a faithful representation of $\mathfrak{A}$ and that $(\mathcal{H}, \pi)$ is strongly locally equivalent to $(\mathcal{H}_0, \pi_0)$ for a double cone $\mathcal{O}_1$. Suppose, further, that duality holds in $(\mathcal{H}, \pi)$ for all regions $\mathcal{O}$ with $\mathcal{O}_1 \subset \mathcal{O}$, i.e.,*

$$\pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}^s))'$$

for all $\mathcal{O}$ with $\mathcal{O}_1 \subset \mathcal{O}$. Then there is an automorphism γ localized in $\mathcal{O}_1$ such that $(\mathcal{H}, \pi)$ is unitarily equivalent to $(\mathcal{H}_0, \pi_0 \circ \gamma)$.

Proof. By theorem 7.8 there is a monomorphism γ, say, localized in $\mathcal{O}_1$ such that $(\mathcal{H}, \pi)$ is unitarily equivalent to $(\mathcal{H}_0, \pi_0 \circ \gamma)$. We shall show that γ is an automorphism of $\mathfrak{A}$. It is enough to show that γ maps $\mathfrak{A}$ onto $\mathfrak{A}$. With the notation of theorem 7.8, but with γ replacing ρ, we have

$$\gamma(A) = V^*\, \pi(A)\, V$$

for all $A \in \mathfrak{A}$. Let $\mathcal{O} \supset \mathcal{O}_1$. Then

$$\pi(\mathfrak{A}(\mathcal{O}^s))' = \{\, V\, \pi_0(\mathfrak{A}(\mathcal{O}^s))\, V^* \,\}' \quad \text{since } \mathcal{O}^s \subset \mathcal{O}_1$$

$$= V\,\pi_0(\mathfrak{A}(\mathcal{O}^s))\,V^*$$
$$= V\,\pi_0(\mathfrak{A}(\mathcal{O}))\,V^* \quad \text{by duality for } \pi_0.$$

But by hypothesis (duality), $\pi(\mathfrak{A}(\mathcal{O}^s))' = \pi(\mathfrak{A}(\mathcal{O}))$. Hence

$$V^*\,\pi(\mathfrak{A}(\mathcal{O}))\,V = \pi_0(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$$

for all $\mathcal{O} \supset \mathcal{O}_1$. In other words, $\gamma(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$ for all $\mathcal{O} \supset \mathcal{O}_1$. By isotony, such $\mathfrak{A}(\mathcal{O})$ are dense in $\mathfrak{A}$ and we conclude that $\gamma(\mathfrak{A}) = \mathfrak{A}$ and therefore $\gamma \in \operatorname{Aut}\mathfrak{A}$. ∎

The converse is also true.

Theorem 7.11. *Let $\gamma \in \Gamma(\mathcal{O}_1)$ for some double cone $\mathcal{O}_1$ and set $\pi = \pi_0 \circ \gamma$. Then $(\mathcal{H}_0, \pi \restriction \mathfrak{A}(\mathcal{O}_1^s)) \simeq (\mathcal{H}_0, \pi_0 \restriction \mathfrak{A}(\mathcal{O}_1^s))$ and for all $\mathcal{O} \supset \mathcal{O}_1$ we have*

$$\pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}^s))'.$$

Proof. The automorphism γ acts as the identity on $\mathcal{O}_1^s$ and so $\pi \restriction \mathfrak{A}(\mathcal{O}_1^s)$ and $\pi_0 \restriction \mathfrak{A}(\mathcal{O}_1^s)$ define the same representation of $\mathfrak{A}(\mathcal{O}_1^s)$ on $\mathcal{H}_0$, so are trivially unitarily equivalent.

Now let $\mathcal{O} \supset \mathcal{O}_1$ be a double cone. We claim that $\gamma(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O})$. Indeed, we have

$$[\gamma(\mathfrak{A}(\mathcal{O})), \mathfrak{A}(\mathcal{O}^s)] = \gamma[\mathfrak{A}(\mathcal{O}), \gamma^{-1}(\mathfrak{A}(\mathcal{O}^s))]$$
$$= \gamma[\mathfrak{A}(\mathcal{O}), \mathfrak{A}(\mathcal{O}^s)] = 0$$

since $\mathfrak{A}(\mathcal{O}^s) \subset \mathfrak{A}(\mathcal{O}_1^s)$ and $\gamma^{-1} \in \Gamma(\mathcal{O}_1)$. Hence, by duality in π_0,

$$\gamma(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O}^s)' = \mathfrak{A}(\mathcal{O}).$$

Applying the same reasoning to γ^{-1}, we deduce that

$$\gamma(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$$

for all $\mathcal{O} \supset \mathcal{O}_1$. Therefore, for any $\mathcal{O} \supset \mathcal{O}_1$, we have

$$\pi(\mathfrak{A}(\mathcal{O})) = \gamma(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$$
$$= \mathfrak{A}(\mathcal{O}^s)' \quad \text{by duality in } \pi_0$$
$$= \gamma(\mathfrak{A}(\mathcal{O}^s))' \quad \text{since } \mathcal{O}^s \subset \mathcal{O}_1^s \text{ and } \gamma \in \Gamma(\mathcal{O}_1)$$
$$= \pi(\mathfrak{A}(\mathcal{O}^s))'$$

and the proof is complete. ∎

Thus we can say that those sectors which are strongly locally equivalent to the vacuum sector and which satisfy duality correspond to those sectors given by localized automorphisms.

It has been shown by Doplicher, Haag and Roberts (1969a) that in the case of a non-abelian gauge group duality cannot hold any any sector other than the vacuum sector. This means that these sectors will be given by localized morphisms but not automorphisms.

Proposition 7.12. *Let* $(\mathcal{H}, \pi)$ *be a representation given by some* $\gamma \in \operatorname{Aut} \mathfrak{A}$. *Then* $(\mathcal{H}, \pi)$ *is irreducible.*

Proof. By hypothesis, $(\mathcal{H}, \pi) \simeq (\mathcal{H}_0, \pi_0 \circ \gamma)$. Since $\gamma \in \operatorname{Aut} \mathfrak{A}$, $\gamma(\mathfrak{A}) = \mathfrak{A}$ and so $\pi_0(\gamma(\mathfrak{A}))$ and $\pi_0(\mathfrak{A})$ are equal as sets of operators in $\mathcal{B}(\mathcal{H}_0)$. The result is therefore a direct consequence of the irreducibility of $(\mathcal{H}_0, \pi_0)$. ■

If $(\mathcal{H}, \pi)$ is given by an automorphism, then it is clearly faithful. Thus automorphisms always give rise to faithful irreducible representations of $\mathfrak{A}$.

Definition 7.13. An automorphism of $\mathfrak{A}$ is said to be inner if it takes the form $A \mapsto \sigma_U(A) = U A U^*$ for some unitary U in $\mathfrak{A}$.

Denote by $\mathcal{I}$ the group of inner automorphisms of $\mathfrak{A}$ which are localized and denote the group $\bigcup_{\mathcal{O}} \Gamma(\mathcal{O})$ by Γ.

Clearly, if $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\Gamma(\mathcal{O}_1) \subset \Gamma(\mathcal{O}_2)$. We also note that $\mathcal{I}$ is a normal subgroup of Γ. In fact, for $\gamma \in \Gamma$, $\sigma_U \in \mathcal{I}$, we have

$$\gamma\, \sigma_U\, \gamma^{-1} = \sigma_{\gamma(U)} \in \mathcal{I}\,.$$

Proposition 7.14. *Let* $\gamma_1, \gamma_2 \in \Gamma$. *Then* $\pi_0 \circ \gamma_1 \simeq \pi_0 \circ \gamma_2$ *if and only if* $\gamma_1\, \gamma_2^{-1} \in \mathcal{I}$.

Proof. It is clear that if $\gamma_1\, \gamma_2^{-1} \in \mathcal{I}$ then $\pi_0 \circ \gamma_1 \simeq \pi_0 \circ \gamma_2$.

Conversely, suppose that $U : \mathcal{H}_0 \to \mathcal{H}_0$ is unitary and that

$$\pi_0(\gamma_2(A)) = U\, \pi_0(\gamma_1(A))\, U^*$$

for all $A \in \mathfrak{A}$. Let $\mathcal{O}$ be such that $\gamma_1, \gamma_2 \in \Gamma(\mathcal{O})$. Then, for any $A \in \mathfrak{A}(\mathcal{O}^s)$, we have $\pi_0(A) = U\, \pi_0(A)\, U^*$. Hence $U \in \pi_0(\mathfrak{A}(\mathcal{O}^s))' = \pi_0(\mathfrak{A}(\mathcal{O}))$, by duality. It follows that

$$\gamma_2(A) = U\, \gamma(A)\, U^* = \sigma_U\, \gamma_1(A)$$

for all $A \in \mathfrak{A}$, i.e., $\gamma_2 = \sigma_U \gamma_1$. ■

Remark 7.15. The element U is determined by σ_U up to a phase. Indeed, if $\sigma_U = \sigma_V$, then $\sigma_U\, \sigma_{V^*}$ is the identity which means that $UV^*\, A = A\, UV^*$ for all $A \in \mathfrak{A}$. By the irreducibility of $\mathfrak{A}$ (on $\mathcal{H}_0$) it follows that $U = e^{i\theta} V$ for some $\theta \in \mathbb{R}$.

Suppose that $\widehat{\pi}$ denotes the unitary equivalence class (i.e., sector) containing the representation π. Then $\gamma \mapsto (\pi_0 \circ \gamma)^{\widehat{}}$ establishes a one-one correspondence between the sectors obtained from localized automorphisms and $\Gamma/\mathcal{I}$. Since $\mathcal{I}$ is a normal subgroup of Γ, $\Gamma/\mathcal{I}$ becomes a group in the obvious way. This means that the family of sectors given by localized automorphisms inherits this group structure.

A sector is said to be covariant if it carries a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$ implementing the corresponding automorphisms of $\mathfrak{A}$. Denote by Γ_c those members of Γ which lead to covariant sectors. Then one can show the following. (For proofs see Doplicher, Haag and Roberts (1969b).)

Proposition 7.16. *Suppose that $\gamma_1, \gamma_2 \in \Gamma_c$ are automorphisms localized in space-like separated regions. If they determine the same sector then they commute.*

Proposition 7.17. *Suppose that $\gamma_1, \gamma_2 \in \Gamma_c$ lead to the same sector and let U be such that $\sigma_U \gamma_1 = \gamma_2$ (as discussed above). If γ_1 and γ_2 are localized in space-like separated regions then*

$$\gamma_1(U) = \pm U.$$

The sign depends only on the sector and not explicitly on γ_1 or γ_2.

Remark 7.18. Those sectors corresponding to the plus sign are called Bose sectors, those corresponding to the minus sign are called Fermi sectors. As a consequence of this result, charge-carrying fields fall into one of two classes - Bose or Fermi. This analysis requires the γs to be automorphisms. In the general case of monomorphisms, an analogous investigation leads to parastatistics.

The proof of this proposition relies on an ability to swap round two regions in space-time whilst keeping them relatively space-like separated. In 2 space-time dimensions, this cannot be done. Indeed, in 2 space-time dimensions the result is no longer true (see Streater, Wilde (1970)).

Proposition 7.19. *The collection Γ_c forms a group and $\Gamma_c/\mathcal{I}$ is abelian.*

These results imply that the "superselection quantum numbers" of those covariant sectors strongly locally equivalent to the vacuum representation and satisfying duality form an abelian group.

It can also been shown that the energy-momentum spectrum in these sectors (determined by $\gamma \in \Gamma_c$) lies in the closed forward light-cone. For proofs and further discussion, see Doplicher, Haag and Roberts (1969b). The general case of monomorphisms is treated by Doplicher, Haag and Roberts (1971, 1974). It is shown that there is a particle-antiparticle structure and also a spin and statistics theorem.