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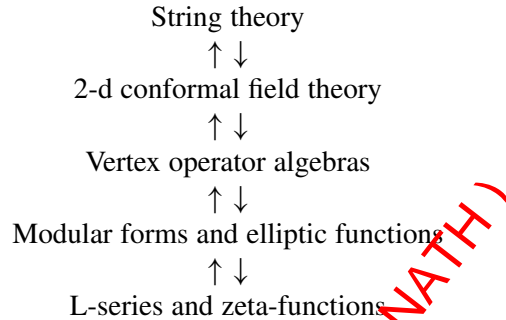
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Part I. Correlation functions and Eisenstein series

1. The big picture



The *leitmotif* of these notes is the idea of a *vertex operator algebra* (VOA) and the relationship between VOAs and *elliptic functions and modular forms*. This is to some extent analogous to the relationship between a finite group and its irreducible characters; the algebraic structure determines a set of numerical invariants, and arithmetic properties of the invariants provides feedback in the form of restrictions on the algebraic structure. One of the main points of these notes is to explain how this works, and to give some reasonably interesting examples.

VOAs may be construed as an axiomatization of 2-dimensional conformal field theory, and it is via this connection that vertex operators enter into physical theories. A sketch of the VOA-CFT connection via the Wightman Axioms can be found in the introduction to [K1]. Although we make occasional comments to relate our development of VOA theory to physics, no technical expertise in physics is necessary to understand these notes. As mathematical theories go, the one we are discussing here is relatively new. There are a number of basic questions which are presently unresolved, and we will get far enough in the notes to explain some of them.

To a modular form one may attach (via the Mellin transform) a Dirichlet series, or *L*-function, and Weil's Converse Theorem says that one can go the other way too. So there is a close connection between modular forms and certain *L*-functions, and this is one way in which our subject matter relates to the contents of other parts of this book. Nevertheless, as things stand at present, it is the Fourier series of a modular form, rather than its Dirichlet series, that is important in VOA theory. As a result, *L*-functions will not enter into our development of the subject.

The notes are divided into three parts. In Part I we give some of the foundations of VOA theory, and explain how modular forms on the full modular group (Eisenstein series in particular) and elliptic functions naturally intervene in the description of n -point correlation functions. This is a general phenomenon and the simplest VOAs, namely the free boson (Heisenberg VOA) and the Virasoro VOA, suffice to illustrate the computations. For this reason we delay the introduction of more complicated VOAs until Part II, where we describe several families of VOAs and their representations. We also cover some aspects of vector-valued modular forms, which is the appropriate language to describe the modular properties of C_2 -cofinite and rational VOAs. We give some applications to holomorphic VOAs to illustrate how modularity impinges on the algebraic structure of VOAs. In Part III we describe two current areas of active research of the authors. The first concerns the development of VOA theory on a genus-two Riemann surface and the second is concerned with the relationship between exceptional VOAs and Lie algebras and the Virasoro algebra.

There are a number of exercises at the end of each subsection. They provide both practice in the ideas and also a subtext to which we often refer during the course of the notes. Some of the exercises are straightforward, others less so. Even if the reader is not intent on working out the exercises, he or she should read them over before proceeding.

These notes constitute an expansion of the lectures we gave at MSRI in the summer of 2008 during the Workshop *A window into zeta and modular physics*. We thank the organizers of the workshop, in particular Klaus Kirsten and Floyd Williams, for giving us the opportunity to participate in the program.

2. Vertex operator algebras

2.1. Notation and conventions. \mathbb{Z} is the set of integers, \mathbb{R} the real numbers, \mathbb{C} the complex numbers, \mathfrak{H} the complex upper half-plane

$$\mathfrak{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}.$$

All linear spaces V are defined over \mathbb{C} ; linear transformations are \mathbb{C} -linear; $\text{End}(V)$ is the space of *all* endomorphisms of V . For an indeterminate z ,

$$V[[z, z^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n z^n \mid v_n \in V \right\}, \quad V[[z]][z^{-1}] = \left\{ \sum_{n=-M}^{\infty} v_n z^n \mid v_n \in V \right\}.$$

These are linear spaces with respect to the obvious addition and scalar multiplication. The formal residue is

$$\text{Res}_z \sum_{n \in \mathbb{Z}} v_n z^n = v_{-1}.$$

For integers m, n with $n \geq 0$,

$$\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{n!}.$$

For indeterminates x, y we adopt the convention that

$$(x+y)^m = \sum_{n \geq 0} \binom{m}{n} x^{m-n} y^n, \quad (1)$$

i.e., for $m < 0$ we formally expand in the second parameter y .

We use the following q -convention:

$$q_x = e^x, \quad q = q_{2\pi i \tau} = e^{2\pi i \tau} \quad (\tau \in \mathbb{R}), \quad (2)$$

where x is anything for which e^x makes sense.

2.2. Local fields. We deal with formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \text{End}(V) \llbracket z, z^{-1} \rrbracket. \quad (3)$$

$a(z)$ defines a linear map $V \rightarrow V \llbracket z, z^{-1} \rrbracket$ by the rule

$$v \mapsto \sum_{n \in \mathbb{Z}} a_n(v) z^{-n-1}.$$

The endomorphisms a_n are called the *modes* of $a(z)$. We often refer to the elements in V as *states*, and call V the *state-space* or *Fock space*.

REMARK 2.1. The convention for powers of z in (3) is standard in mathematics. A different convention is common in the physics literature. Whenever a mathematician and physicist discuss fields, they should first agree on their conventions.

DEFINITION 2.2. $a(z) \in \text{End}(V) \llbracket z, z^{-1} \rrbracket$ is a *field* if it satisfies the following truncation condition $\forall v \in V$:

$$a(z)v \in V \llbracket z \rrbracket \llbracket z^{-1} \rrbracket.$$

That is, for $v \in V$ there is an integer N (depending on v) such that $a_n(v) = 0$ for all $n > N$.

Set

$$\mathfrak{F}(V) = \{a(z) \in \text{End}(V) \llbracket z, z^{-1} \rrbracket \mid a(z) \text{ is a field}\}.$$

$\mathfrak{F}(V)$ is the field-theoretic analog of $\text{End}(V)$; it's a subspace of $\text{End}(V) \llbracket z, z^{-1} \rrbracket$.

The introduction of a *second* indeterminate facilitates the study of products and commutators of fields. Set

$$\left[\sum_m a_m z_1^{-m-1}, \sum_n b_n z_2^{-n-1} \right] = \sum_{m,n} [a_m, b_n] z_1^{-m-1} z_2^{-n-1},$$

which lies in $\text{End}(V)[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$. The idea of *locality* is crucial.

DEFINITION 2.3. Two elements $a(z), b(z) \in \text{End}(V)[[z, z^{-1}]]$ are *mutually local* if there is a nonnegative integer k such that

$$(z_1 - z_2)^k [a(z_1), b(z_2)] = 0. \quad (4)$$

If (4) holds, we write $a(z) \sim_k b(z)$ and say that $a(z)$ and $b(z)$ are *mutually local of order k* . Write $a(z) \sim b(z)$ if k is not specified. $a(z)$ is a *local field* if $a(z) \sim a(z)$. (4) means that the coefficient of each monomial $z_1^{r-1} z_2^{s-1}$ in the expansion of the left hand side vanishes. Explicitly this means that

$$\sum_{j=0}^k (-1)^j \binom{k}{j} [a_{k-j-r}, b_{j-s}] = 0. \quad (5)$$

Locality defines a *symmetric relation* which is generally neither reflexive nor transitive.

Fix a *nonzero* state $\mathbf{1} \in V$. We say that $a(z) \in \mathfrak{F}(V)$ is *creative* (with respect to $\mathbf{1}$) and *creates the state a* if

$$a(z)\mathbf{1} = a + \cdots \in V[[z]].$$

We sometimes write this in the form $a(z)\mathbf{1} = a + O(z)$. In terms of modes,

$$a_n \mathbf{1} = 0, \quad n \geq 0, \quad a_{-1} \mathbf{1} = a.$$

EXERCISE 2.4. Let $\partial a(z) = \sum_n (-n-1) a_n z^{-n-2}$ be the formal derivative of $a(z)$. Suppose that $a(z), b(z) \in \mathfrak{F}(V)$ and $a(z) \sim_k b(z)$. Prove that $\partial a(z) \in \mathfrak{F}(V)$ and $\partial a(z) \sim_{k+1} b(z)$.

EXERCISE 2.5. (Locality-truncation relation) Suppose that $a(z), b(z)$ are creative fields with $a(z) \sim_k b(z)$. By choosing $s = 1$ and $r = k - n$ for $n \geq k$ in (5), show that $a_n b = 0$ for all $n \geq k$ i.e., the order of truncation N is $k - 1$.

2.3. Axioms for a vertex algebra. For various approaches to the contents of this subsection, see [B], [FHL], [FLM], [Go], [K1], [LL], [MN].

DEFINITION 2.6. A *vertex algebra* (VA) is a quadruple $(V, Y, \mathbf{1}, D)$, where

$$Y : V \rightarrow \mathfrak{F}(V), \quad v \mapsto Y(v, z) = \sum v_n z^{-n-1} \text{ is a linear map,}$$

$$\mathbf{1} \in V, \quad \mathbf{1} \neq 0,$$

$$D \in \text{End}(V), \quad D\mathbf{1} = 0,$$

and the following hold for all $u, v \in V$:

$$\text{locality: } Y(u, z) \sim Y(v, z),$$

$$\text{creativity: } Y(u, z)\mathbf{1} = u + \theta(z),$$

$$\text{translation covariance: } [D, Y(u, z)] = \partial Y(u, z).$$

We often refer to the Fock space V itself, rather than $(V, Y, \mathbf{1}, D)$, as a vertex algebra. The element $\mathbf{1}$ is called the *vacuum state* and Y is the *state-field correspondence*. The physical interpretation of creativity is that $Y(u, z)$ *creates* the state u from the vacuum. This set-up models the creation and annihilation of bosonic states from the vacuum. Most of the subtlety is tied to locality and its consequences.

There are a number of equivalent formulations of these axioms. We discuss some of them. Another approach, via so-called *rationality* ([FHL]) is also discussed in Section 10.1. The *Jacobi Identity* of [FLM] is equivalent to the identity

$$\sum_{i \geq 0} \binom{p}{i} (u_{r+i} v)_{p+q-i} = \sum_{i \geq 0} (-1)^i \binom{r}{i} \{u_{p+r-i} v_{q+i} - (-1)^r v_{q+r-i} u_{p+i}\}, \quad (6)$$

which holds in a VA for all $u, v \in V$ and all $p, q, r \in \mathbb{Z}$. Conversely, if we have creative fields $Y(v, z) \in \mathfrak{F}(V)$, $v \in V$, with respect to $\mathbf{1}$ and they satisfy (6), then $(V, Y, \mathbf{1}, D)$ is a vertex algebra with $Du = u_{-2}\mathbf{1}$.

Specializing (6) in various ways leads to some particularly useful identities first written down in [B]:

$$\text{commutator: } [u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i},$$

$$\text{associator: } (u_m v)_n = \sum_{i \geq 0} (-1)^i \binom{m}{i} \{u_{m-i} v_{n+i} - (-1)^m v_{m+n-i} u_i\},$$

$$\text{skew-symmetry: } u_m v = \sum_{i \geq 0} (-1)^{m+i+1} \frac{1}{i!} D^i v_{m+i} u.$$

These identities may be stated more compactly using vertex operators, and it is often more efficacious to use the vertex operator format. We state one more consequence of (6), the *associativity* formula, in the operator format. For large enough k , and recalling convention (1),

$$(z_1 + z_2)^k Y(u, z_1 + z_2)Y(v, z_2)w = (z_1 + z_2)^k Y(Y(u, z_1)v, z_2)w. \quad (7)$$

THEOREM 2.7 [FKRW], [MP]. *Let V be a linear space with $0 \neq \mathbf{1} \in V$ and $D \in \text{End}(V)$. Suppose $S \subseteq \mathfrak{F}(V)$ is a set of mutually local, creative, translation-covariant fields which generates V in the sense that*

$$V = \text{span}\{a_{-n_1}^1 \dots a_{-n_k}^k \mathbf{1} \mid a^i(z) \in S, n_1, \dots, n_k \geq 1, k \geq 0\}.$$

Then there is a unique vertex algebra $(V, Y, \mathbf{1}, D)$ such that $Y(a_{-1}^i \mathbf{1}, z) = a^i(z)$.

EXERCISE 2.8. Prove that the state-field correspondence is *injective*.

EXERCISE 2.9. Prove that

$$Y(u, z)\mathbf{1} = q_z^D u \quad (\text{which equals } \sum_{n \geq 0} \frac{z^n}{n!} D^n u).$$

EXERCISE 2.10. Deduce the commutator, associator and skew-symmetry formulas from (6).

EXERCISE 2.11. Assume V is a linear space and

$$\{Y(v, z) \mid v \in V\} \subseteq \mathfrak{F}(V)$$

are mutually local fields such that $Y(v, z)$ is creative (with respect to $\mathbf{1} \neq 0$) and creates v . Prove that (6) and the associator formula are *equivalent*.

EXERCISE 2.12. A is a commutative, associative algebra with identity element 1 and derivation D . Show that there is a vertex algebra $(A, Y, 1, D)$ with

$$Y(a, z)b = \sum_{n \leq -1} \frac{(D^{-n-1}a)b}{(-n-1)!} z^{-n-1}.$$

EXERCISE 2.13. $(V, Y, \mathbf{1}, D)$ is a VA. Assume either (a) $Y(v, z) \in \text{End}(V)[[z]]$ for $v \in V$, (b) D is the zero map, or (c) $\dim V$ is finite. Prove in each case that V is of the type described in Exercise 2.12.

EXERCISE 2.14. Show that the commutator formula is equivalent to the identity $[u_m, Y(v, z)] = \sum_{i \geq 0} \binom{m}{i} Y(u_i v, z) z^{m-i}$.

EXERCISE 2.15. Show that the skew-symmetry formula is equivalent to the identity $Y(u, z)v = q_z^D Y(v, -z)u$.

EXERCISE 2.16. Show that $q_y^D Y(u, x)q_y^{-D} = Y(u, x + y)$.

2.4. Heisenberg algebra. In this and the following Subsection we will use Theorem 2.7 to construct two fundamental examples of VAs. We must look for generating sets S of mutually local, creative, translation-covariant fields. In our two examples, S consists of a *single* field. The construction relies on some basic techniques from Lie theory (universal enveloping algebras, Poincaré–Birkhoff–Witt Theorem, and so on) which are reviewed in the Appendix.

Let $A = \mathbb{C}a$ be a 1-dimensional linear space. The *affine algebra*

$$\hat{A} = A[t, t^{-1}] \oplus \mathbb{C}K$$

is the Lie algebra with central element K and bracket

$$[a \otimes t^m, a \otimes t^n] = m\delta_{m,-n}K. \quad (8)$$

REMARK 2.17. Set $p_m = \frac{1}{\sqrt{m}}a \otimes t^m$ ($m > 0$) and $q_{-m} = \frac{1}{\sqrt{-m}}a \otimes t^m$ ($m < 0$). Then (8) reads

$$[p_m, q_n] = \delta_{m,n}K. \quad (9)$$

These are essentially the *canonical commutation relations* of quantum mechanics.

Set $\hat{A}^{\geq} = \langle a \otimes t^n, K \mid n \geq 0 \rangle$, $\hat{A}^- = \langle a \otimes t^n \mid n < 0 \rangle$. These are a Lie ideal and Lie subalgebra of \hat{A} respectively. Let $\mathbb{C}v_h$ be the 1-dimensional \hat{A}^{\geq} -module defined for a scalar h via

$$K.v_h = v_h, \quad (a \otimes t^n).v_h = h\delta_{n,0}v_h \quad (n \geq 0).$$

The induced (Verma) module is

$$M_h = \text{Ind}_{\mathcal{U}(\hat{A}^{\geq})}^{\mathcal{U}(\hat{A})} \mathbb{C}v_h = \mathcal{U}(\hat{A}) \otimes_{\mathcal{U}(\hat{A}^{\geq})} \mathbb{C}v_h = \mathcal{U}(\hat{A}^-) \otimes \mathbb{C}v_h \quad (10)$$

where $\mathcal{U}(\cdot)$ denotes universal enveloping algebra and the third equality in the last display is just a linear isomorphism.

Let $a_n \in \text{End}(M_h)$ be the induced action of $a \otimes t^n$ on M_h , with $a(z) = \sum_n a_n z^{-n-1}$. In what follows we identify v_h with $1 \otimes v_h$. Let

$$v = a_{-n_1} \dots a_{-n_k}.v_h,$$

with $n_1 \geq \dots \geq n_k \geq 1$. For $n > n_1$, a_n commutes with each a_{-n_i} by (8). Therefore, $a_n.v = a_{-n_1} \dots a_{-n_k} a_n.v_h = 0$. This shows that $a(z) \in \mathfrak{F}(M_h)$. As for locality,

$$\begin{aligned} \sum_{j=0}^2 (-1)^j \binom{2}{j} [a_{2-j-r}, a_{j-s}] &= \sum_{j=0}^2 (-1)^j \binom{2}{j} (2-j-r)\delta_{2-j-r,s-j}K \\ &= \{(2-r) - 2(1-r) - r\}\delta_{r+s,2}K = 0. \end{aligned} \quad (11)$$

By (5) this shows that $a(z) \sim_2 a(z)$. Because

$$a(z)v_h = hv_h z^{-1} + \sum_{n \leq -1} a_n \cdot v_h z^{-n-1},$$

we see that $a(z)$ is creative with respect to v_h if (and only if) $h = 0$. In this case, Theorem 2.7 and what we have shown imply:

THEOREM 2.18. *There is a unique vertex algebra (M_0, Y, v_0, D) generated by $a(z)$ with $Y(a, z) = a(z)$ with $a = a_{-1}v_0 \in M_0$, and $Da_n \cdot v_0 = -na_{n-1}v_0$.*

REMARK 2.19. In terms of operators on M_0 , (9) reads $[p_m, q_n] = \delta_{m,n} \text{Id}$. These relations may be realized by taking $p_m = \partial/\partial x_{-m}$, $q_n = x_{-n}$ acting on the Fock space $\mathbb{C}[x_{-1}, x_{-2}, \dots]$. This affords an alternate way to understand M_0 .

M_0 is variously called the (rank 1) Heisenberg VA, Heisenberg algebra, or *free boson*. In CFT it models a single free boson. (As opposed to standard mathematical usage, *free* here means that the particle is not interacting with other particles.)

2.5. Virasoro algebra. The Virasoro algebra is the Lie algebra with underlying linear space

$$\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}K$$

and bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m,-n} K. \tag{12}$$

Set $\text{Vir}^{\geq} = \langle L_n, K \mid n \geq 0 \rangle$, $\text{Vir}^- = \langle L_n \mid n < 0 \rangle$, and let $\mathbb{C}v_{c,h}$ be the 1-dimensional Vir^{\geq} -module defined via

$$K \cdot v_{c,h} = cv_{c,h}, \quad L_n \cdot v_{c,h} = \delta_{n,0} h v_{c,h} \quad (n \geq 0).$$

with arbitrary scalars c, h . The induced (Verma) module is then

$$M_{c,h} = \mathcal{U}(\text{Vir}) \otimes_{\mathcal{U}(\text{Vir}^{\geq})} \mathbb{C}v_{c,h} = \mathcal{U}(\text{Vir}^-) \otimes \mathbb{C}v_{c,h}. \tag{13}$$

By analogy with Theorem 2.18, Exercise 2.22 (below) suggests that there is a VA with Fock space $M_{c,0}$ and vacuum¹ $v_{c,0}$, with L_{-1} playing the rôle of D . This cannot be true as it stands because $\omega(z) \cdot v_{c,0} = L_{-1} \cdot v_{c,0} z^{-1} + \dots$ is *not* creative. To cure this ill requires (at the very least) that we take a quotient of $M_{c,0}$ by a Vir-submodule that contains $L_{-1} \cdot v_{c,0}$, and indeed it suffices to quotient out the cyclic Vir-submodule generated by this state. We will abuse notation by identifying states, operators and fields associated with $M_{c,0}$ with the corresponding

¹As in the case of the Heisenberg algebra, we identify $v_{c,0}$ and $1 \otimes v_{c,0}$.

states, operators and fields induced on the quotient $M_{c,0}/\mathcal{U}(\text{Vir})L_{-1}.v_{c,0}$. We then arrive at

THEOREM 2.20. *Set*

$$\text{Vir}_c = M_{c,0}/\mathcal{U}(\text{Vir})L_{-1}.v_{c,0},$$

$\omega = L_{-2}.v_{c,0}$, and $Y(\omega, z) = \omega(z)$. Then $(\text{Vir}_c, Y, v_{c,0}, L_{-1})$ is a vertex algebra generated by $Y(\omega, z)$.

Vir_c is called the Virasoro VA of *central charge* c .

EXERCISE 2.21. Show that L_{-1} , L_0 and L_1 span a Lie subalgebra of Vir . What Lie algebra is it?

EXERCISE 2.22. Identify elements of Vir with the endomorphisms they induce on $M_{c,h}$ and set $\omega(z) = \sum L_n z^{-n-2} \in \text{End}(M_{c,h})[[z, z^{-1}]]$. Prove that $\omega(z)$ is a local field of order 4, and $[L_{-1}, \omega(z)] = \partial\omega(z)$.

EXERCISE 2.23. Give the details of the proof of Theorem 2.20.

2.6. Axioms for a vertex operator algebra. There is no consensus as to nomenclature for the many variants of vertex algebra. Our definition of a *vertex operator algebra* (VOA) is the one used by many practitioners of the art, but not all.

DEFINITION 2.24. A VOA is a quadruple $(V, Y, \mathbf{1}, \omega)$, where $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is a \mathbb{Z} -graded linear space and

$$Y : V \rightarrow \mathfrak{F}(V), \quad v \mapsto Y(v, z) = \sum v_n z^{-n-1},$$

$$\mathbf{1}, \omega \in V, \quad \mathbf{1} \neq 0.$$

The fields $Y(v, z)$ are assumed to be mutually local and creative, and certain conditions must be satisfied:

- $Y(\omega, z) = \sum L_n z^{-n-2}$ with a constant c such that

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}c \text{Id}_V;$$

- $V_n = \{v \in V_n \mid L_0 v = n v\}$;
- $\dim V_n < \infty$, $V_n = 0$ for $n \ll 0$;
- $Y(L_{-1}u, z) = \partial Y(u, z)$.

In effect, a VOA is a vertex algebra with a dedicated Virasoro field. This is the field determined by the distinguished state ω , called the *conformal* or *Virasoro vector*. The modes of ω are operators L_n satisfying the Virasoro relations (12) with $K = c \text{Id}_V$. As in Theorem 2.20 we call c the *central charge* of V . The mode L_0 of ω , called the *degree operator*, is required to be semisimple, to

have eigenvalues lying in a subset of \mathbb{Z} that is bounded below, and to have finite-dimensional eigenspaces. We often write $\text{wt}(v) = n$ if v is an eigenstate for L_0 with eigenvalue n , the conformal weight. It is not hard to see that $[L_{-1}, Y(u, z)] = \partial Y(u, z)$, so that $(V, Y, \mathbf{1}, L_{-1})$ is a vertex algebra.

It should come as no surprise that the vertex algebra Vir_c has the structure of a VOA of central charge c with vacuum vector $v_{c,0}$ and conformal vector ω . To see this, note that $\{L_{-n_1} \dots L_{-n_k} \cdot v_{c,0} \mid n_1 \geq \dots \geq n_k \geq 2\}$ is a basis of the Fock space. We have

$$L_0 L_{-n_1} \dots L_{-n_k} \cdot v_{c,0} = n_1 L_{-n_1} \dots L_{-n_k} \cdot v_{c,0} + L_{-n_1} L_0 L_{-n_2} \dots L_{-n_k} \cdot v_{c,0}.$$

Now an easy induction shows that

$$L_0 \cdot L_{-n_1} \dots L_{-n_k} \cdot v_{c,0} = \left(\sum_i n_i \right) \cdot L_{-n_1} \dots L_{-n_k} \cdot v_{c,0},$$

so that

$$\text{wt}(L_{-n_1} \dots L_{-n_k} \cdot v_{c,0}) = \sum n_i.$$

The needed properties of L_0 required for the next result follow easily, and we obtain the following extension of Theorem 2.20.

THEOREM 2.25. *Vir_c is a VOA of central charge c .*

A pair of VOAs V, V' are called *isomorphic* if there is a linear isomorphism $\varphi : V \rightarrow V', v \mapsto v'$ such that $\varphi(\omega) = \omega'$, and $\varphi Y(v, z) \varphi^{-1} = Y'(v', z)$.

In the following exercises, V is a VOA.

EXERCISE 2.26. Complete the proof of Theorem 2.25.

EXERCISE 2.27. Prove the following: $Y(\mathbf{1}, z) = \text{Id}, \mathbf{1} \in V_0, \omega \in V_2, L_n \mathbf{1} = 0$ for $n \geq -1, (L_{-1} v)_n = -n v_{n-1}$.

EXERCISE 2.28. Suppose that $v \in V$ satisfies $L_{-1} v = 0$. Prove that $v \in V_0$.

EXERCISE 2.29. Suppose that $V_0 = \mathbb{C} \mathbf{1}$ (cf. Exercise 2.27). Prove that $V_n = 0$ for $n < 0$.

EXERCISE 2.30. Show that $\dim V$ is finite if, and only if, $\omega = 0$ (cf. Exercise 2.13).

EXERCISE 2.31. Show that the Heisenberg theory M_0 (cf. Section 2.4) is a VOA with vacuum $\mathbf{1} = v_0, \omega = \frac{1}{2} a_{-1}^2 \mathbf{1} = \frac{1}{2} a_{-1} a$ and central charge $c = 1$. This is the theory of *one free boson*.

EXERCISE 2.32. Let U, V be linear spaces. Show that there is a natural injection $\mathfrak{F}(U) \otimes \mathfrak{F}(V) \rightarrow \mathfrak{F}(U \otimes V)$. Suppose in addition that U and V are Fock spaces for VOAs with vacuum vectors $\mathbf{1}, \mathbf{1}'$ and conformal vectors ω, ω' respectively. Show how to construct the *tensor product VOA* $(U \otimes V, Y, \mathbf{1} \otimes \mathbf{1}', \omega \otimes \omega')$. What is the central charge of this VOA?

EXERCISE 2.33. Let $\varphi : V \rightarrow V'$ be an isomorphism of VOAs. Prove the following: (i) V and V' have the *same* central charge; (ii) $\varphi(\mathbf{1}) = \mathbf{1}'$.

2.7. VOAs on the cylinder and the square bracket formalism. There is a sense in which we may think of a VOA as being ‘on the sphere’. This is closely related to the axiomatic approach via rationality (cf. [FHL] and Section 10.1). Here we want to describe the corresponding VOA that lives ‘on the cylinder’. Roughly, this corresponds to a change of variable $z \rightarrow q_z - 1$ which we call the *square bracket formalism*. The main purpose is to construct vertex operators that are automatically periodic in z with period $2\pi i$. Let $V = (V, Y, \mathbf{1}, \omega)$ be a VOA of central charge c . For $v \in V$ introduce²

$$Y[v, z] = Y(q_z^{L_0} v, q_z - 1) = \sum_{n \in \mathbb{Z}} v[n] z^{-n-2}. \quad (14)$$

Here, $q_z^{L_0}$ is the operator

$$q_z^{L_0} : V \rightarrow V[[z]], \quad v \mapsto q_{kz} v \quad (v \in V_k), \quad (15)$$

and our q -convention (2) is in force. Similar expressions will occur frequently in what follows. The $v[n]$ are new operators on V , and for $v \in V_k$ are given by

$$v[m] = m! \sum_{i \geq m} c(k, i, m) v_i \quad (16)$$

for $m \geq 0$, with

$$\binom{k-1+x}{i} = \sum_{m=0}^i c(k, i, m) x^m. \quad (17)$$

From (16) and (17) we find

$$\sum_{i=0}^k \binom{k}{i} v_i = \sum_{m \geq 0} \frac{(k+1-k)^m}{m!} v[m]. \quad (18)$$

These identities are proved in Section 13 (Appendix). We also have a new conformal vector

$$\tilde{\omega} = \omega - \frac{c}{24} \mathbf{1}, \quad (19)$$

with corresponding square bracket modes

$$Y[\tilde{\omega}, z] = \sum_n L[n] z^{-n-2}.$$

In particular, $L[0]$ provides us with an alternative \mathbb{Z} -grading structure on V :

$$V = \bigoplus_n V_{[n]}, \quad V_{[n]} = \{u \in V \mid L[0]u = nu\}.$$

²We write modes in the square bracket formalism as $v[n]$ rather than $v_{[n]}$.

We write $\text{wt}[v] = n$ if $v \in V_{[n]}$. The following can be proved.

THEOREM 2.34. *The quadruple $(V, Y[\cdot, \cdot], \mathbf{1}, \tilde{\omega})$ is a VOA of central charge c .*

Given a VOA V , we say that its alter ego $(V, Y[\cdot, \cdot], \mathbf{1}, \tilde{\omega})$ is ‘on the cylinder’. VOAs on the cylinder play an important rôle in forging the connections with modular forms.

EXAMPLE 2.35. In the square bracket formalism, the VOA $(M_0, Y[\cdot, \cdot], \mathbf{1}, \tilde{\omega})$ is generated by a state a with $\text{wt}[a] = 1$. It has a basis of Fock vectors of the form $a[-n_1] \dots a[-n_k] \mathbf{1}$, $n_1 \geq \dots \geq n_k \geq 1$ satisfying

$$[a[m], a[n]] = m\delta_{m+n,0}\text{Id}.$$

EXERCISE 2.36. Show that $L[-1] = L_{-1} + L_0$.

EXERCISE 2.37. A state v in a VOA V is called *primary* of weight k with respect to the original Virasoro algebra $\{L_n\}$ if, and only if, it satisfies $L_nv = k\delta_{n,0}v$ for $n \geq 0$. Prove that v is primary of weight k with respect to $\{L_n\}$ if, and only if, it is primary of weight k with respect to $\{L[n]\}$.

EXERCISE 2.38. Prove the assertions of Example 2.35 in the more precise form that $(M_0, Y, \mathbf{1}, \omega)$ and $(M_0, Y[\cdot, \cdot], \mathbf{1}, \tilde{\omega})$ are *isomorphic* Heisenberg VOAs.

3. Modular and quasimodular forms

In this section we compile some relevant background involving *elliptic modular forms*. This is a standard part of analytic number theory, and there are many excellent texts dealing with the subject, such as [Kn], [O], [Se], [Sc]. Because it is so central to our cause, we describe what we need here, referring the reader elsewhere for more details and further development.

3.1. Modular forms on $\text{SL}_2(\mathbb{Z})$. The (homogeneous) modular group is

$$\Gamma = \text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

with standard generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The complex upper half-plane \mathfrak{H} carries a left Γ -action by Möbius transformations

$$(\gamma, \tau) \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \quad (20)$$

In particular, $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$. For $k \in \mathbb{Z}$, there is a right action of Γ on meromorphic functions in \mathfrak{H} given by

$$f|_k\gamma(\tau) = (c\tau + d)^{-k} f(\gamma\tau). \quad (21)$$

A *weak modular form* of weight k on Γ is an *invariant* of this action. Thus $f|_k\gamma(\tau) = f(\tau)$ for $\gamma \in \Gamma$, which amounts to

$$f(\tau + 1) = f(\tau), \quad f(-1/\tau) = \tau^k f(\tau).$$

By a standard argument the first of these equalities implies that $f(\tau)$ has a *q-expansion*, or *Fourier expansion at ∞* ,

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n, \quad (22)$$

with constants a_n called the *Fourier coefficients* of $f(\tau)$. Here we are using our *q-convention* (2).

We say that $f(\tau)$ is a *meromorphic modular form of weight k* if its *q-expansion* has the form

$$f(\tau) = \sum_{n \geq n_0} a_n q^n \quad (23)$$

for some n_0 . Assume that $f(\tau) \neq 0$ with $a_{n_0} \neq 0$. We then say that $f(\tau)$ has a pole of order n_0 at ∞ if $n_0 \leq 0$ or a zero of order n_0 if $n_0 \geq 0$. In the latter situation we also say that $f(\tau)$ is *holomorphic at ∞* . $f(\tau)$ is a *holomorphic modular form of weight k* if it is holomorphic in $\mathfrak{H} \cup \{\infty\}$. $f(\tau)$ is *almost holomorphic* if it is holomorphic in \mathfrak{H} (the behaviour at ∞ being unspecified beyond being at worst a pole). Modular forms of weight 0 are often called *modular functions*, though we will not be consistent on this point. Let \mathfrak{M}_k be the set of holomorphic modular forms of weight k . It is a \mathbb{C} -linear space, possibly equal to 0.

EXERCISE 3.1. Show that the *kernel* of the Γ -action (20) is the *center* of Γ and consists of $\pm I$, where I is the 2×2 identity matrix. (The quotient group $PSL_2(\mathbb{Z}) = \tilde{\Gamma} / \{\pm I\}$ is the *inhomogeneous modular group*.)

EXERCISE 3.2. (a) Show that torsion elements in $\tilde{\Gamma}$ have order *at most* 3. (b) Show that $\tilde{\Gamma}$ has a *unique* conjugacy class of subgroups of order 2 or 3.

EXERCISE 3.3. Let $z \in \mathfrak{H}$ with $\text{Stab}_{\tilde{\Gamma}}(z) = \{\gamma \in \tilde{\Gamma} \mid \gamma.z = z\}$ the *stabilizer* of z in $\tilde{\Gamma}$. Prove the following: (a) $\text{Stab}_{\tilde{\Gamma}}(z)$ is a finite cyclic subgroup, (b) each nontrivial torsion element in $\tilde{\Gamma}$ stabilizes a *unique* point in \mathfrak{H} .

EXERCISE 3.4. Show that $\tilde{\Gamma}$ acts *properly discontinuously* on \mathfrak{H} in the following sense: every $z \in \mathfrak{H}$ has an open neighborhood N_z with the property that if $\gamma \in \tilde{\Gamma}$ then $\gamma(N_z) \cap N_z = \emptyset$ if $\gamma \notin \text{Stab}_{\tilde{\Gamma}}(z)$ and $\gamma(N_z) \cap N_z = N_z$ otherwise. Conclude that the *orbit space* $\Gamma \backslash \mathfrak{H}$ is a topological 2-manifold (a Hausdorff space such that each point has an open neighborhood homeomorphic to \mathbb{R}^2).

EXERCISE 3.5. Suppose that $f(\tau)$ is a nonzero weak modular form of weight k . Show that k is even.

EXERCISE 3.6. Let E be the set of meromorphic modular functions of weight zero. Show that E is a field³ containing \mathbb{C} .

EXERCISE 3.7. Show that pointwise multiplication defines a bilinear product $\mathfrak{M}_k \otimes \mathfrak{M}_l \rightarrow \mathfrak{M}_{k+l}$, with respect to which $\mathfrak{M} = \bigoplus_k \mathfrak{M}_k$ is a \mathbb{Z} -graded commutative \mathbb{C} -algebra.

EXERCISE 3.8. Suppose that $f(\tau)$ is a meromorphic modular form of weight zero. Show that $f'(\tau)$ is a meromorphic modular form of weight 2.

3.2. Eisenstein series on $SL_2(\mathbb{Z})$. Beyond the fact that constants in \mathbb{C} are modular functions of weight 0 (cf. Exercise 3.6), it is not so easy to construct nonconstant modular functions of weight 0 or any nonzero modular form of nonzero weight. We content ourselves with the description of some examples chosen because of their relevance to VOA theory.

The most accessible nonconstant modular forms are the *Eisenstein series*. For an integer $k \geq 2$, set

$$\begin{aligned} E_k(\tau) &= -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1-q^n} \\ &= -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n. \end{aligned} \tag{24}$$

Here, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k is the k -th Bernoulli number defined by⁴

$$\frac{z}{qz-1} = \sum_{k \geq 0} \frac{B_k}{k!} z^k = 1 - \frac{1}{2}z + \frac{1}{12}z^2 + \dots \tag{25}$$

The well-known identity of Euler

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2(k!)} \quad (k \geq 2 \text{ even}), \tag{26}$$

permits us to reexpress the constant term of (24) in terms of zeta-values. The basic fact is this: *Let $k \geq 3$. Then $E_k(\tau)$ is a holomorphic modular form of weight k ; it is identically zero if, and only if, k is odd.* We will see one way to prove this in Section 5. We emphasize that $E_2(\tau)$ is not a modular form.

The normalization employed in (24) is related to elliptic functions (Section 5). In fact B_{2k} never vanishes, so we can renormalize so that the q -expansion

³Of course, field here is in the algebraic sense.

⁴Several different conventions are used to define Bernoulli numbers in the literature.

begins $1 + \dots$. We single out the first three Eisenstein series corresponding to $k = 2, 4, 6$ renormalized in this way, and rename them (following Ramanujan)

$$P = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n, \quad Q = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n, \quad R = 1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n.$$

P, Q, R are algebraically independent, so that they generate a weighted polynomial algebra, denoted

$$\mathfrak{Q} = \mathbb{C}[P, Q, R], \quad (27)$$

where P, Q, R naturally have weights (degree) 2, 4, 6 respectively. \mathfrak{Q} is the algebra of quasimodular forms. \mathfrak{Q} contains every holomorphic modular form. Indeed, we have (cf. Exercise 3.7)

THEOREM 3.9. *The graded algebra $\mathfrak{M} = \bigoplus \mathfrak{M}_k$ of holomorphic modular forms on Γ is the graded subalgebra $\mathbb{C}[Q, R]$ of \mathfrak{Q} .*

Theorem 3.9 follows from a careful study of the singularities (zeros and poles) of modular forms, but we will not discuss this here. The Theorem contains a lot of information about holomorphic modular forms. For example, there are no such nonzero forms of negative weight or weight 2, holomorphic forms of weight zero are necessarily constant, and $\dim \mathfrak{M}_k < \infty$. Indeed, inasmuch as Q and R are free generators in weights 4 and 6 respectively, the Hilbert–Poincaré series of \mathfrak{M} is

$$\begin{aligned} \sum_{k \geq 0} (\dim \mathfrak{M}_k) t^k &= \frac{1}{(1-t^4)(1-t^6)} \\ &= 1 + t^4 + t^6 + t^8 + t^{10} + 2t^{12} + t^{14} + 2t^{16} + \dots \end{aligned} \quad (28)$$

As we already mentioned, $E_2(\tau)$ is not a modular form. Indeed, it satisfies the transformation law

$$E_2|_2 \gamma(\tau) = E_2(\tau) - \frac{c}{2\pi i(c\tau + d)}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (29)$$

The importance of $E_2(\tau)$ for us stems from its relation to derivatives of modular forms. Suppose that $f(\tau)$ is a meromorphic modular form of weight k . We define the modular derivative of $f(\tau)$ by

$$D_k f(\tau) = Df(\tau) = (\theta + kE_2(\tau)) f_k(\tau). \quad (30)$$

where $\theta = q d/dq$. One can show without difficulty (cf. Exercise 3.13) that $Df_k(\tau)$ is modular of weight $k + 2$, and is holomorphic if $f_k(\tau)$ is.

EXERCISE 3.10. Prove that

$$\frac{q_z}{(1-q_z)^2} = \frac{1}{z^2} - \sum_{k \geq 2} \frac{B_k}{k!} (k-1)z^{k-2}.$$

Deduce that $B_k = 0$ for odd $k \geq 3$.

EXERCISE 3.11. Prove that $E_8 = \frac{3}{7}E_4^2$ and $E_{10} = \frac{5}{11}E_4E_6$.

EXERCISE 3.12. Show that (28) is equivalent to the formula $\dim \mathfrak{M}_{2k} = [k/6]$ if $k \equiv 1 \pmod{6}$ and $1 + [k/6]$ otherwise.

EXERCISE 3.13. Prove that $(D_k f)|_{k+2}(\tau) = D_k(f|_k \gamma)(\tau)$ for any meromorphic function $f(\tau)$. Conclude that D_k induces a linear map $\mathfrak{M}_k \rightarrow \mathfrak{M}_{k+2}$.

EXERCISE 3.14. Prove that $DE_4 = 14E_6$ and $DE_6 = \frac{60}{7}E_4^2$.

EXERCISE 3.15. Let $D : \mathfrak{M} \rightarrow \mathfrak{M}$ be the linear map whose restriction to \mathfrak{M}_k is D_k . Prove that D is a *derivation* of the algebra \mathfrak{M} .

3.3. Cusp-forms and modular functions on $SL_2(\mathbb{Z})$. Thanks to Theorem 3.9, every holomorphic modular form of weight k is equal to a unique homogeneous polynomial in Q and R . In this subsection we describe some important examples of particular relevance to VOAs. We start with the *discriminant*, defined by

$$\Delta(\tau) = \frac{Q^3 - R^2}{12^3} = q - 24q^2 + \dots \tag{31}$$

$\Delta(\tau)$ is evidently a holomorphic modular form of weight 12. It may alternatively be described by a q -product which goes back to Kronecker, namely

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}. \tag{32}$$

This formula finds its natural place in the theory of elliptic functions. From our present vantage point, the fact that (31) and (32) coincide is miraculous. Beyond the product formula, the properties that make $\Delta(\tau)$ important for us are the following: it *does not vanish* in \mathfrak{H} , and (up to scalars) it is the unique nonzero holomorphic modular form of least weight that *vanishes at* ∞ . The nonvanishing property has a natural explanation in the theory of elliptic functions. Concerning the second property, we introduce *cusp forms* defined by

$$\mathfrak{S}_k = \{f(\tau) \in \mathfrak{M}_k \mid f \text{ vanishes at } \infty\}, \quad \mathfrak{S} = \bigoplus_k \mathfrak{S}_k.$$

Our assertions then say that $\mathfrak{S}_k = 0$ for $k < 12$ and $\mathfrak{S}_{12} = \mathbb{C}\Delta(\tau)$. Using (31), it follows that $\Delta(\tau)^{-1}$ is an almost holomorphic modular form of weight -12 with a pole of order 1 at ∞ . Applications of these facts are given in Exercise 3.16.

Closely related to $\Delta(\tau)$ is the *Dedekind η -function*, whose q -expansion is the 24th root of that for $\Delta(\tau)$:

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n). \tag{33}$$

Note that $\eta(\tau)$ is not a modular form in our sense. It satisfies identity

$$\begin{aligned}\eta(\tau)^{-1} &= q^{-1/24} \sum_{n \geq 0} p(n)q^n \\ &= q^{-1/24}(1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots),\end{aligned}\tag{34}$$

where $p(n)$ is the *unrestricted partition function*. This identity goes back to Euler.

Our next example is the famous j -function, defined by

$$j(\tau) = \frac{Q^3}{\Delta(\tau)} = q^{-1} + 744 + 196884q + \cdots\tag{35}$$

As the quotient of two modular forms of weight 12, $j(\tau)$ has weight zero, and because Δ^{-1} is almost holomorphic, so too is $j(\tau)$. In the notation of Exercise 3.6, $j(\tau) \in E$. Let $\Gamma \backslash \mathfrak{H}$ be the orbit space for the action of Γ on \mathfrak{H} (cf. Exercise 3.4). Because the weight is zero, we see from (2) that j induces a map

$$j : \Gamma \backslash \mathfrak{H} \rightarrow \mathbb{C}$$

which turns out to be a *homeomorphism*. It can be shown that $E = \mathbb{C}(j)$ is exactly the field of rational functions in j .

EXERCISE 3.16. Considered as a subspace of the algebra \mathfrak{M} , show that \mathfrak{S} is the principal ideal generated by Δ .

EXERCISE 3.17. Show that $\mathfrak{M}_k = \mathfrak{S}_{2k} \oplus \mathbb{C}E_{2k}$ for $k \geq 2$.

EXERCISE 3.18. Prove that $\theta\eta(\tau) = -\frac{1}{2}\eta(\tau)E_2(\tau)$. Conclude that $D_{12}\Delta(\tau) = 0$. Give another proof of this by using Theorem 3.9.

EXERCISE 3.19. Regard D as a derivation of \mathfrak{M} as in Exercise 3.15. Show that the *space of D -constants* (i.e., the subspace of \mathfrak{M} annihilated by D) is the polynomial algebra $\mathbb{C}[\Delta]$.

EXERCISE 3.20. Prove that the *ring of almost holomorphic modular functions of weight zero* on Γ is the space $\mathbb{C}[j]$ of polynomials in $j(\tau)$.

4. Characters of vertex operator algebras

Fix a VOA $(V, Y, \mathbf{1}, \omega)$ with \mathbb{Z} -graded Fock space $V = \bigoplus V_n$ and central charge c . In this section we introduce the idea of the *character* of V as a sort of analog of the character of a group representation. This is essentially the theory of *1-point correlation functions* on V .

4.1. Zero modes. We start with a useful calculation. Suppose that $v \in V_k, w \in V_m, n \in \mathbb{Z}$. Remembering that $L_i = \omega_{i+1}$, we have

$$\begin{aligned} L_0 v_n w &= ([\omega_1, v_n] + v_n L_0)w = \left(\sum_i \binom{1}{i} (L_{i-1} v)_{n+1-i} + v_n L_0 \right) w \\ &= ((L_{-1} v)_{n+1} + (L_0 v)_n + v_n L_0)w = (m + k - n - 1)v_n w. \end{aligned}$$

Here, we used the commutator formula (cf. Section 2.3) for the second equality and the last identity of Exercise 2.27 for the fourth equality. What we take from this is that *modes of homogeneous states are graded operators on V* :

$$v \in V_k \Rightarrow v_n : V_m \rightarrow V_{m+k-n-1}. \tag{36}$$

In particular, let us define⁵ the *zero mode* $o(v)$ of a state $v \in V_k$ to be v_{k-1} , and extend this definition to V additively. From (36) we then have for all integers m and states v that

$$o(v) : V_m \rightarrow V_m. \tag{37}$$

The point of all this is that as an operator on V_n we can *trace* the zero mode and form a generating function (cf. (15))

$$Z(v, q) = \text{Tr}_V o(v) q^{L_0 - c/24} = q^{-c/24} \sum_n \text{Tr}_{V_n} o(v) q^n. \tag{38}$$

This is to be regarded as a *formal q -expansion* at this point. Apart from memorializing the central charge, the factor $q^{-c/24}$ may seem somewhat arbitrary. This feeling will pass. Because the homogeneous spaces V_n vanish for small enough n , we see that

$$Z(v, q) \in q^{-c/24} \mathbb{C}[[q]][[q^{-1}]].$$

$Z = Z_V$ defines the *character* of V , i.e., the linear map

$$Z : V \rightarrow q^{-c/24} \mathbb{C}[[q]][[q^{-1}]], \quad v \mapsto Z(v, q).$$

EXERCISE 4.1. Let $U \otimes V$ be the tensor product of VOAs U, V (Exercise 2.32). Prove that $Z_{U \otimes V} = Z_U Z_V$.

EXERCISE 4.2. Suppose that V is a VOA with $v \in V$. Prove the identity $q_x^{L_0} Y(v, z) q_x^{-L_0} = Y(q_x^{L_0} v, q_x z)$.

EXERCISE 4.3. Let a be the generating state of weight 1 for the Heisenberg VOA (Section 2.4). Prove that the zero mode $o(a)$ is *zero*.

EXERCISE 4.4. Suppose that $V_0 = \mathbb{C}\mathbf{1}$ (cf. Exercise 2.29). Prove using (36) that for $a \in V_{n_1}$ and $b \in V_{n_2}$ we have $a_n b = 0$ for all $n \geq n_1 + n_2$. Using Exercise 2.5, deduce that $Y(a, z) \sim_k Y(b, z)$ with order of locality $k \leq n_1 + n_2$.

⁵The zero mode of v is generally not the *zeroth* mode v_0 but rather the mode which has weight zero as an operator. However, in the convention used for modes in CFT as practiced by physicists, it is the zero mode.

4.2. Graded dimension. The most prominent Z -value is that obtained by tracing the zero mode of the vacuum. From Exercise 2.26 we have $Y(\mathbf{1}, z) = \text{Id}_V$ and $\mathbf{1} \in V_0$. So the zero mode of $\mathbf{1}$ is Id_V , whence

$$Z_V(\mathbf{1}) = \text{Tr}_V q^{L_0 - c/24} = q^{-c/24} \sum_n \dim V_n q^n. \quad (39)$$

This is variously called the *graded dimension*, *q-dimension*, *0-point function*, or *partition function* of V .

The graded dimensions of our two main examples M_0 and Vir_c are readily computed. This is because the Fock spaces are Verma modules, or closely related to them, and these are easy to handle. Let us start with the Fock space M_0 for the free boson, which has central charge $c = 1$ (Theorem 2.18 and Exercise 2.32). In the notation of (10), M_0 (considered as a \mathbb{Z} -graded linear space) coincides with $\mathcal{U}(\hat{A}^-)$ equipped with the natural product grading for which $a \otimes t^{-n}$ has weight n . Because of the PBW Theorem, the universal enveloping algebra is itself isomorphic as graded space to the symmetric algebra $S(\prod_{n \geq 1} \mathbb{C}x_{-n})$ with x_{-n} having weight n . (In other words, M_0 'is' a polynomial algebra in variables x_{-n} . Compare with Remark 2.19.) As graded algebras, symmetric algebras are multiplicative over direct sums. It follows that

$$\begin{aligned} Z_{M_0}(\mathbf{1}) &= q^{-1/24} \prod_{n=1}^{\infty} (\text{q-dimension of } \mathbb{C}[x_{-n}]) \\ &= q^{-1/24} \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \dots) = q^{-1/24} \prod_{n=1}^{\infty} (1 - q^n)^{-1}, \end{aligned}$$

which is none other than the inverse eta-function (33), (34). Thus we have

$$Z_{M_0}(\mathbf{1}) = \eta(q)^{-1}. \quad (40)$$

For an integer $n \geq 1$, let $M_0^{\otimes n}$ be the n -fold tensor product of M_0 considered as a VOA as described in Exercise 2.32. This is the theory of n free bosons. Using Exercise 3.1 we deduce from (40) that

$$Z_{M_0^{\otimes n}}(\mathbf{1}) = \eta(q)^{-n}. \quad (41)$$

In particular, the graded dimension of the VOA $M_0^{\otimes 24}$ of 24 free bosons (the *bosonic string*) is the inverse discriminant $\Delta(\tau)^{-1}$.

The calculation of the graded dimension of Vir_c is similar. Indeed, the Fock space $M_{c,0}$ (13) is *isomorphic* as \mathbb{Z} -graded linear space to M_0 . We must quotient out the graded submodule $\mathcal{U}(\text{Vir})L_{-1}.v_{c,0}$, and this is isomorphic to $M_{c,0}[1]$,

that is $M_{c,0}$ with an overall shift of $+1$ in the grading, because $L_{-1} \cdot v_{c,0}$ has weight 1 as an element of $M_{c,0}$. We find that

$$Z_{\text{Vir}_c}(\mathbf{1}) = \frac{q^{-c/24}}{\prod_{n \geq 2} (1 - q^n)}, \tag{42}$$

which is *not* the q -expansion of a modular form.

Next we consider the character value $Z_V(\omega)$ for a VOA V . Because the zero mode of the conformal vector is L_0 , which acts on V_n as multiplication by n , we have

$$Z_V(\omega) = q^{-c/24} \sum_n n \dim V_n q^n.$$

This is almost, but not quite, equal to $\theta Z_V(\mathbf{1})$ (θ as in (50)). If instead we use the square bracket conformal vector $\tilde{\omega} \in V_{[2]}$ (19) we find

$$\begin{aligned} Z_V(\tilde{\omega}) &= q^{-c/24} \sum_n \left(n - \frac{c}{24}\right) \dim V_n q^n \\ &= \theta(Z_V(\mathbf{1})). \end{aligned}$$

In the case of M_0 , for example, we obtain using Exercise 3.18 that

$$Z_{M_0}(\tilde{\omega}) = \theta \eta(\tau)^{-1} = \frac{E_2(\tau)}{2\eta(\tau)}.$$

This suggests that ‘nicer’ character values obtain by evaluating Z_V on states which are homogeneous in the square bracket formalism, i.e., lie in $V_{[k]}$ for some k .

4.3. The character of the Heisenberg algebra. It is generally a difficult problem to compute the 1-point functions $Z_V(v)$ of a VOA V for a complete basis of states. We describe the solution for the Heisenberg algebra M_0 ([MT1]). It well illustrates the principle suggested at the end of the previous Subsection.

THEOREM 4.5. *Let $M_{[0]} = \bigoplus_{n \geq 0} (M_0)_{[n]}$ be the Fock space for M_0 equipped with the square bracket grading (cf. Section 2.7). Let \mathfrak{Q} be the graded algebra of quasimodular forms (27). There is a surjection of graded linear spaces*

$$M_{[0]} \rightarrow \mathfrak{Q}, \quad v \mapsto Q_v(\tau)$$

such that $Z_{M_0}(v) = Q_v(\tau)/\eta(\tau)$.

Up to a normalizing factor $\eta(\tau)^{-1}$ then, every 1-point function is a quasimodular form, and every quasimodular form of weight k arises in this way from a state $v \in (M_0)_{[k]}$ (cf. Exercise 4.7). There is an *explicit* description of the quasimodular

form $Q_v(\tau)$ attached to a state v with $\text{wt}[v] = k$ which goes as follows. A basis of states for $(M_0)_{[k]}$ is given by

$$v_\lambda = a[-k_1] \dots a[-k_n] \mathbf{1} \quad (43)$$

where $k = k_1 + \dots + k_n$ and $1 \leq k_1 \leq \dots \leq k_n$ range over the parts of a partition λ of k . The quasimodular form $Q_{v_\lambda}(\tau)$ is given by

$$Q_{v_\lambda}(\tau) = \sum_{\varphi=\dots(rs)\dots} \prod_{(rs)} (-1)^{r+1} \frac{(r+s-1)!}{(r-1)!(s-1)!} E_{r+s}(\tau), \quad (44)$$

where the notation is as follows. Let $\Phi = \{k_1, \dots, k_n\}$ be the parts of the partition λ . Then φ ranges over all fixed-point-free involutions in the symmetric group $\Sigma(\Phi)$, so that φ can be represented as a product of transpositions $\dots(rs)\dots$ with (r, s) a pair of parts of λ . For each such φ , the product ranges over the transpositions whose product (in $\Sigma(\Phi)$) is φ . We will indicate how (44) can be proved in the next Section. A detailed proof appears later in Section 11.1.

Assume formula (44) in the following exercises.

EXERCISE 4.6. Show that $Q_{v_\lambda}(\tau)$ vanishes if λ has either an odd number of parts or an odd number of odd parts.

EXERCISE 4.7. Assume that λ has both an even number of parts and an even number of odd parts. Prove that $Q_{v_\lambda}(\tau)$ has a *nonzero* constant term, and in particular does not vanish.

EXERCISE 4.8. With the same assumptions as the previous Exercise, prove that $Q_{v_\lambda}(\tau) \in \mathfrak{M}$ if, and only if, λ has *at most one part equal to 1*.

EXERCISE 4.9. Prove the assertion that *every* quasimodular form arises as the trace of a state in \mathfrak{M}_0 .

5. Elliptic functions and 2-point functions

There is an extension of the idea of 1-point functions to n -point functions for any (nonnegative) n . We mainly restrict ourselves here to the case of 2-point correlation functions, which are related to *elliptic functions*.

5.1. Elliptic functions. Throughout this section, *lattice* means an additive subgroup $\Lambda \subseteq \mathbb{C}$ of rank 2. As such it is the \mathbb{Z} -span of an \mathbb{R} -basis (ω_1, ω_2) of \mathbb{C} . An *elliptic function* is a function $f(z)$ which is meromorphic in \mathbb{C} and satisfies $f(z + \lambda) = f(z)$ for all λ in some lattice Λ . Equivalently, $f(z + \omega_i) = f(z)$ for basis vectors ω_1, ω_2 of Λ . Λ is the *period lattice* of $f(z)$. Note that \mathbb{C}/Λ

has the structure of a *complex torus* (aka *complex elliptic curve*) and that $f(z)$ induces a map

$$f : \mathbb{C}/\Lambda \rightarrow \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}.$$

The set of all meromorphic functions with period lattice Λ is a field M_Λ (the function field of the torus). We have $\mathbb{C} \subseteq M_\Lambda$ where \mathbb{C} is identified with the constants, moreover $f'(z) \in M_\Lambda$ whenever $f(z) \in M_\Lambda$.

Two lattices Λ_1, Λ_2 are *homothetic* if there is $\alpha \in \mathbb{C}$ with $\alpha\Lambda_1 = \Lambda_2$. It is usually enough to deal with some fixed lattice in a homothety class (the corresponding complex tori are *isomorphic*), and every Λ is homothetic to a lattice with basis $(2\pi i, 2\pi i\tau)$ and $\tau \in \mathfrak{H}$. We let Λ_τ denote this lattice.

The classical *Weierstrass \wp -function* is⁶

$$\wp(z, \tau) = \frac{1}{z^2} + \sum'_{m,n \in \mathbb{Z}} \left(\frac{1}{(z - \omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2} \right). \tag{45}$$

Here, $\omega_{m,n} = 2\pi i(m\tau + n)$. The double sum is independent of the order of summation and absolutely convergent. It defines a function with the following properties: (a) double pole at each point of Λ_τ , (b) holomorphic in $(\mathbb{C} \times \mathfrak{H}) \setminus \Lambda_\tau$, (c) *even* in z , (d) period lattice Λ_τ . In particular, for fixed $\tau \in \mathfrak{H}$ the \wp -function $\wp(z, \tau)$ lies in the field M_{Λ_τ} . It turns out that

$$M_{\Lambda_\tau} = \mathbb{C}(\wp(z, \tau), \wp'(z, \tau))$$

is a *function field in one variable*. Indeed the set of *even* elliptic functions is a simple transcendental extension $\mathbb{C}(\wp)$ and $M_{\Lambda_\tau} \supseteq \mathbb{C}(\wp)$ a *quadratic extension*.

There is a natural left action of Γ on $\mathbb{C} \times \mathfrak{H}$ extending (20). It is given by

$$\gamma : (z, \tau) \mapsto \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{46}$$

corresponding to a base change $2\pi i(\tau, 1) \mapsto 2\pi i(a\tau + b, c\tau + d)$ of Λ_τ followed by the homothety (conformal rescaling) $z \mapsto z/(c\tau + d)$. Then it follows that

$$\wp(\gamma(z, \tau)) = (c\tau + d)^2 \wp(z, \tau). \tag{47}$$

This says that $\wp(z, \tau)$ is *Jacobi form* of weight 2 [EZ], though we will neither explain nor pursue this idea here.

What we need is that $\wp(z, \tau)$ is invariant under $\tau \mapsto \tau + 1$ as well as $z \mapsto z + 2\pi i$ (from the elliptic property). It follows that $\wp(z, \tau)$ has a Fourier expansion

⁶Here and below, a prime appended to a summation indicates that terms rendering the sum meaningless, in this case $(m, n) = (0, 0)$, are to be omitted.

in both q and q_z . (Compare with the development in Section 3.1.) To describe this we define

$$P_1(z, \tau) = \sum'_{n \in \mathbb{Z}} \frac{q_z^n}{1 - q^n} - \frac{1}{2}, \quad (48)$$

$$P_2(z, \tau) = \frac{d}{dz} P_1(z, \tau) = \sum'_{n \in \mathbb{Z}} \frac{n q_z^n}{1 - q^n}. \quad (49)$$

The extra term $-\frac{1}{2}$ in (48) ensures that $P_1(z, \tau)$ is odd in z . For nonzero z in the fundamental parallelogram defined by the basis $(2\pi i, 2\pi i \tau)$ of Λ_τ we have $-2\pi \operatorname{Im} \tau < \operatorname{Re} z < 0$, so that $|q| < |q_z| < 1$. $P_1(z, \tau)$ and its z -derivatives are absolutely convergent in this domain. We can now give the Fourier expansion of the \wp -function, which reveals a fundamental relationship with the Eisenstein series of Section 3.2.

THEOREM 5.1. *We have*

$$\wp(z, \tau) = P_2(z, \tau) - E_2(\tau) = \frac{1}{z^2} + \sum_{k \geq 2} (2k-1) E_{2k}(\tau) z^{2k-2}.$$

We sketch the proof, which uses a key identity (cf. Exercise 5.3 below):

$$\sum_{n \in \mathbb{Z}} \frac{1}{(x - 2\pi i n)^2} = \frac{q_x}{(1 - q_x)^2} \quad (50)$$

for $x \neq 0$. In the exceptional case,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(2\pi i n)^2} = \frac{2\zeta(2)}{(2\pi i)^2} = -\frac{1}{12}. \quad (51)$$

Now

$$\wp(z, \tau) + \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} \frac{1}{\omega_{m,n}^2} \right) = \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} \frac{1}{(z - \omega_{m,n})^2} \right), \quad (52)$$

where the convergent nested double sums depend on the order of summation. For the lhs, use (50) with $x = 2\pi i m \tau \neq 0$, (51) and $|q| < 1$ to obtain

$$\sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} \frac{1}{\omega_{m,n}^2} \right) = -\frac{1}{12} + \sum_{0 \neq m \in \mathbb{Z}} \frac{q^m}{(1 - q^m)^2} = -\frac{1}{12} + 2 \sum_{m, n > 0} n q^{mn} = E_2(\tau)$$

(cf. (24)). For the rhs of (52), use (50) with $x = z - 2\pi i m \tau$ and argue similarly using $|q_z q^m|, |q_z^{-1} q^m| < 1$ for $m > 0$ to get

$$\begin{aligned}
 \sum_{m \in \mathbb{Z}} \frac{q_z q^m}{(1 - q_z q^m)^2} &= \frac{q_z}{(1 - q_z)^2} + \sum_{m > 0} \left(\frac{q_z q^m}{(1 - q_z q^m)^2} + \frac{q_z q^{-m}}{(1 - q_z q^{-m})^2} \right) \\
 &= \frac{q_z}{(1 - q_z)^2} + \sum_{m > 0} \sum_{n > 0} n (q_z^n + q_z^{-n}) q^{nm} \\
 &= \frac{q_z}{(1 - q_z)^2} + \sum_{n > 0} n (q_z^n + q_z^{-n}) \frac{q^n}{1 - q^n} \\
 &= \sum_{n > 0} n \left(\frac{q_z^n}{1 - q^n} - \frac{q_z^{-n}}{1 - q^{-n}} \right) = P_2(z, \tau). \tag{53}
 \end{aligned}$$

This proves the first equality in the theorem. From (45) we see that

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{k \geq 3} (k-1) \tilde{E}_k(\tau) z^{k-2},$$

with

$$\tilde{E}_k(\tau) = \sum'_{m, n \in \mathbb{Z}} \frac{1}{\omega_{m, n}^k} = \frac{1}{(2\pi i)^k} \sum'_{m, n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}. \tag{54}$$

We can use (50) to identify \tilde{E}_k with the corresponding Eisenstein series $E_k(\tau)$ (24), in particular $\tilde{E}_k(\tau)$ is identically zero for k odd. This completes our discussion of Theorem 5.1.

We note that P_1 is not an elliptic function (cf. Exercise 5.6). Higher z -derivatives $P_1^{(m)}(z, \tau)$ for $m \geq 1$ are elliptic functions, and are derivatives of $\wp(z, \tau)$ for $m \geq 2$. We have

$$\begin{aligned}
 P_1^{(m)}(z, \tau) &= \sum'_{n \in \mathbb{Z}} \frac{n^m q_z^n}{1 - q^n} \\
 &= m! \left(\frac{(-1)^{m+1}}{z^{m+1}} + \sum_{k \geq m+1} \binom{k-1}{m} E_k(\tau) z^{k-m-1} \right). \tag{55}
 \end{aligned}$$

EXERCISE 5.2. Prove directly from the definition that an elliptic function which is *holomorphic* is necessarily constant.

EXERCISE 5.3. Verify (50) by comparing poles.

EXERCISE 5.4. Prove that for even $k \geq 4$, $\tilde{E}_k(\tau)$ coincides with $E_k(\tau)$. (Use (26).)

EXERCISE 5.5. Deduce from (47) that $E_k(\tau) \in \mathfrak{M}_k$ for even $k \geq 4$.

EXERCISE 5.6. Prove that $P_1(z + 2\pi i \tau, \tau) = P_1(z, \tau) - 1$.

5.2. 2-Point correlation functions. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA of central charge c . For an integer $n \geq 0$, the n -point correlation function for states $u^1, \dots, u^n \in V$ is the formal expression

$$F_V((u^1, z_1), \dots, (u^n, z_n), q) = \text{Tr}_V Y(q_1^{L_0} u^1, q_1) \dots Y(q_n^{L_0} u^n, q_n) q^{L_0 - c/24}, \quad (56)$$

where $q_i = q_{z_i}$ for variables z_1, \dots, z_n . For $n = 0$ this reduces to the graded dimension $\text{Tr}_V q^{L_0 - c/24}$ as discussed in Section 4.2. If $n = 1$ and $u^1 \in V_k$, the expression in (56) equals

$$\begin{aligned} \text{Tr}_V Y(q_1^{L_0} u^1, q_1) q^{L_0 - c/24} &= q_1^k \sum_m \text{Tr}_V u_m^1 q_1^{-m-1} q^{L_0 - c/24} \\ &= \text{Tr}_V o(u^1) q^{L_0 - c/24} = Z(u^1, q), \end{aligned}$$

where we used (36) to get the second equality. So for $n = 1$, (56) is the 1-point function of Section 4, as expected. There are similar modal expressions for all n -point functions, but for $n \geq 2$ they are unhelpful. Here we focus on the 2-point function

$$F_V((u, z_1), (v, z_2), \tau) = \text{Tr}_V Y(q_1^{L_0} u, q_1) Y(q_2^{L_0} v, q_2) q^{L_0 - c/24}. \quad (57)$$

We want to re-express the 2-point function as a *1-point function*, and for this we need be able to manipulate vertex operators. More precisely, we need to manipulate expressions involving vertex operators which are *traced* over V . In such a context the locality of operators (4) simplifies in the sense that

$$\text{Tr}_V Y(u, z_1) Y(v, z_2) q^{L_0} = \text{Tr}_V Y(v, z_2) Y(u, z_1) q^{L_0},$$

where the additional factor $(z_1 - z_2)^k$ (loc. cit.) has conveniently disappeared. Similar comments apply to the associativity formula (7), where we have

$$\text{Tr}_V Y(u, z_1 + z_2) Y(v, z_2) q^{L_0} = \text{Tr}_V Y(Y(u, z_1)v, z_2) q^{L_0}.$$

These and similar assertions fall under the heading of *duality* in CFT, which is discussed in [FHL]. We shall use them below without further comment. Thus with some changes of variables together with Exercise 4.2, we have

$$\begin{aligned} F_V((u, z_1), (v, z_2), \tau) &= \text{Tr}_V Y(Y(q_1^{L_0} u, q_1 - q_2) q_2^{L_0} v, q_2) q^{L_0 - c/24} \\ &= \text{Tr}_V Y(q_2^{L_0} Y((q_{z_{12}}^{L_0} u, q_{z_{12}} - 1)v, q_2) q^{L_0 - c/24} \\ &= Z_V(Y[u, z_{12}]v, \tau), \end{aligned} \quad (58)$$

where $z_{12} = z_1 - z_2$. This is the desired 1-point function. Similarly,

$$\begin{aligned} F_V((u, z_1), (v, z_2 + 2\pi i \tau), \tau) &= q^{-c/24} \operatorname{Tr}_V Y(q_1^{L_0} u, q_1) Y(q_2^{L_0} q_2^{L_0} v, q_2) q^{L_0} \\ &= q^{-c/24} \operatorname{Tr}_V Y(q_1^{L_0} u, q_1) q^{L_0} Y(q_2^{L_0} v, q_2) \\ &= q^{-c/24} \operatorname{Tr}_V Y(q_2^{L_0} v, q_2) Y(q_1^{L_0} u, q_1) q^{L_0} \\ &= F_V((u, z_1), (v, z_2), \tau), \end{aligned}$$

Thus F_V is periodic in z_2 with period $2\pi i \tau$, and the same holds for z_1 . It is obvious that F_V is also periodic in each z_i with period $2\pi i$. It follows that, at least formally, the 2-point function F_V (alias the 1-point function (58)) is elliptic in the variable z_{12} with period lattice Λ_τ .

5.3. First Zhu recursion formula We continue to pursue the ellipticity of the 2-point function F_V . It is the analytic of F_V which needs to be established. To this end we develop a recursion formula of Zhu (see [Z]), which finds a number of applications.

THEOREM 5.7. *We have*

$$F_V((u, z_1), (v, z_2), \tau) = \operatorname{Tr}_V o(u) o(v) q^{L_0 - c/24} - \sum_{m \geq 1} \frac{\binom{L_0 - m}{m}}{m!} P_1^{(m)}(z_{12}, \tau) Z_V(u[m]v, \tau). \quad (59)$$

The sum in (59) is finite since $u[n]v = 0$ for m sufficiently large, and from Section 5.1 $P_1^{(m)}(z_{12}, \tau)$ is elliptic for $m \geq 1$. Thus the ellipticity of F_V is reduced to the convergence of $\operatorname{Tr}_V o(u) o(v)$ and the 1-point functions $Z_V(u[m]v, \tau)$. This Theorem makes clear the deep connection between elliptic functions (and therefore also modular forms) and VOAs. There is an analogous recursion for all n -point functions.

To prove Theorem 5.7 we may assume that $u \in V_k$, whence

$$F_V((u, z_1), (v, z_2), \tau) = \sum_{n \in \mathbb{Z}} q_1^{-n-1+k} \operatorname{Tr}_V (u_n Y(q_2^{L_0} v, q_2) q^{L_0 - c/24}). \quad (60)$$

Using (36), Exercise 4.2 and (18) we have

$$\begin{aligned} [u_n, Y(q_2^{L_0} v, q_2)] &= \sum_{i \geq 0} \binom{n}{i} Y(u_i q_2^{L_0} v, q_2) q_2^{n-i} = q_2^r Y(q_2^{L_0} \sum_{i \geq 0} \binom{n}{i} u_i v, q_2) \\ &= q_2^r \sum_{m \geq 0} \frac{r^m}{m!} Y(q_2^{L_0} u[m]v, q_2), \end{aligned}$$

where $r = n + 1 - k$.

Hence

$$\begin{aligned} & \text{Tr}_V(u_n Y(q_2^{L_0} v, q_2) q^{L_0 - c/24}) \\ &= \text{Tr}_V([u_n, Y(q_2^{L_0} v, q_2)] q^{L_0 - c/24}) + \text{Tr}_V(Y(q_2^{L_0} v, q_2) u_n q^{L_0 - c/24}) \\ &= q_2^r \sum_{m \geq 0} \frac{r^m}{m!} Z_V(u[m]v, \tau) + q^r \text{Tr}_V(Y(q_2^{L_0} v, q_2) q^{L_0 - c/24} u_n). \end{aligned}$$

From this we obtain

$$q_2^r \sum_{m \geq 0} \frac{r^m}{m!} Z_V(u[m]v, \tau) = (1 - q^r) \text{Tr}_V(u_n Y(q_2^{L_0} v, q_2) q^{L_0 - c/24}),$$

so that for $r \neq 0$ we have

$$\text{Tr}_V(u_n Y(q_2^{L_0} v, q_2) q^{L_0 - c/24}) = \frac{q_2^r}{1 - q^r} \sum_{m \geq 1} \frac{r^m}{m!} Z_V(u[m]v, \tau).$$

Finally, the term corresponding to $r = 0$ in (60) is $\text{Tr}_V(o(u)o(v)q^{L_0 - c/24})$. Substituting into (60), we find

$$\begin{aligned} & F_V((u, z_1), (v, z_2), \tau) \\ &= \text{Tr}_V(o(u)o(v)q^{L_0 - c/24}) + \sum_{m \geq 1} \frac{1}{m!} Z_V(u[m]v, \tau) \sum_{n \in \mathbb{Z}} \frac{r^m q_{z_{12}}^{-n}}{1 - q^n}, \end{aligned}$$

and the theorem follows upon comparison with (55).

EXERCISE 5.8. Let a be the generating state for the Heisenberg VOA M_0 (cf. Section 2.4). Prove that $F_{M_0}((a, z_1), (a, z_2), \tau) = P_2(z_{12}, \tau)/\eta(\tau)$.

EXERCISE 5.9. For states u, v in a VOA V , show that $Z_V(u[0]v, q) = 0$.

5.4. Second Zhu recursion formula. Theorem 5.7 allows us to obtain a related recursion formula for 1-point functions.

THEOREM 5.10. For $n \geq 1$,

$$\begin{aligned} Z_V(u[-n]v, \tau) &= \delta_{n,1} \text{Tr}_V(o(u)o(v)q^{L_0 - c/24}) \\ &+ \sum_{m \geq 1} (-1)^{m+1} \binom{n+m-1}{m} E_{n+m}(\tau) Z_V(u[m]v, \tau). \quad (61) \end{aligned}$$

To see this, note from (58) that

$$F_V((u, z_1), (v, z_2), \tau) = \sum_{n \in \mathbb{Z}} Z_V(u[-n]v, \tau) z_{12}^{-n-1}.$$

Now compare this with the z_{12} -expansion of the rhs of (59) using (55). Taking $n \geq 1$ we obtain (61). (For $n \leq 0$ we get no information.)

One can apply Theorem 5.10 in a number of contexts. If we work with states u, v, \dots in V that are homogeneous with respect to the square bracket Virasoro operator $L[0]$, then the 1-point functions occurring on the rhs of (61) are those of states $u[m]v$ whose (square bracket) weight is *strictly less* than that of $u[-n]v$ for $n \geq 1$. Thus one might hope to proceed inductively (with respect to square bracket weights) to show that 1-point functions are holomorphic in \mathfrak{H} . To illustrate, we introduce the important class of VOAs V of *CFT-type* defined by the property that the zero weight space V_0 is *nondegenerate*, i.e., *spanned* by the vacuum vector. This implies (Exercise 2.29) that

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus \dots \tag{62}$$

Using Theorem 5.10 and the remarks following Theorem 5.7 we obtain:

LEMMA 5.11. *Suppose that V is a VOA of CFT-type, and let S be a generating set for V as in Theorem 2.7. Assume that $\text{Tr}_V o(u)o(v)q^{L_0 - c/24}$ is holomorphic in \mathfrak{H} for all $u \in S$ and $v \in V$, and that the graded dimension $Z_V(\mathbf{1})$ is holomorphic in \mathfrak{H} . Then every 1-point function for V is holomorphic in \mathfrak{H} , and every 2-point function for V is elliptic.*

By way of example, consider the Heisenberg algebra M_0 , which is certainly of CFT-type. It is generated by a single state a in weight 1, and $o(a) = 0$ (cf. Exercise 4.3). Furthermore $Z_{M_0}(\mathbf{1})$ is the inverse η -function (40) and hence holomorphic in \mathfrak{H} . So the conditions of the Lemma apply to M_0 , so that all 1- and 2-point functions for M_0 have the desired analytic properties. Indeed, the vanishing of the zero mode for a means that in the recursion (61), the anomalous first term on the rhs is not present (taking $u = a$, as we may). We get a recursion for 1-point functions which may be solved with some effort, and this is how one proves Theorem 4.5 and (44). The details are described in Section 11.1.

EXERCISE 5.12. Give the details for the proof of Lemma 5.11.

EXERCISE 5.13. Show that the analysis of 1-point and 2-point functions associated to the Heisenberg algebra goes through with the same conclusions for the Virasoro algebra Vir_c .

Part II. Modular-invariance and rational vertex operator algebras

The representation theory of a VOA V , i.e., the study of V -modules and their characters (correlation functions) is fundamental. In this Section we introduce some of the ideas in this subject.

6. Modules over a vertex operator algebra

6.1. Basic definitions. Let $V = (V, Y, \mathbf{1}, \omega)$ be a VOA of central charge c . As one might expect, a V -module is (roughly speaking) linear space M admitting fields associated to states of V which satisfy axioms analogous to those satisfied by the fields $Y(v, z)$. It is useful to introduce various types of modules, the most basic of which is the following.

DEFINITION 6.1. A weak V -module is a pair (M, Y_M) where

$$Y_M : V \rightarrow \mathfrak{F}(M), \quad v \mapsto Y_M(v, z) = \sum_n a_n^M z^{-n-1} \text{ is a linear map,}$$

and the following hold for all $u, v \in V, w \in M$:

$$\text{vacuum : } Y_M(\mathbf{1}, z) = \text{Id}_M,$$

$$\text{locality : } Y_M(u, z) \sim Y_M(v, z),$$

associativity : for large enough k ,

$$(z_1 + z_2)^k Y_M(u, z_1 + z_2) Y_M(v, z_2) w = (z_1 + z_2)^k Y_M(Y(u, z_1)v, z_2) w.$$

There is no notion of creativity or translation covariance *per se* for V -modules. It is not sufficient to assume only locality of operators here; the associativity axiom (the analog of (7)) is crucial. Locality and associativity are jointly equivalent to the analog of (6), namely

$$\sum_{i \geq 0} \binom{p}{i} (u_{r+i} v)_{p+q-i}^M = \sum_{i \geq 0} (-1)^i \binom{r}{i} (u_{p+r-i} v_{q+i}^M - (-1)^r v_{q+r-i}^M u_{p+i}^M).$$

As before, this is the modal version of the *Jacobi Identity*. For further details, see [FHL] and [LL]. A weak V -module is essentially a module for a vertex algebra.

DEFINITION 6.2. An *admissible* V -module is a weak V -module (M, Y_M) equipped with an \mathbb{N} -grading $M = \bigoplus_{n \geq 0} M_n$ such that

$$v \in V_k \Rightarrow v_n^M : M_m \rightarrow M_{m+k-n-1}. \quad (63)$$

Admissible modules are also called \mathbb{N} -gradable modules. Note that (63) is the analog of (36). There is *no* requirement that the homogeneous spaces M_n have finite dimension. An overall shift in the grading does not affect (63), so we may, and usually shall, assume that $M_0 \neq 0$ if $M \neq 0$. We then refer to M_0 as the *top level*.

DEFINITION 6.3. A V -module is a weak V -module (M, Y_M) equipped with a grading $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ such that

$$\begin{aligned} \dim M_\lambda &< \infty, \\ \forall \lambda, M_{\lambda+n} &= 0 \text{ for } n \ll 0, \\ L_0 m &= \lambda m, \quad m \in M_\lambda. \end{aligned}$$

We frequently call a V -module as in Definition 6.3 an *ordinary* V -module if we want to emphasize that it is not merely a weak or admissible module. There are containments

$$\{\text{weak } V\text{-modules}\} \supseteq \{\text{admissible } V\text{-modules}\} \supseteq \{\text{ordinary } V\text{-modules}\},$$

which amounts to saying that ordinary V -modules can be equipped with an \mathbb{N} -grading making them admissible (cf. Exercise 6.6). A (weak, admissible, or ordinary) V -module M is *irreducible* if no proper, nonzero subspace of V is invariant under all modes v_n^M . More generally, we can define *submodules* of M in the usual way, though we will not go much into this here.

A VOA V is *ipso facto* an ordinary V -module in which $Y = Y_M$. It is called the *adjoint* module. If the adjoint module is irreducible then we say that V is *simple*. This is consistent with standard algebraic usage: it can be shown using skew-symmetry that if $U \subseteq V$ is a submodule of the adjoint module V then U is a (2-sided) *ideal* in a natural sense, and that V/U has a well-defined structure of VOA. See Exercise 6.11 for further details.

We want to define the *partition function* and *character* of a V -module M along the lines of that for V itself, as discussed in Section 4. This makes no sense unless M is equipped with a suitable grading. An important case when this can be carried through is when M is an irreducible, ordinary V -module. In this case, if $\lambda \in \mathbb{C}$ satisfies $M_{\lambda+n} \neq 0$ for some integer n then $\bigoplus_{n \in \mathbb{Z}} M_{\lambda+n}$ is invariant under all modes v_n^M and hence coincides with M thanks to irreducibility. Relabeling, the grading on a (nonzero) irreducible, ordinary V -module M takes the shape

$$M = \bigoplus_{n \geq 0} M_{h+n}. \tag{64}$$

M_h is the *top level* and h a uniquely determined scalar called the *conformal weight* of M . It is an important numerical invariant of the module.

The *zero mode* $o^M(v)$ for $v \in V$ is the mode of $Y_M(v, z)$ which has weight zero as an operator on M ; it is defined because of (63). We can now define the *character* Z_M of an irreducible V -module M of conformal weight h in the expected manner, namely

$$Z_M(v) = \text{Tr}_M o^M(v) q^{L_0^M - c/24} = q^{h-c/24} \sum_{n \geq 0} \text{Tr}_{M_{h+n}} o^M(v) q^n. \quad (65)$$

The partition function of M is

$$Z_M(\mathbf{1}) = \text{Tr}_M q^{L_0^M - c/24} = q^{h-c/24} \sum_{n \geq 0} \dim M_{h+n} q^n$$

where, naturally, L_0^M is the corresponding zero mode for the Virasoro element.

The set of *all* irreducible modules over the Heisenberg VOA M_0 is readily described. As usual, let a be the weight one state that generates M_0 . In Section 2.4 we defined, for each $h \in \mathbb{C}$, the Verma module M_h and constructed a field $a(z) \in \mathfrak{F}(M_h)$. It is more precise to denote this by $a^h(z)$. Much as in the case $h = 0$, one finds that each M_h is an irreducible M_0 -module of conformal weight h with $Y_{M_h}(a, z) = a^h(z)$. In particular, M_0 is a simple VOA. The Stone-von Neumann Theorem is essentially the converse: for each h , M_h is the unique (up to isomorphism⁷) irreducible module over M_0 of conformal weight h . See [FLM] for a proof. The construction of the Verma module M_h shows that

$$Z_{M_h}(\mathbf{1}) = q^h / \eta(q). \quad (66)$$

The characters Z_{M_h} can be understood along the same lines as the special case of M_0 that we described in Sections 4 and 5. As illustrated by (66), results identical to Theorem 4.5 and (44) hold for M_h , except that an extra factor q^h must be included. Our development of the theory of 1- and 2-point functions may be carried out, with essentially no change, for general V -modules rather than just adjoint modules. It should be pointed out, however, that the extra factor *spoils* the quasimodularity of the character values.

While ordinary V -modules are perhaps natural, the reader may be wondering how and why admissible V -modules are relevant. Here we will limit ourselves here to a some general comments, and continue the discussion below. See Exercises 6.8-6.10 for some details, and [DLM3], [Z] for complete proofs. One considers certain subspaces $O_0 \subseteq O_1 \subseteq \cdots \subseteq V$, the quotient spaces $A_n(V) = V/O_n(V)$, and the inverse limit

$$A(V) = \lim_{\leftarrow} A_n(V).$$

⁷We have not defined morphisms of V -modules, but readers should be able to formulate it for themselves without difficulty.

Each $A_n(V)$ has natural structure of *associative algebra* such that the canonical projection $A_{n+1} \rightarrow A_n(V)$ is an algebra morphism. So $A(V)$ is also an associative algebra. $A_0(V)$ is called the *Zhu algebra* of V .

The representation theory of these algebras is intimately related to that of V itself. There are functors

$$\Omega_n : \text{Adm } V\text{-Mod} \rightarrow A_n(V)\text{-Mod}$$

from the category of admissible V -modules to the category of $A_n(V)$ -modules, and because of the details of the construction the quotient functor Ω_n/Ω_{n-1} makes sense (Ω_{-1} is trivial). For an admissible V -module M ,

$$\Omega_n(M)/\Omega_{n-1}(M)$$

is an $A_n(V)$ -module that is *not* the lift of an $A_{n-1}(V)$ -module. $A_n(V)$ is designed in such a way that it acts naturally on the sum of the first n graded pieces of an admissible V -module, and this is how the functor Ω_n is defined. It turns out that there is another functor

$$L_n : A_n(V)\text{-Mod} \rightarrow \text{Adm } V\text{-Mod} \tag{67}$$

which is a *right inverse* of the functor Ω_n/Ω_{n-1} , and which is harder to describe. This is a key point. It is the existence of L_n which motivates the introduction of admissible V -modules. L_n and Ω_n/Ω_{n-1} induce *bijections* between (isomorphism classes of) irreducible, admissible V -modules and irreducible $A_n(V)$ -modules which are not lifts of $A_{n-1}(V)$ -modules. For $n = 0$, this is just the set of irreducible $A_0(V)$ -modules. To a large extent these functors reduce the study of admissible V -modules to that of modules over the associative algebras $A_n(V)$, which are more familiar objects, and they have led to a number of theoretical advances. On the other hand, the computation of the Zhu algebra $A_0(V)$, not to mention the higher $A_n(V)$'s, is usually difficult. The complete structure has been elucidated in only a relatively few cases, and computer calculations have often been important.

Needless to say, there is much more that can be said about modules over a VOA. There is a notion of *dual* module ([B], [FHL] and Section 10.3). There is also a theory of *tensor products* of modules that is important. This is an extensive subject in its own right, and we can do no more than refer the reader to the literature (e.g., [HL], [HLZ]) for further details.

EXERCISE 6.4. Let (M, Y_M) be a weak V -module. Prove that $Y_M(L_{-1}v, z) = \partial Y_M(v, z)$.

EXERCISE 6.5. Show that

$$[L_m^M, L_n^M] = (m - n)L_{m-n}^M + (m^3 - m)/12\delta_{m,-n}c\text{Id}_M.$$

(Thus, a weak module for V is *ipso facto* a module over the Virasoro algebra with the *same* central charge as V .)

EXERCISE 6.6. Show that an ordinary V -module M is an admissible V -module as follows. Let $\Lambda \subseteq \mathbb{C}$ consist of those λ for which $M_{\lambda+k} = 0$ whenever k is a negative integer, and let $M_n = \bigoplus_{\lambda \in \Lambda} M_{\lambda+n}$. Show that $M = \bigoplus_{n \geq 0} M_n$ is an \mathbb{N} -grading on M satisfying (63).

EXERCISE 6.7. Give a complete proof that the Verma modules M_h are irreducible modules over the Heisenberg algebra M_0 .

EXERCISE 6.8. Let M be an admissible V -module. Prove that for each $v \in V$, the zero mode $o^M(L[-1]v)$ annihilates M . (Use Exercises 6.4 and 2.36.)

EXERCISE 6.9. For $n \geq 0$, $u \in V_k$, $v \in V$ define

$$u \circ_n v = \text{Res}_z Y(u, z) v \frac{(1+z)^{k+n}}{z^{2n+2}}.$$

Let $O_n(V)$ be the span of all states $u \circ_n v$ and $L[-1]u$.

- (a) Prove that if $n = 0$, the span of the states $u \circ_0 v$ already contains $L[-1]u$.
 (b) Prove that $O_0(V) \subseteq O_1(V) \subseteq \dots$.

EXERCISE 6.10. With the notation of Exercise 6.9, introduce the product

$$u * _n v = \sum_{m=0}^n \binom{m+n}{n} \text{Res}_z Y(u, z) v \frac{(1+z)^{k+n}}{z^{n+m+1}}.$$

- (a) Show that $O_n(V)$ is a 2-sided ideal with respect to the product $*_n$.
 (b) Show that $*_n$ induces a structure of associative algebra on the quotient space $A_n(V) = V/O_n(V)$.

EXERCISE 6.11. V is a VOA and $U \subseteq V$ a submodule of the adjoint module, so that $v_n u \in U$ for all $u \in U, v \in V, n \in \mathbb{Z}$. Prove that $u_n v \in V$, and deduce that if $U \neq V$ then V/U inherits the structure of VOA.

6.2. C_2 -cofinite, rational and regular vertex operator algebras. We are going to focus on some important classes of VOAs V which have the property that they have only *finitely many* (inequivalent) irreducible modules. The reader might well be surprised that there are any such VOAs at all beyond those of finite dimension (cf. Exercises 2.12 and 2.13). We will also make the simplifying assumption that V is of *CFT-type* (62) throughout the rest of these notes, although for many of the results to be discussed this assumption is not necessary.

DEFINITION 6.12. (a) V is *rational* if every admissible V -module is completely reducible, i.e., a direct sum of irreducible, admissible V -modules.

- (b) V is *regular* if every weak V -module is a direct sum of irreducible, ordinary V -modules.
- (c) V is C_2 -*cofinite* if the graded subspace $C_2(V) = \langle u_{-2}v \mid u, v \in V \rangle$ has finite codimension in V .

Based on what we said in the previous Subsection, it is easy to see that a regular VOA V is a rational VOA. Indeed, an admissible V -module is a weak module, hence a direct sum of irreducible, ordinary modules and *ipso facto* a direct sum of irreducible admissible modules. It is also known (see [ABD], [Li]) that regularity is *equivalent* to the conjunction of rationality and C_2 -cofiniteness.

While (a) and (b) of Definition 6.12 both assert that certain module categories are semisimple, (c) is rather different. (a) and (b) are *external conditions* that can be difficult to verify, whereas (c) is an *internal condition* that is easier to deal with. On the other hand, regular VOAs have better modular invariance properties than those which are C_2 -cofinite.

THEOREM 6.13. *Suppose that V is a C_2 -cofinite VOA.*

- (a) *Each $A_n(V)$ is finite-dimensional.*
- (b) *Every weak V -module is an admissible module.*
- (c) *V has only finitely many isomorphism classes of irreducible, admissible modules.*

Note that for a *finitely generated* VOA V , (b) is *equivalent* to C_2 -cofiniteness.

For further discussion of (a), see [Z], [My1], [GN], [Bu]; (b) is proved in [My1]. The approach in [GN] produces a sort of weak analog of the PBW Theorem in Lie theory (cf. Appendix) which applies to weak modules. This idea is very useful, and is used in [ABD], [My1], [Bu] and elsewhere in the literature. (c) follows from (a) and the properties of the functors L_n and Ω_n discussed in Section 6.1.

The following *omnibus* result collects some of the main facts about rational VOAs.

THEOREM 6.14 [DLM1], [DLM3]. *Suppose that V is a rational VOA.*

- (a) $A_0(V)$ is semisimple.
- (b) *Each $A_n(V)$ is finite-dimensional.*
- (c) *V has only finitely many isomorphism classes of irreducible, admissible V -modules,*
- (d) *every irreducible, admissible V -module is an ordinary V -module.*

Note that (b) is *equivalent* to rationality (loc. cit.)

Whether a rational VOA is necessarily C_2 -cofinite is presently one of the main open questions in the representation theory of VOAs. If this is so, then there would be no difference between rational and regular VOAs. In the early history of VOA theory it was possible to believe that rationality and C_2 -cofiniteness

were *equivalent*. That, however, has turned out to be a chimera. There are VOAs which are C_2 -cofinite but have admissible (in fact ordinary) modules which are *not* completely reducible. These are *logarithmic field theories*, a name that we will justify in Section 9.

EXERCISE 6.15. Give two proofs that the Heisenberg VOA M_0 is *not* a rational VOA: (a) by using Theorem 6.14, and (b) by explicitly constructing an admissible M_0 -module that is *not* completely reducible.

EXERCISE 6.16. For any VOA V , show that the quotient space $P(V) = V/C_2(V)$ carries the structure of a *Poisson algebra* in the following sense: the products $\{u, v\} = u_0v, uv = u_{-1}v$ afflict $P(V)$ with (well-defined) structures of Lie algebra and commutative, associative algebra respectively, moreover $\{uv, w\} = u\{v, w\} + \{u, v\}w$.

EXERCISE 6.17. Calculate the Poisson algebra $P(M_0)$ associated to the Heisenberg VOA.

7. Examples of regular vertex operator algebras

It is time to describe some further examples of VOAs beyond the Heisenberg and Virasoro theories. In particular, we want to have available a selection of regular VOAs. Our examples are fairly standard, but require some effort to construct. For this reason, we will mainly limit ourselves to a description of the underlying Fock spaces and generating fields.

7.1. Vertex algebras associated to Lie algebras. The reader might want to look over Appendix 1 before reading this Subsection. We can construct a VOA from a pair $(\mathfrak{g}, (\cdot, \cdot))$ consisting of a Lie algebra \mathfrak{g} equipped with a symmetric, invariant, bilinear form $(\cdot, \cdot) : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$. The details amount to an elaboration of the case of the Heisenberg algebra discussed in Section 2.4, which is the 1-dimensional case. The *affine Lie algebra* or *Kac–Moody algebra* associated to $(\mathfrak{g}, (\cdot, \cdot))$ is the linear space

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K = \bigoplus_n \mathfrak{g} \otimes t^n \oplus \mathbb{C}K$$

with brackets

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m(a, b)\delta_{m, -n}K, \quad (a, b \in \mathfrak{g}), \quad [\hat{\mathfrak{g}}, K] = 0.$$

The element $\hat{\mathfrak{g}}$ has a triangular decomposition with $\hat{\mathfrak{g}}^\pm = \bigoplus_{\pm n > 0} \mathfrak{g} \otimes t^n$ and $\hat{\mathfrak{g}}^0 = \mathfrak{g} \oplus \mathbb{C}K$. Here and below, we identify \mathfrak{g} with $\mathfrak{g} \otimes t^0$. Fix a \mathfrak{g} -module X and a scalar l . We extend X to a $\hat{\mathfrak{g}}^+ \oplus \hat{\mathfrak{g}}^0$ -module by letting $\hat{\mathfrak{g}}^+$ annihilate X

and letting K act as multiplication by l called the *level*. We have the induced $\hat{\mathfrak{g}}$ -module

$$V_{\mathfrak{g}}(l, X) = \text{Ind}(X) \tag{68}$$

(notation as in (208)). Following the Heisenberg case discussed in Section 2.4, we can define fields on $V_{\mathfrak{g}}(l, X)$ for each $a \in \mathfrak{g}$ by setting

$$Y_{V_{\mathfrak{g}}(l, X)}(a, z) = \sum_n a_n z^{-n-1}$$

where a_n is the induced action of $a \otimes t^n$. As in (11) we obtain

$$\begin{aligned} \sum_{j=0}^2 (-1)^j \binom{2}{j} [a_{2-j-r}, b_{j-s}] \\ &= \sum_{j=0}^2 (-1)^j \binom{2}{j} ([a, b]_{2-r-s} + (2-j-r)(a, b)l\delta_{2-j-r, s-j} \text{Id}) \\ &= ((2-r) - 2(1-r) - r)(a, b)l\delta_{r+s, 2} \text{Id} = 0, \end{aligned}$$

so that the fields $\{Y_{V_{\mathfrak{g}}(l, X)}(a, z) \mid a \in \mathfrak{g}\}$ are mutually local of order two. Taking $X = \mathbb{C}\mathbf{1}$ to be the *trivial* 1-dimensional \mathfrak{g} -module, one shows via Theorem 2.7 that the corresponding fields generate a vertex algebra with Fock space $V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})$. Moreover, each $V_{\mathfrak{g}}(l, X)$ is an admissible module.

To describe a conformal vector in $V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})$ and thereby obtain the structure of VOA, it is convenient at this point to specialize to the case that \mathfrak{g} is a *finite-dimensional, simple Lie algebra* of dimension d , say. We will also take (\cdot, \cdot) to be the *Killing form*, appropriately normalized.⁸ Note that this takes us out of the regime of the Heisenberg theory, to which we return in Section 7.3. An approach that covers both cases is described in [LL]. With our assumptions, one shows that

$$\omega = \frac{1}{2} \frac{1}{l + h^\vee} \sum_{i=1}^d u_i(-1)u^i \tag{69}$$

is the desired conformal vector with central charge $c = ld/(l + h^\vee)$. Here, $\{u_i\}$ is a basis of \mathfrak{g} , $\{u^i\}$ the basis dual to $\{u_i\}$ with respect to the form (\cdot, \cdot) , and h^\vee the *dual Coxeter number* of \mathfrak{g} . This is usually called the *Sugawara construction*. Needless to say, we must also assume that $l + h^\vee \neq 0$.

The L_0 -grading on $V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})$ that obtains from the Sugawara construction is the natural one in which the state $a_n\mathbf{1}$ has weight $-n$ for $a \in \mathfrak{g}$ and $n \leq 0$. In particular the zero weight space is $V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})_0 = \mathbb{C}\mathbf{1}$, and the VOA is of CFT-type. Because an ideal in the adjoint module is a graded submodule (cf. the discussion

⁸The normalization is an important detail, of course, but we will not need it.

in Section 6.1), any *proper* ideal necessarily lies in $\bigoplus_{n \geq 2} V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})_n$. It follows that there is a *unique maximal proper ideal*, call it J , and the quotient space

$$L_{\mathfrak{g}}(l, 0) = V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})/J$$

is a simple VOA.

More generally, take X to be a finite-dimensional irreducible \mathfrak{g} -module. As such it is a highest-weight module $L(\lambda)$ indexed by an element λ in the weight lattice of \mathfrak{g} . The top level of $V_{\mathfrak{g}}(l, L(\lambda))$ is naturally identified with $L(\lambda)$, and because this is an irreducible \mathfrak{g} -module then there is a unique maximal proper submodule $J \subseteq V_{\mathfrak{g}}(l, L(\lambda))$ (considered as $\hat{\mathfrak{g}}$ -module). The quotient spaces

$$L_{\mathfrak{g}}(l, \lambda) = V_{\mathfrak{g}}(l, L(\lambda))/J$$

are ordinary, irreducible $V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})$ -modules, and they are inequivalent for distinct choices of highest weight λ . Thus the VOA $V_{\mathfrak{g}}(l, \mathbb{C}\mathbf{1})$ has infinitely many inequivalent ordinary, irreducible modules, and in particular it cannot be rational (Theorem 6.14). Concerning the question of regularity of these VOAs, we collect the main facts ([FZ], [DL], [DM1], [DLM2], [DLM4]):

THEOREM 7.1. *Let \mathfrak{g} be a finite-dimensional simple Lie algebra. The simple VOA $L_{\mathfrak{g}}(l, 0)$ is rational if, and only if, l is a positive integer. In this case it is regular, and the ordinary, irreducible modules are the spaces $L_{\mathfrak{g}}(l, \lambda)$ where λ satisfies $\lambda(\theta) \leq l$ and θ is the longest positive root.*

These theories are called WZW models in the physics literature.

7.2. Discrete series Virasoro algebras. Here we discuss some quotients of Virasoro VOAs Vir_c (cf. Theorems 2.20 and 2.25) that turn out to be regular. As in the last Subsection, it is the underlying Lie structure that makes the calculations manageable. The details are quite different, however, and depend on the *Kac determinant* (e.g. [KR]) and the structure of the Verma modules (13) $M_{c,h}$ over the Virasoro algebra (these are Vir_c -modules) ([FF]). There is no space to describe these results systematically here, although we discuss some examples of Kac determinants in Subsection 10.4. So we give less detail compared to the WZW models. The theories we are going to describe in this Subsection find important applications in the physics of phase transitions and critical phenomena. See [FMS] for further background.

The Virasoro VOA Vir_c may, or may not, be a simple VOA, but there is a *unique maximal proper submodule* J and $L_c = \text{Vir}_c/J$ is a simple vertex operator algebra of central charge c . It turns out that Vir_c is *never* rational (cf. Exercise 7.4). As for the rationality of L_c , we have the following omnibus result:

THEOREM 7.2. *The following are equivalent:*

- (a) L_c is a rational VOA.
- (b) $J \neq 0$.
- (c) c lies in the so-called discrete series, i.e., there are coprime integers $p, q \geq 2$ such that

$$c = c_{pq} = 1 - \frac{6(p-q)^2}{pq}. \tag{70}$$

In this case L_c is regular, the conformal weights of the ordinary irreducible modules are

$$h_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}, \quad 1 \leq r \leq q - 1, 1 \leq s \leq p - 1$$

(taking only one value of h for each pair $h_{r,s}, h_{q-r,p-s}$) and two ordinary irreducible modules are isomorphic if, and only if, they have the same conformal weight.⁹ Thus there are just $(p - 1)(q - 1)/2$ inequivalent ordinary irreducible modules over L_c .

See [Wa] for the proof of rationality (also [DMZ]), where the idea is to compute the Zhu algebra $A_0(L_c)$. Regularity is shown in [DLM2]. The origin of the values $c_{p,q}$ is discussed in Section 10.4

Apart from the trivial case when $p = 2, q = 3$, the two ‘smallest’ cases, i.e., those with the fewest number of ordinary irreducible modules, correspond to $(p, q) = (2, 5)$ and $(3, 4)$. In the first case (the Yang–Lee model in physics) we have $c = -22/5$ and conformal weights $0, -1/5$. In the second case (the Ising model) $c = 1/2$ with conformal weights $0, 1/2, 1/16$.

EXERCISE 7.3. Prove that Vir_c has a unique maximal proper submodule J .

EXERCISE 7.4. Suppose that $J = 0$. Prove that Vir_c is not a rational VOA.

EXERCISE 7.5. Opine on the statement that the case $p = 2, q = 3$ is ‘trivial’.

7.3. Lattice theories. Lattice theories ([B], [FLM]) are VOAs whose connections with Lie algebras are of lesser importance compared to the examples in the last two subsections. Their basic properties are of a more combinatorial nature, and reflect features that one may expect in general rational theories. Because of this and the fact that they are amenable to computation, lattice theories occupy a central position in current VOA theory.

Let d be a positive integer and $\mathfrak{h} = \mathbb{C}^d$ a rank d linear space equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) . Consideration of \mathfrak{h} as an abelian Lie algebra leads to the affine algebra $\hat{\mathfrak{h}}$ as in Section 7.1. Let

$$M_0^d = V_{\hat{\mathfrak{h}}}(1, \mathbb{C}\mathbf{1}) \tag{71}$$

⁹Generally, a VOA may have inequivalent irreducible modules with the same conformal weight.

be the corresponding vertex algebra of *level 1*. The conformal vector ω is defined as in (69) with $l = h^\vee = 1$. The resulting VOA has central charge $c = d$. This is nothing more than a slightly different approach to the *rank d Heisenberg VOA*, as discussed in Section 2.4 (cf. Exercise 7.7).

The irreducible \mathfrak{h} -modules are 1-dimensional and indexed by a weight in the dual space of \mathfrak{h} . Identifying \mathfrak{h} with its dual via (\cdot, \cdot) , we obtain M_0^d -modules (68) with underlying linear space

$$V_{\mathfrak{h}}(1, \alpha) = S(\hat{\mathfrak{h}}^-) \otimes e^\alpha \quad (\alpha \in \mathfrak{h}).$$

Here $\mathbf{1} \otimes e^\alpha$ (or just e^α) is notation for the spanning vector of the (1-dimensional) top level of $V_{\mathfrak{h}}(1, \alpha)$, and

$$\beta \otimes e^\alpha = \beta_0 \cdot e^\alpha = (\beta, \alpha) \mathbf{1} \otimes e^\alpha, \quad \beta \in \mathfrak{h} = \mathfrak{h} \otimes t^0. \quad (72)$$

In order to describe the Fock spaces of lattice theories we need a bit more structure. Namely, we assume that $(\mathfrak{h}, (\cdot, \cdot))$ is the scalar extension of a Euclidean space. Thus, $E = \mathbb{R}^d = \mathfrak{h}_{\mathbb{R}}$ is a real space equipped with a *positive-definite quadratic form* $Q : E \rightarrow \mathbb{R}$, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} E$, and (\cdot, \cdot) is the \mathbb{C} -linear extension of the bilinear form on E defined by Q , also denoted by (\cdot, \cdot) . In particular, $Q(\alpha) = (\alpha, \alpha)/2$ for $\alpha \in E$. A *lattice* $L \subseteq E$ is the additive subgroup spanned by a *basis* of E . L is an *even lattice* if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$, i.e., the restriction of Q to L is integral.

For an even lattice $L \subseteq E$ we introduce the linear space

$$V_L = \bigoplus_{\alpha \in L} V_{\mathfrak{h}}(1, \alpha). \quad (73)$$

Identifying $\bigoplus_{\alpha} \mathbb{C}e^\alpha$ with the *group algebra*¹⁰ $\mathbb{C}[L]$ of the lattice, we can write (73) more compactly as

$$V_L = S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L]. \quad (74)$$

There is a natural grading on V_L that turns out to be the one defined by the L_0 operator. We take the tensor product grading on (74) in which $S(\hat{\mathfrak{h}}^-)$ has the grading of the Fock space of the rank d Heisenberg algebra that it is, and where e^α has weight $Q(\alpha)$. Using (41), the partition function of V_L is

$$Z_{V_L}(\mathbf{1}) = \frac{\sum_{\alpha \in L} q^{Q(\alpha)}}{\eta(q)^d}. \quad (75)$$

The numerator here is the *theta function* of L , a topic to which we shall return in Section 8.

So far then, we have described the Fock space V_L as a sum of Heisenberg modules. We define $Y(v, z)$ for $v \in M_0^d$ to be the operator whose restriction to

¹⁰We only explicitly use the linear structure of $\mathbb{C}[L]$, although the algebra structure also plays a rôle.

$V_{\mathfrak{h}}(1, \alpha)$ is just $Y_{V_{\mathfrak{h}}(1, \alpha)}(v, z)$. In order to impose the structure of VOA on V_L , we must construct fields for all of the states in the Fock space (74). Because of Theorem 2.7 it suffices to define $Y(e^\alpha, z)$ for $\alpha \in L$ and establish locality, but nothing that has come so far has prepared us for this. The generating fields we have considered in detail for the Heisenberg, WZW and Virasoro theories have modes a_n that are closely related to some Lie algebra, but in theories such as V_L this will generally not be the case. We content ourselves with the prescription for $Y(e^\alpha, z)$, referring the reader to [FLM], [K1] for further background and motivation:

$$Y(e^\alpha, z) = \exp\left(\sum_{n>0} \frac{\alpha-n}{n} z^n\right) \exp\left(\sum_{n<0} \frac{\alpha-n}{n} z^n\right) e^\alpha z^\alpha. \tag{76}$$

Beyond the modes α_n of $Y(\alpha, z)$, z^α is a shift operator $z^\alpha : v \otimes e^\beta \mapsto z^{(\alpha, \beta)} v \otimes e^\beta$ ($v \in \hat{\mathfrak{h}}^-$), and $e^\alpha : v \otimes e^\beta \mapsto \varepsilon(\alpha, \beta) v \otimes e^{\alpha+\beta}$ for a certain bilinear 2-cocycle $\varepsilon : L \otimes L \rightarrow \{\pm 1\}$ (loc. cit.)

The ordinary, irreducible modules over V_L are constructed in [Do]. The underlying Fock spaces are very similar to (74), and are indexed by the *cosets* of L in its \mathbb{Z} -dual L^0 (cf. Exercise 7.13). Precisely, they are

$$V_{L+\lambda} = \bigoplus_{\alpha \in L} V_{\mathfrak{h}}(1, \alpha + \lambda) = \hat{\mathfrak{h}}^- \otimes \mathbb{C}[L + \lambda]$$

for $\lambda \in L^0$, with partition functions

$$Z_{V_{L+\lambda}}(\mathbf{1}) = \frac{\sum_{\alpha \in L} q^{Q(\alpha+\lambda)}}{\eta(q)^d}. \tag{77}$$

The fields $Y_{V_{L+\lambda}}(v, z)$ are similarly analogous to (76) (loc. cit.) Indeed, one can usefully combine all of these fields and Fock spaces into a bigger and better edifice. For this, see [DL]. For rationality and C_2 -cofiniteness, see [Do] and [DLM4] respectively. Summarizing,

THEOREM 7.6. *Let L be an even lattice. Then V_L is a regular VOA, and its ordinary, irreducible modules are the Fock spaces $V_{L+\lambda}$. It thus has just $|L^0 : L|$ distinct ordinary, irreducible modules.*

In the following exercises, $L \subseteq E$ is an even lattice in Euclidean space as above.

EXERCISE 7.7. Show that the VOA (71) is *isomorphic* to the tensor product $M_0^{\otimes d}$ of d copies of the Heisenberg VOA M_0 (cf. Exercise 2.32).

EXERCISE 7.8. Show that M_0^d is a *simple* VOA.

EXERCISE 7.9. Verify that if $\alpha \in L$ then $\mathbf{1} \otimes e^\alpha$ has L_0 -weight $Q(\alpha)$.

EXERCISE 7.10. In the definition of V_L , what is the purpose of requiring L to be an *even* lattice? What about positive-definiteness?

EXERCISE 7.11. Let $\alpha \in L$.

- (a) Prove that $Y(e^\alpha, z)$ is a creative field in $\mathfrak{F}(V_L)$.
- (b) Prove that $Y(e^\alpha, z)$ and $Y(v, z)$ are mutually local ($v \in \hat{\mathfrak{h}}^-$).

EXERCISE 7.12. Let L be an even lattice with $L_0 = \{\alpha \in L \mid Q(\alpha) = 1\}$. Prove that L_0 is a *semisimple root system* with components of type *ADE*.

EXERCISE 7.13. The *dual lattice* of L is defined via

$$L^0 = \{\beta \in E \mid (\alpha, \beta) \in \mathbb{Z} \text{ for all } \alpha \in L\}.$$

Prove that $L \subseteq L^0$ is a subgroup of *finite index*.

EXERCISE 7.14. Let \mathfrak{g} be a finite-dimensional simple Lie algebra of type *ADE*.

- (a) Show that the WZW model $L_{\mathfrak{g}}(1, 0)$ of level 1 is (isomorphic to) the lattice theory V_L where L is the root lattice associated to \mathfrak{g} .
- (b) Compute the number of inequivalent ordinary, irreducible modules over $L_{\mathfrak{g}}(1, 0)$ both by using Theorem 7.1, and by using Theorem 7.6.

EXERCISE 7.15. Let L_1, L_2 be a pair of even lattices.

- (a) Show that the *orthogonal direct sum* $L_1 \perp L_2$ is an even lattice.
- (b) Prove that $V_{L_1 \perp L_2} \cong V_{L_1} \otimes V_{L_2}$ (cf. Exercise 2.32).

8. Vector-valued modular forms

In order to formulate *modular invariance* for C_2 -cofinite and regular VOAs, the idea of a *vector-valued modular form* is useful. This generalizes the theory of modular forms that we discussed in Section 3, and includes as a special case the theory of modular forms on a finite-index subgroup of $SL_2(\mathbb{Z})$. We use the notation of Section 3.

8.1. Basic definitions. Fix an integer k and let \mathfrak{F}_k be the space of holomorphic functions¹¹ in \mathfrak{H} regarded as a right Γ -module with respect to the action defined in (20), (21). A *weak vector-valued modular form of weight k* may be taken to be a finite-dimensional Γ -submodule $V \subseteq \mathfrak{F}_k$. Let¹² $F(\tau) = (f_1(\tau), \dots, f_p(\tau))^t$ where the component functions $f_i(\tau)$ are a set of (not necessarily linearly independent) generators for V . There is then a representation $\rho : \Gamma \rightarrow GL_p(\mathbb{C})$ such that

$$\rho(\gamma)F(\tau) = F|_k\gamma(\tau), \quad \gamma \in \Gamma, \quad (78)$$

¹¹We could equally well deal with *meromorphic functions*.

¹²Superscript t denotes *transpose*.

where $|_k$ is the obvious extension of the stroke operator to vector-valued functions. We also call the pair (F, ρ) a weak vector-valued modular form of weight k . Given a pair (F, ρ) satisfying (78), we recover V as the span of the component functions of $F(\tau)$. The classical modular forms of Section 3 correspond to the case when ρ is the trivial 1-dimensional representation of Γ .

To describe the extension of (22) to the vector-valued case, decompose V into a direct sum of T -invariant indecomposable subspaces

$$V = V_1 \oplus \cdots \oplus V_r$$

corresponding to the Jordan decomposition of the action $T : f(\tau) \mapsto f(\tau + 1)$. The characteristic polynomial on V_i is $(x - e^{2\pi i \mu_i})^{\dim V_i}$. The basic fact is

THEOREM 8.1. *There are q -expansions $g_j(\tau) = q^{\mu_i} \sum_{n \in \mathbb{Z}} a_{ijn} q^n$, $(0 \leq j \leq n_i - 1)$ such that the functions*

$$g_0(\tau) + g_1(\tau) \log q + \cdots + g_m(\tau) (\log q)^m, \quad 0 \leq m \leq n_i - 1, \quad (79)$$

are a basis of V_i . In particular, V has a basis of functions of this form.

We call (79) a *logarithmic*, or *polynomial*,¹³ q -expansion.

Suppose that (F, ρ) is a weak vector-valued modular form. Then the component functions of $F(\tau)$ are linear combinations of polynomial q -expansions (79). We say that (F, ρ) , or simply $F(\tau)$, is *almost holomorphic* if the component functions are holomorphic in \mathfrak{H} and if the q -expansions $g_j(\tau)$ are *left-finite* or *meromorphic at ∞* , i.e., for all i, j the Fourier coefficients a_{ijn} vanish for $n \ll 0$. Similarly, $F(\tau)$ is *holomorphic* if it is almost holomorphic and if $a_{ijn} = 0$ whenever $\text{Re}(\mu_i) + n < 0$. These definitions are independent of the choice of $g_j(\tau)$.

Fix an integer $N \geq 1$. We set

$$\Delta(N) = \langle \gamma T^N \gamma^{-1} \mid \gamma \in \Gamma \rangle.$$

This is the smallest normal subgroup of Γ that contains T^N . We say that a subgroup $G \subseteq \Gamma$ has level N if $\Delta(N) \subseteq G$. A representation $\rho : \Gamma \rightarrow GL_p(\mathbb{C})$ has *level N* if $\ker \rho$ has level N (equivalently, $\rho(T)$ has finite order dividing N). A vector-valued modular form (F, ρ) has level N if ρ has level N . Now recall that finite-order operators are *diagonalizable*. It follows from Theorem 8.1 that if (F, ρ) has level N then the component functions of $F(\tau)$ have q -expansions that are free of logarithmic terms. Indeed, the eigenvalues of $\rho(T)$ are N -th roots of unity, so that the q -expansions (79) reduce to a single q -expansion of the form

$$g_j(\tau) = q^{r/N} \sum_{n \geq 0} a_{jn} q^n \quad (80)$$

¹³We may rewrite (79) using powers of τ , or other polynomials in τ , instead of powers of $\log q$.

for some integer r .

The *principal congruence subgroup of level N* is the subgroup of Γ given by

$$\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv I_2 \pmod{N}\}.$$

We have $\Delta(N) \trianglelefteq \Gamma(N) \trianglelefteq \Gamma$. While $\Gamma(N)$ always has finite index in Γ , $\Delta(N)$ has finite index if, and only if $N \leq 5$ ([KLN], [Wa]). A subgroup $G \subseteq \Gamma$ is a *congruence subgroup* if $\Gamma(N) \subseteq G$ for some N ; ρ and (F, ρ) are called *modular* if $\ker \rho$ is a congruence subgroup. It follows that (F, ρ) is modular if, and only if, the component functions $g_j(\tau)$ of $F(\tau)$ are such that $g_j|_k \gamma(\tau)$ has a q -expansion of shape (80) for every $\gamma \in \Gamma$. This is precisely the definition of a *classical modular form* of weight k and level N (we are assuming holomorphy in \mathfrak{H} for convenience). The case of level 1 again reduces to the theory discussed in Section 3.

Because $\Gamma(N)$ has finite index in Γ it follows that the image $\rho(\Gamma)$ is *finite* whenever ρ is modular. However, the *converse is false*: it may be that the image $\rho(\Gamma)$ is *finite*, so that $\ker \rho$ has finite index in Γ and therefore has some finite level, yet it is not a congruence subgroup. The existence of such subgroups goes back to Klein and Fricke. In this case, a vector-valued modular form (F, ρ) will have some finite level N and its component functions have q -expansions (80), however not all of them will be classical modular forms in the previous sense. This is essentially the theory of modular forms on *noncongruence subgroups*. Modular forms on noncongruence subgroups, and more generally component functions of vector valued modular forms, share many properties in common with classical modular forms and the differences between them can be subtle. It can be difficult to determine whether a given vector-valued modular form (F, ρ) is modular. A fundamental problem in this direction is the following:

Conjecture: Let (F, ρ) be a vector-valued modular form of level N and weight k , and suppose that the component functions of $F(\tau)$ are linearly independent¹⁴ and have rational integers Fourier coefficients. Then (F, ρ) is modular.

We shall see how this fits into VOA theory in the next Section.

EXERCISE 8.2. Prove the following: (a) $\Delta(N) \subseteq \Gamma(N) \trianglelefteq \Gamma$, (b) if $G \subseteq \Gamma$ is a subgroup of finite index then $\Delta(N) \subseteq G$ for some N .

EXERCISE 8.3. Let $\rho: \Gamma \rightarrow GL_p(\mathbb{C})$ be a representation of level N . Show that ρ is modular if, and only if, $\Gamma(N) \subseteq \ker \rho$.

EXERCISE 8.4. : Let $\tilde{\Gamma}$ be the inhomogeneous modular group (Exercise 3.1) and let $\tilde{\Gamma}(N)$ be the image of $\Gamma(N)$ under the natural projection $\Gamma \rightarrow \tilde{\Gamma}$. Prove that $\tilde{\Gamma}(N)$ is *torsion-free* if, and only if, $N \geq 2$.

¹⁴This condition is harmless in practice, but is necessary to avoid trivial counterexamples, e.g. when $F = 0$.

EXERCISE 8.5. It is known that Γ can be abstractly defined by generators and relations $\langle x, y \mid x^4 = y^6 = x^2y^{-3} = 1 \rangle$. Use this to prove the following: (a) $\Gamma/\Gamma' \cong \mathbb{Z}_{12}$, (b) Γ' is a congruence subgroup of level 12. (Γ' is the *commutator subgroup* of Γ .)

EXERCISE 8.6. Let $V \subseteq \mathfrak{F}_k$ be a finite-dimensional Γ -submodule and let (f_1, \dots, f_p) be a sequence of functions in V that contains a basis. Prove the existence of a representation ρ satisfying (78). (Hint: first do the case that (f_1, \dots, f_p) is a linearly independent set.)

8.2. Examples of vector-valued modular forms. One can construct a slew of almost holomorphic vector-valued modular forms using *modular linear differential equations* (MLDE) [M]. We briefly explain this. Let k, n be integers with n positive. The n -th iterate D_k^n of the differential operator (30) is the intertwining map

$$D_k^n = D_{k+2n-2} \circ \dots \circ D_{k+2} \circ D_k : \mathfrak{F}_k \rightarrow \mathfrak{F}_{k+2n}.$$

For justification of the notation, see Exercise 3.13. A modular linear differential equation is a differential equation of the form

$$(D_k^n + g_2(\tau)D_k^{n-2} + \dots + g_{2n}(\tau))f = 0, \quad g_i(\tau) \in \mathfrak{M}_{2i}. \quad (81)$$

Using (30) one can write (81) as an ordinary differential equation with coefficients in the algebra of quasimodular forms \mathfrak{Q} . We can also write everything in terms of the variable q (in the interior of the unit disk in the q -plane)

$$(\theta^n + h_1(q)\theta^{n-1} + \dots + h_{2n}(q))f = 0, \quad h_i(q) \in \mathfrak{Q}_{2i}, \quad (82)$$

where we recall that $\theta = q d/dq$. Then one sees that $q = 0$ is a regular singular point ([H], [I]). By the theory of ODE, the space of solutions is an n -dimensional linear space, and because the coefficients are holomorphic in \mathfrak{H} , so too are the solutions. One sees that the space of solutions is a Γ -submodule of \mathfrak{F}_{k+2n} , and the theory of Frobenius–Fuchs (loc. cit.) shows that the solutions have q -expansions which are meromorphic at ∞ in the sense of Section 8.1. A disadvantage of this approach is that it is hard to get information about the representation of Γ furnished by the space of solutions.

We have seen that vector-valued modular forms naturally incorporate the classical theory of level N modular forms. We complete this Subsection with a discussion of an important class of such forms, namely *theta functions*. Let L be an even lattice of rank d with associated positive-definite quadratic form Q (Section 7.3). The theta function of L is defined by

$$\theta_L(\tau) = \sum_{\alpha \in L} q^{Q(\alpha)} = \sum_{n \geq 0} |L_n| q^n$$

where $L_n = \{\alpha \in L \mid Q(\alpha) = n\}$ (cf. (75)). Hecke and Schoeneberg proved ([O], [Se], [Sc]) that if d is even then $\theta_L(\tau)$ is a holomorphic modular form of weight $d/2$ and a certain level N . A precise description of the level would take us too far afield, but it divides twice the exponent of the finite abelian group L^0/L (cf. Exercise 7.13). In particular, suppose that L is *self-dual* in the sense that $L = L^0$. Then the level is 1, and as we explained this means that $\theta_L(\tau)$ is a holomorphic modular form of weight $d/2$ on the full group Γ .

There are various ways to prove the modularity of $\theta_L(\tau)$. One method that is useful in many other contexts is that of *Poisson summation* ([O], [Se]). The approach in ([Sc]) shows that the space spanned by the theta functions corresponding to the *cosets* of L in L^0 , i.e., the numerators of the expressions on the rhs of (77), is a Γ -submodule of $\mathfrak{F}_{d/2}$. Note that the theta functions of such cosets arise as the numerator in the expression (77) of the character of an ordinary, irreducible module over a lattice VOA.

The reader may be wondering about the case when the rank d of L is *odd*. One still has holomorphic theta functions as above, however they are of *half-integral weight* and do not qualify as modular forms as we have defined them. Odd powers of the eta function also have half-integral weight. These and other examples demonstrate the significance of half-integer weight (vector-valued) modular forms to our subject, but there is no time to develop the subject here.

EXERCISE 8.7. Use your knowledge of the theory of ODEs to verify the details of the assertions following (81) leading to the result that the solution space is a Γ -submodule of \mathfrak{F}_{k+2n} .

EXERCISE 8.8. Why is there no term $g_1(\tau)D_k^{n-1}$ in (81)?

EXERCISE 8.9. For a positive-definite, even lattice L of rank d , prove the estimate $|L_n| = O(n^{d/2})$, and deduce that $\theta_L(\tau)$ is *holomorphic* in \mathfrak{H} .

EXERCISE 8.10. Show that E_8 is the *only* finite dimensional simple Lie algebra whose root lattice is even and self-dual.

EXERCISE 8.11. Show that the theta function $\theta_{E_8}(\tau)$ of the E_8 root lattice coincides with the Eisenstein series Q of Section 3.

EXERCISE 8.12. Show that the partition functions of a lattice theory V_L and its ordinary, irreducible modules are *classical, almost holomorphic, modular functions of weight zero* of some level N .

9. Vertex operator algebras and modular invariance

In this section we describe some of the main results concerning the connections between (vector-valued) modular forms and VOAs. We are concerned here

exclusively with regular and C_2 -cofinite VOAs as discussed in Sections 6 and 7. We recall that V is always assumed to be of CFT-type.

9.1. The regular case. It is convenient to assume at the outset that V is a C_2 -cofinite (but not necessarily rational) VOA of central charge c . By Theorem 6.13 there are only finitely many inequivalent, ordinary, irreducible V -modules, and we denote them $V = M^1, M^2, \dots, M^r$. Let the conformal weight of M^i be h^i (cf. (64) and attendant discussion), and let Z_i be the character of M^i (65).

The first basic fact is that *1-point functions are holomorphic in \mathfrak{H}* . For example, it follows from this and Theorems 5.7 and 5.10 that the 2-point functions $F_V((u_1, z_1), (u_2, z_2))$ are elliptic functions. There are two approaches to the holomorphy of 1-point functions. The first ([Z]) is to find a modular linear differential equation (82) satisfied by $f = Z_i(v, q)$. In this case, because the coefficients of the MLDE are holomorphic in \mathfrak{H} , then so are the solutions (cf. Exercise 8.7). The second approach ([GN]) uses the PBW-type bases that we already mentioned in Section 6.2.

We now take V to be regular. The main properties *vis-à-vis* modular invariance are as follows:

THEOREM 9.1. *Let the notation be as above, and assume that V is regular.*

- (a) *The central charge c and conformal weights h^i are rational numbers.*
- (b) *There is a representation $\rho : \tilde{\Gamma} \rightarrow GL_r(\mathbb{C})$ of the inhomogeneous modular group (cf. Exercise 3.1) with the following property: if $v \in V$ has $L[0]$ -weight k and we set $F_v = (Z_1(v), \dots, Z_r(v))$, then (F_v, ρ) is an almost holomorphic vector-valued modular form of weight k and finite level N .*

We have already discussed the holomorphy of $Z_i(v)$. The heart of the matter - that there is ρ such that (F_v, ρ) is a vector-valued modular form of weight k - is more difficult. It ultimately depends on the complete reducibility of admissible V -modules into ordinary irreducible V -modules. See [Z], [DLM4] for details. The argument shows that the representation ρ is *independent of the state v* . Once the vector-valued modular form is available, one uses the theory of ODEs with regular singular points [MA] to show that (a) holds. The argument, which is arithmetic in nature, makes use of the fact that if v is taken to be the vacuum vector then the component functions $Z_i(\mathbf{1})$ of $F_{\mathbf{1}}$ are just the partition functions of the ordinary irreducible modules over V , and as such have *integral* Fourier coefficients. Also, because $F_{\mathbf{1}}$ has weight zero (because $\mathbf{1} \in V_{[0]}$), $\ker \rho$ contains $\pm I_2$ and so ρ descends to a representation of $\tilde{\Gamma}$. The rationality of conformal weights and central charge implies that (F_v, ρ) has finite level N (e.g., one can take N to be the gcd of the denominators of the rational numbers $h_i - c/24$). There is a basic open problem here:

Modularity conjecture. *In the context of Theorem 9.1, (F_v, ρ) is modular.*

This is an article of faith in the physics literature. There are compelling arguments (e.g., [Ba1], [Ba2], [FMS]) which, however, are not (yet) mathematically rigorous. Note that this Conjecture follows from the conjectured modularity of vector-valued modular forms of level N with integral Fourier coefficients stated at the end of Section 8.1. There are other avenues via which the modularity of (F_v, ρ) might be established, in particular using the theory of tensor products of modules over a VOA and tensor categories (cf. [HL]).

It hardly needs to be said that all known regular VOAs satisfy the Modularity Conjecture. The case of lattice theories follows from Exercise 8.12. The case of WZW models was studied prior to the advent of VOA theory using Lie theory (cf. [KP], [K2]). A discussion of this case as well as that of the simple Virasoro VOAs L_c in the discrete series may be found in [FMS].

9.2. The C_2 -cofinite case. One desires an analog of Theorem 9.1 for the more general case of C_2 -cofinite VOAs, but any generalization must deal with the fact that the span of the partition functions $Z_i(\mathbf{1})$ of the ordinary irreducible modules is generally *not* a Γ -module unless V is a regular VOA. Miyamoto's solution [My1] (see also [Fl]) involves generalized or *pseudo trace functions*. The idea is to utilize the admissible V -modules $L_n(X)$ constructed from a finite-dimensional module X over the algebra $A_n(V)$ (67). C_2 -cofiniteness implies that $A_n(V)$ is finite-dimensional (Theorem 6.13), and this leads to the fact that each of the homogeneous pieces $L_n(X)_m$ are also finite-dimensional. Because $L_n(X)$ is admissible then the zero mode $o(\omega) = L_0$ of the conformal vector operates on these homogeneous pieces (63). However, in the present context L_0 may not be the degree operator. Indeed L_0 may not be a *semisimple operator*.

We decompose $L_n(X)_m$ into a direct sum of Jordan blocks for the action of L_0 . On such a block B there is an L_0 -eigenvector with eigenvalue $m + \lambda$, $\lambda \in \mathbb{C}$, $L_0 - (m + \lambda)I$ is *nilpotent*, and the exponential operator

$$q^{L_0} = q^{m+\lambda} \sum_{t \geq 0} \frac{(2\pi i \tau (L_0 - m - \lambda))^t}{t!} \quad (83)$$

on B reduces to a *finite* sum. If X is indecomposable, λ is determined by the action of ω , which (when regarded as an element of $A_n(V)$) turns out to be a central element and thus acts on X as a scalar. One can piece together the exponentials (83) and incorporate zero modes $o(v)$ of other states as before. However, the details are subtle, as one needs pseudotraces [My1], which is a type of symmetric function on $A_n(V)$ which replaces the usual trace.

The upshot of the analysis sketched above is this: we can define¹⁵ (pseudo) trace functions $\text{Tr}_{L_n(X)}^\phi o(v) q^{L_0 - c/24}$. Once these gadgets are introduced, one

¹⁵ ϕ denotes 'pseudo'.

can use the arguments in the regular case described in the previous Subsection together with additional arguments (to account for the failure of $A_n(V)$ to be semisimple) to show that for each n and for $v \in V_{[k]}$, the pseudo trace functions define a (finite-dimensional) almost holomorphic vector-valued modular form of weight k . Alternatively, they span a finite-dimensional Γ -submodule of \mathfrak{F}_k (notation as in Section 8.1). In particular, the pseudo characters $\text{Tr}_{L_n(X)}^\phi q^{L_0 - c/24}$ are seen to be linear combinations of characters of ordinary, irreducible V -modules with coefficients in $\mathbb{C}[\tau]$. That is, they are polynomial q -expansions in the sense of Section 8.1. This is, of course, fully consistent with Theorem 8.1. Furthermore, one finds as in the regular case that the central charge and conformal weights of the ordinary, irreducible V -modules again lie in \mathbb{Q} .

It would take us too far afield to try to describe any VOAs for which the pseudo trace functions actually involve log terms. Such theories are, naturally, called *logarithmic field theories* in the physics literature. For some examples, see e.g., [GK], [A] and references therein.

EXERCISE 9.2. Prove that the (image of) the conformal vector ω is a central element of $A_n(V)$ (cf. Exercises 6.9, 6.10).

9.3. The holomorphic case. We call a simple, regular VOA V *holomorphic* if it has a *unique* irreducible module, namely the adjoint module V . It seems likely that a simple VOA with a unique ordinary irreducible module is necessarily regular, and therefore holomorphic, but this appears to be unknown. Be that as it may, in the case of holomorphic VOAs Theorem 9.1 can be refined, and in particular the Modularity Conjecture of Section 9.1 holds in this case. This is because if a vector-valued modular form of weight k has a single component $f(\tau)$ then it affords a 1-dimensional representation of Γ and so there is a character $\alpha : \Gamma \rightarrow \mathbb{C}^*$ such that

$$f|_k \gamma(\tau) = \alpha(\gamma)f(\tau), \quad \gamma \in \Gamma. \tag{84}$$

Since Γ' is a congruence subgroup of level 12 (Exercise 8.5) it follows that $f(\tau)$ is a classical modular form of level dividing 12. Thanks to Theorem 9.1 all of this applies with $f = Z_V(v, q)$, indeed a bit more is true in this case: the group of characters of Γ is cyclic of order 12 (Exercise 8.5) hence that of $\tilde{\Gamma}$ is cyclic of order 6; and one can argue (cf. Exercise 9.4) that $S \in \ker \alpha$, so that in fact α has order dividing 3 and each $\alpha(\gamma)$ in (84) is a cube root of unity. We thus arrive at

THEOREM 9.3. *Suppose that V is a holomorphic VOA of central charge c .*
 (a) *If $v \in V_{[k]}$ then $Z_V(v, \tau)$ is an almost holomorphic modular form of weight k and level 1 or 3.*

(b) c is an integer divisible by 8. It is divisible by 24 if, and only if, $Z_V(v, q)$ has level 1.

Lattice theories provide a large number of holomorphic VOAs. From Theorem 7.6 it is immediate that V_L is holomorphic if, and only if, $L = L^0$ is self-dual. The partition function is $\theta_L(\tau)/\eta^c(\tau)$ where c is the rank of L (75), and in this case the modularity of the partition function follows directly from comments in Section 8.2.

We also mention that the modules over a tensor product $U \otimes V$ of VOAs (Exercise 2.32) are just the tensor products $M \otimes N$ of modules M over U and N over V ([FHL]). In particular, if U, V are holomorphic then so too is $U \otimes V$.

EXERCISE 9.4. Let V be a holomorphic VOA, and let α be the character of Γ satisfying $(*) Z_V(\mathbf{1})|_0 \gamma(\tau) = \alpha(\gamma) Z_V(\mathbf{1})$. Prove that $\alpha(S) = 1$. (Hint: take $\gamma = S$ and evaluate $(*)$ at $\tau = i$.) Using this, give the details of the proofs of (a) and (b) in Theorem 9.3.

EXERCISE 9.5. Let V be a holomorphic VOA of central charge c , and let $v \in V_{[k]}$. Prove that $Z_V(v, \tau) = g(\tau)/\eta^c(\tau)$ where $g(\tau)$ is an almost holomorphic modular form on Γ of weight $k + c/2$.

9.4. Applications of modular invariance. Theorem 9.1 places strong conditions on the 1-point trace functions of a regular VOA, and in particular on the partition function. If V is a holomorphic VOA the conditions are even stronger. In this subsection we give a few illustrations of how modular invariance can be used to study the structure of holomorphic VOAs.

By Exercise 9.5, $Z_V(\mathbf{1}) = g(\tau)/\eta^c(\tau)$ where $g(\tau) = 1 + \dots \in \mathfrak{M}_{c/2}$ is a holomorphic modular form on Γ of weight $c/2$. There are no (nonzero) such forms of negative weight, so we have $c \geq 0$. If $c = 0$ then $g(\tau) = 1 = Z_V(\tau)$, corresponding to the 1-dimensional VOA $\mathbb{C}\mathbf{1}$ (cf. Exercise 2.30) which is indeed holomorphic.

Since $8|c$, the next two cases are $c = 8, 16$, when $g(\tau)$ has weight 4 and 8 respectively. Because of the structure of the algebra \mathfrak{M} of modular forms on Γ (Theorem 3.9 and (28)) there is only one choice for $g(\tau)$ in these cases, namely $g(\tau) = Q$ or Q^2 , so the partition function is *uniquely determined* as $Z_V(\mathbf{1}) = Q/\eta^8(\tau)$ or $Q^2/\eta^{16}(\tau) = (Q/\eta^8(\tau))^2$ (Exercise 8.11 is relevant here). We have already seen holomorphic VOAs with these partition functions in Section 9.3, namely the lattice theories V_{E_8} and $V_{E_8 \perp E_8} \equiv V_{E_8}^{\otimes 2}$ (E_8 refers to the root lattice of type E_8). In fact, there is a second even, self-dual lattice L_2 of rank 16 not isometric to $E_8 \perp E_8$ and we obtain in this way a second holomorphic VOA V_{L_2} . It turns out that these are the *only* holomorphic VOAs (up to isomorphism) with $c = 8$ or 16. This result requires additional techniques based on applications of

the recursion in Theorem 5.10 and analytic properties of vector-valued modular forms ([DM2], [DM3]). To summarize:

THEOREM 9.6. *Suppose that V is a holomorphic VOA of central charge $c \leq 16$. Then one of the following holds:*

- (a) $c = 0$ and $V = \mathbb{C}\mathbf{1}$.
- (b) $c = 8$ and $V = V_{E_8}$ is the E_8 -lattice theory.
- (c) $c = 16$ and $V = V_{E_8 \perp E_8}$ or V_{L_2} is a lattice theory.

We now consider holomorphic VOAs V of central charge $c = 24$. In some ways, this is the most interesting case. If $c \geq 32$ the number of isometry classes of even, self-dual lattices of rank c is very large (see [Se] for further comments), so there are a correspondingly large number of isomorphism classes of holomorphic VOAs. For rank 24 there are just 24 isometry classes of even, self-dual lattices (cf. [CS], [Se]), so one might hope that there are not too many holomorphic VOAs with $c = 24$. In fact, Schellekens has conjectured that there are just 71 such theories [Sch]. Now $Z_V(\mathbf{1}) = q^{-1} + \dots$ is an almost holomorphic modular function of weight zero and level 1 by Theorem 9.3. As such it is a polynomial in the modular function $j(\tau) = q^{-1} + 744 + \dots$ (cf. (35) and Exercise 3.20). So there is an integer d such that

$$Z_V(\mathbf{1}) = j(\tau) + (d - 744)q^{-1} + d + 196884q + \dots$$

and the partition function is *determined uniquely* by d . Obviously $d = \dim V_1$, so it is a nonnegative integer, but one cannot say more about d on the basis of modular invariance alone because $j(\tau) + c'$ is a modular function for *any* constant c' . It can in fact be proved that there are only finitely many choices of d that correspond to possible holomorphic VOAs¹⁶. The arguments use Lie algebra theory, starting with the Lie algebra structure on V_1 (Exercise 9.7) as well as modular forms (see [DM1], [DM2], [DM3], [Sch]). Of the 71 conjectured holomorphic $c = 24$ VOAs, it seems that only 39 are known to exist. Beyond the 24 lattice theories, the other 15 are constructed as so-called \mathbb{Z}_2 -orbifold models of lattice theories [DGM]. The first construction of this type [FLM] leads to the famous *Moonshine Module*, about which we will shortly say a bit more. It is a major problem to decide whether the others also exist, and to develop construction techniques when they do.

As a final example, we mention some recent work of E. Witten [Wi] where certain holomorphic vertex operator algebras $V^{(k)}$ are posited to exist which are related, via the *AdS-CFT correspondence*, to phenomena concerning gravity with a negative cosmological constant. $V^{(k)}$ has central charge $c_k = 24k$, $k = 1, 2, \dots$ and a *minimal structure* compatible with the requirements of modular

¹⁶No more than a few hundred.

invariance imposed by Theorem 9.3. To explain what this is supposed to mean, recall (cf. Theorem 7.2) that $\text{Vir}_{c_k} = L_{c_k}$ is simple, and the L_{c_k} -submodule of $V^{(k)}$ generated by $\mathbf{1}$ is a graded subspace U naturally identified as the Fock space for L_{c_k} . By (42), the graded dimension of U is

$$q^{-k} \prod_{n \geq 2} (1 - q^n)^{-1} = q^{-k} \sum_{n=0}^k d_n q^n + O(q)$$

for integers d_0, \dots, d_k . The posited minimal structure of $V^{(k)}$ means that the partition function of $V^{(k)}$ also satisfies

$$Z_{V^{(k)}}(\mathbf{1}) = q^{-k} \sum_{n=0}^k d_n q^n + O(q).$$

In other words, the first $k+1$ graded subspaces $V_n^{(k)}$ ($0 \leq n \leq k$) of $V^{(k)}$ coincide with the corresponding graded pieces of U , so that they are as small as they can be. We know that $Z_{V^{(k)}}(\mathbf{1})$ is a monic polynomial $\Phi_k(j)$ of degree k in $j(\tau)$, and it is clear that Φ_k is *uniquely determined* by d_0, \dots, d_k , and hence by k .

As in the case of the ‘missing’ holomorphic $c = 24$ theories, the main question here for the VOA theorist is whether $V^{(k)}$ exists or not. The answer is unknown for any k with the notable exception of the *Moonshine module* V^{\natural} ([B], [FLM], [DGM], [My2]) corresponding to $k = 1$. In this case the graded dimension of U is $q^{-1} + O(q)$, the partition function of V^{\natural} is

$$Z_{V^{\natural}}(q) = j(q) - 744 = q^{-1} + 0 + 196884q + \dots,$$

and the minimal structure is reflected in the vanishing of the constant term. In this case the Lie algebra structure on the weight 1 subspace is absent, and one must exploit instead the *Griess algebra*, i.e., the commutative algebra structure on V_2^{\natural} (cf. Exercise 9.9).

One of the main features of the Moonshine Module is its automorphism group, which is the Monster sporadic simple group ([FLM], [G1], [G2]). In order to develop this aspect of V^{\natural} as well as the \mathbb{Z}_2 -orbifold construction that we mentioned above and other features of VOAs, it would be necessary to develop the theory of *automorphism groups* of VOAs. This will have to wait for another day. A brief description of some of the connections between automorphism groups and generalized modular forms can be found in [KM].

EXERCISE 9.7. Let V be a VOA of CFT-type. Prove the following:

- (a) The product $[a, b] = a_0 b$ equips V_1 with the structure of a *Lie algebra*.
- (b) $\langle a, b \rangle = a_1 b$ defines a symmetric, invariant, bilinear form on V_1 .

EXERCISE 9.8. Prove that the Fourier coefficients of the q -expansion of $\Phi_k(j)$ are nonnegative integers (a necessary condition for the existence of $V^{(k)}$).

EXERCISE 9.9. Show that the product $a_1 b$ ($a, b \in V_2^{\natural}$) equips the weight 2 subspace of V^{\natural} with the structure of a *commutative, nonassociative* algebra.

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Part III. Two current research areas

10. Some preliminaries

10.1. VOAs and rational matrix elements. As noted in Section 2.6 there are a number of equivalent sets of axioms for VOA theory. Here we discuss one of these equivalent approaches wherein the properties of a VOA are expressed in terms of the properties of matrix elements which turn out to be rational functions of the formal vertex operator parameters. In many ways, this is the closest approach to CFT (see [FMS], for example) in that the formal parameters can be taken to be complex numbers with the matrix elements considered as rational functions on the Riemann sphere.

We begin by defining matrix elements. In order to simplify the discussion, we always assume that the VOA is of CFT-type (62). This condition is satisfied in all examples we consider. We define the *restricted dual space* of V by [FHL]

$$V' = \bigoplus_{n \geq 0} V_n^*, \quad (85)$$

where V_n^* is the dual space of linear functionals on the finite dimensional space V_n . Let $\langle \cdot, \cdot \rangle_d$ denote the canonical pairing between V' and V . Define *matrix elements* for $a' \in V'$, $b \in V$ and vertex operators $Y(u^1, z_1), \dots, Y(u^n, z_n)$ by

$$\langle a', Y(u^1, z_1) \dots Y(u^n, z_n) b \rangle_d. \quad (86)$$

In particular, choosing $b = \mathbf{1}$ and $a' = \mathbf{1}'$ we obtain the (genus zero) n -point correlation function

$$F_V^{(0)}((u^1, z_1), \dots, (u^n, z_n)) = \langle \mathbf{1}', Y(u^1, z_1) \dots Y(u^n, z_n) \mathbf{1} \rangle_d. \quad (87)$$

One can show in general that every matrix element is a homogeneous rational function of z_1, \dots, z_n [FHL], [DGM]. Thus the formal parameters of VOA theory can be replaced by complex parameters on (appropriate subdomains of) the genus zero Riemann sphere $\mathbb{C}\mathbb{P}^1$. We illustrate this by considering matrix elements containing one or two vertex operators. Recall from (36) that, for $u \in V_n$,

$$u_k : V_m \rightarrow V_{m+n-k-1}. \quad (88)$$

Hence it follows that for $a' \in V_{m'}$, $b \in V_m$ and $u \in V_n$ we obtain a monomial

$$\langle a', Y(u, z) b \rangle_d = C_{a'b}^u z^{m'-m-n}, \quad (89)$$

where $C_{a'b}^u = \langle a', u_{m+n-m'-1} b \rangle_d$.

We next consider the matrix element of two vertex operators to find (recalling convention (1)):

THEOREM 10.1. Let $a' \in V'_{m'}$, $b \in V_m$, $u^1 \in V_{n_1}$ and $u^2 \in V_{n_2}$. Then

$$\langle a', Y(u^1, z_1)Y(u^2, z_2)b \rangle_d = \frac{f(z_1, z_2)}{z_1^{m+n_1} z_2^{m+n_2} (z_1 - z_2)^{n_1+n_2}}, \quad (90)$$

$$\langle a', Y(u^2, z_2)Y(u^1, z_1)b \rangle_d = \frac{f(z_1, z_2)}{z_1^{m+n_1} z_2^{m+n_2} (-z_2 + z_1)^{n_1+n_2}}, \quad (91)$$

where $f(z_1, z_2)$ is a homogeneous polynomial of degree $m + m' + n_1 + n_2$.

REMARK 10.2. The matrix elements (90), (91) are thus determined by a unique homogeneous rational function which can be evaluated on $\mathbb{C}\mathbb{P}^1$ in the domains $|z_1| > |z_2|$ and $|z_2| > |z_1|$ respectively.

PROOF. Consider

$$\langle a', Y(u^1, z_1)Y(u^2, z_2)b \rangle_d = \sum_{k \geq 0} \sum_{c \in V_k} \langle a', Y(u^1, z_1)c \rangle_d \langle c', Y(u^2, z_2)b \rangle_d,$$

where c ranges over any basis of V_k and $c' \in V_k^*$ is dual to c . From (89) it follows that

$$\langle a', Y(u^1, z_1)Y(u^2, z_2)b \rangle_d = \frac{z_1^{m'-n_1}}{z_2^{m+n_2}} G\left(\frac{z_2}{z_1}\right),$$

for infinite series

$$G(x) = \sum_{k \geq 0} \sum_{c \in V_k} C_{a'c}^{u^1} C_{c'b}^{u^2} x^k.$$

Hence the matrix element is homogeneous of degree $m' - m - n_1 - n_2$. Similarly

$$\langle a', Y(u^2, z_2)Y(u^1, z_1)b \rangle_d = \frac{z_2^{m'-n_2}}{z_1^{m+n_1}} H\left(\frac{z_1}{z_2}\right),$$

for the infinite series

$$H(y) = \sum_{k \geq 0} \sum_{c \in V_k} C_{a'c}^{u^2} C_{c'b}^{u^1} y^k.$$

But $Y(u^2, z_2)$ and $Y(u^1, z_1)$ are local of order at most $n_1 + n_2$ (cf. Exercise 4.4) and hence

$$\frac{(z_1 - z_2)^{n_1+n_2}}{z_1^{m+n_1} z_2^{m+n_2}} z_1^{m'+m} G\left(\frac{z_2}{z_1}\right) = \frac{(z_1 - z_2)^{n_1+n_2}}{z_1^{m+n_1} z_2^{m+n_2}} z_2^{m'+m} H\left(\frac{z_1}{z_2}\right). \quad (92)$$

It follows that

$$f(z_1, z_2) = z_1^{m'+m} (z_1 - z_2)^{n_1+n_2} G\left(\frac{z_2}{z_1}\right) = z_2^{m'+m} (z_1 - z_2)^{n_1+n_2} H\left(\frac{z_1}{z_2}\right)$$

is a homogeneous polynomial of degree $m + m' + n_1 + n_2$. \square

Properties (90) and (90) are equivalent to locality of $Y(u^1, z_1)$ and $Y(u^2, z_2)$ so that the axioms of a VOA can be alternatively formulated in terms of rational matrix elements [DGM], [FHL]. Theorem 10.1 can also be generalized for all matrix elements. Furthermore, using the vertex commutator property (Exercise 2.14) one can also derive a recursive relationship in terms of rational functions between matrix elements for n vertex operators and $n - 1$ vertex operators that is the genus zero version of Zhu's first recursion formula (Theorem 5.7).

EXERCISE 10.3. Prove (89).

EXERCISE 10.4. Show (92) implies that $f(z_1, z_2)$ is a polynomial.

10.2. Genus-zero Heisenberg correlation functions. We illustrate these structures by considering the example of the rank one Heisenberg VOA M_0 generated by a weight one vector a . Let

$$G_n^{(0)}(z_1, \dots, z_n) \equiv F_{M_0}^{(0)}((a, z_1), \dots, (a, z_n)). \quad (93)$$

denote the n -point correlation function for n Heisenberg vectors. This must be a symmetric rational function in z_i with poles of order two at $z_i = z_j$ for all $i \neq j$ from locality. We now determine its exact form. Since $a_0 \mathbf{1} = 0$ it follows that $G_1^{(0)}(z_1) = 0$. The 2-point function is

$$G_2^{(0)}(z_1, z_2) = \sum_{m \geq 0} z_1^{-m-1} \langle \mathbf{1}', a_m Y(a, z_2) \mathbf{1} \rangle_d,$$

where (88) implies that there is no contribution for $m < 0$. Commuting a_m we find

$$G_2^{(0)}(z_1, z_2) = \sum_{m \geq 0} z_1^{-m-1} \langle \mathbf{1}', [a_m, Y(a, z_2)] \mathbf{1} \rangle_d,$$

using $a_m \mathbf{1} = 0$ for $m \geq 0$. But the Heisenberg commutation relations imply

$$[a_m, Y(a, z_2)] = m z_2^{m-1},$$

so that

$$G_2^{(0)}(z_1, z_2) = \sum_{m \geq 0} m z_1^{-m-1} z_2^{m-1} = \frac{1}{(z_1 - z_2)^2}. \quad (94)$$

The general n -point function is similarly given by

$$G_n^{(0)}(z_1, \dots, z_n) = \sum_{m \geq 0} z_1^{-m-1} \sum_{i=2}^n \langle \mathbf{1}', Y(a, z_2) \dots [a_m, Y(a, z_i)] \dots Y(a, z_n) \mathbf{1} \rangle_d,$$

leading to a recursive identity

$$G_n^{(0)}(z_1, \dots, z_n) = \sum_{i=2}^n \frac{1}{(z_1 - z_i)^2} G_{n-2}^{(0)}(z_2, \dots, \hat{z}_i, \dots, z_n), \quad (95)$$

where \hat{z}_i is deleted. Thus we may recursively solve to find $G_n^{(0)} = 0$ for n odd whereas for n even, $G_n^{(0)}$ is expressed as multiples of rational terms of the form $1/(z_i - z_j)^2$ for all possible pairings z_i, z_j . This can be equivalently described in terms of the subset, denoted by $F(\Phi)$, of the permutations of the label set $\Phi = \{1, \dots, n\}$ consisting of *fixed-point-free involutions*. Thus a typical element $\varphi \in F(\Phi)$ is given by $\varphi = \dots (ij) \dots$, a product of $n/2$ disjoint cycles. We then find (95) implies

THEOREM 10.5. $G_n^{(0)}$ vanishes for n odd, whereas for n even

$$G_n^{(0)}(z_1, \dots, z_n) = \sum_{\varphi \in F(\Phi)} \prod \frac{1}{(z_i - z_j)^2} \tag{96}$$

where the product ranges over all the cycles of $\varphi = \dots (ij) \dots$

REMARK 10.6. Using associativity one can show that $G_n^{(0)}(z_1, \dots, z_n)$ is in fact a generating function for all matrix elements of the Heisenberg VOA.

EXERCISE 10.7. Show that $|F(\Phi)| = (n-1)!! = (n-1).(n-3).(n-5) \dots$

EXERCISE 10.8. For $n = 4$ show that $F(\Phi) = \{(12)(34), (13)(24), (14)(23)\}$ and $G_4^{(0)}(z_1, z_2, z_3, z_4)$ is given by

$$\frac{1}{(z_1 - z_2)^2(z_3 - z_4)^2} + \frac{1}{(z_1 - z_3)^2(z_2 - z_4)^2} + \frac{1}{(z_1 - z_4)^2(z_2 - z_3)^2}.$$

10.3. Adjoint vertex operators. The Virasoro subalgebra $\{L_{-1}, L_0, L_1\}$ generates a natural action on vertex operators associated with $SL(2, \mathbb{C})$ Möbius transformations on z (cf. [B], [DGM], [FHL], [K1] and Exercise 2.21). Thus under the translation $z \mapsto z + \lambda$ generated by L_{-1} we have (cf. Exercise 2.16)

$$q_\lambda^{L_{-1}} Y(u, z) q_\lambda^{-L_{-1}} = Y(u, z + \lambda). \tag{97}$$

Under $z \mapsto q_\lambda z$ generated by L_0 we have (cf. Exercise 4.2)

$$q_\lambda^{L_0} Y(u, z) q_\lambda^{-L_0} = Y(q_\lambda^{L_0} u, q_\lambda z). \tag{98}$$

Finally, under the transformation $z \mapsto z/(1 - \lambda z)$ generated by L_1 we find

$$q_\lambda^{L_1} Y(u, z) q_\lambda^{-L_1} = Y(q_\lambda^{L_1(1-\lambda z)} (1 - \lambda z)^{-2L_0} u, \frac{z}{1 - \lambda z}). \tag{99}$$

Combining these it follows that the transformation $z \mapsto -\lambda^2 z^{-1}$ is described by $T_\lambda \equiv q_\lambda^{L_{-1}} q_{\lambda^{-1}}^{L_1} q_\lambda^{L_{-1}}$ with

$$T_\lambda Y(u, z) T_\lambda^{-1} = Y(q_{-\lambda^2 z^{-1}}^{L_1} (\lambda^{-2} z^2)^{-L_0} u, -\lambda^2 z^{-1}). \tag{100}$$

Taking $\lambda = \sqrt{-1}$ in (100) corresponding to the inversion $z \mapsto z^{-1}$ we find

$$Y^\dagger(u, z) \equiv T_{\sqrt{-1}} Y(u, z) T_{\sqrt{-1}}^{-1} = Y(q_z^{L_1} (-z^2)^{-L_0} u, z^{-1}). \quad (101)$$

We call $Y^\dagger(u, z)$ the *adjoint* vertex operator¹⁷. For u of weight $\text{wt}(u)$ it follows that $Y^\dagger(u, z) = \sum_n u_n^\dagger z^{-n-1}$ has modes

$$u_n^\dagger = (-1)^{\text{wt}(u)} \sum_{k=0}^{\text{wt}(u)} \frac{1}{k!} (L_1^k u)_{2\text{wt}(u)-n-k-2}. \quad (102)$$

For a quasiprimary state u (102) simplifies to

$$u_n^\dagger = (-1)^{\text{wt}(u)} u_{2\text{wt}(u)-n-2}. \quad (103)$$

Thus for a weight one Heisenberg vector a we find

$$a_n^\dagger = -a_{-n}. \quad (104)$$

and for the weight two Virasoro vector ω we find that for $L_n^\dagger \equiv \omega_{n+1}^\dagger$

$$L_n^\dagger = L_{-n}. \quad (105)$$

We also note that the adjoint vertex operators can be used to construct a canonical V -module as follows. Define vertex operators $Y_{V'} : V \rightarrow \mathcal{F}(V')$ by

$$\langle Y'(u, z) a', b \rangle_d = \langle a', Y^\dagger(u, z) b \rangle_d, \quad (106)$$

for $a' \in V'$ and $b, u \in V$. Then $(V', Y_{V'})$ can be shown to be a V -module called the *dual or contragredient module* [FHL].

EXERCISE 10.9. Prove (100).

EXERCISE 10.10. Show for a quasiprimary state u (i.e., $L_1 u = 0$) of weight $\text{wt}(u)$ that under a Möbius transformation $z \rightarrow \phi(z) = (az + b)/(cz + d)$

$$Y(u, z) \rightarrow \left(\frac{d\phi}{dz} \right)^{\text{wt}(u)} Y(u, \phi(z)). \quad (107)$$

EXERCISE 10.11. Hence show for n quasiprimary vectors u^i of weight $\text{wt}(u^i)$ that the rational n -point function (87) is associated with a (formal) Möbius-invariant differential form on $\mathbb{C}\mathbb{P}^1$

$$\mathcal{F}_V^{(0)}(u^1, \dots, u^n) = F_V^{(0)}((u^1, z_1), \dots, (u^n, z_n)) \prod_{1 \leq i \leq n} dz_i^{\text{wt}(u^i)}. \quad (108)$$

¹⁷This terminology differs from that of [FHL]

REMARK 10.12. $\mathcal{F}_V^{(0)}(u^1, \dots, u^n)$ is a conformally invariant global meromorphic differential form on \mathbb{CP}^1 if u^1, \dots, u^n are primary vectors i.e., $L_n u^i = 0$ for all $n > 0$.

EXERCISE 10.13. Prove (102).

EXERCISE 10.14. Show that $(Y^\dagger(u, z))^\dagger = Y(u, z)$.

10.4. Invariant bilinear forms. In this subsection we consider the construction of a canonical bilinear form on V motivated by (106). We say a bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ is *invariant* if for all $a, b, u \in V$

$$\langle Y(u, z)a, b \rangle = \langle a, Y^\dagger(u, z)b \rangle, \quad (109)$$

with $Y^\dagger(a, z)$ the adjoint operator of (101). In terms of modes, (109) reads

$$\langle u_n a, b \rangle = \langle a, u_n^\dagger b \rangle, \quad (110)$$

using (102). Applying (105) it follows that

$$\langle L_0 a, b \rangle = \langle a, L_0 b \rangle. \quad (111)$$

Thus for homogeneous a and b then $\langle a, b \rangle = 0$ for $\text{wt}(a) \neq \text{wt}(b)$.

Next consider a, b with $\text{wt}(a) = \text{wt}(b)$. Invariance and skew-symmetry (see Exercise 2.15) give

$$\begin{aligned} \langle \mathbf{1}, Y^\dagger(a, z)b \rangle &= (-z^2)^{-\text{wt}(a)} \langle \mathbf{1}, Y(q_z^{L_1} a, z^{-1})b \rangle \\ &= (-z^2)^{-\text{wt}(b)} \langle \mathbf{1}, q_z^{L_{-1}} Y(b, -z^{-1})q_z^{L_1} a \rangle \\ &= \langle \mathbf{1}, q_z^{L_{-1}} Y^\dagger(q_z^{L_1} b, -z)q_z^{L_1} a \rangle. \end{aligned}$$

But (105) implies this is

$$\langle q_z^{L_{-1}} \mathbf{1}, Y^\dagger(q_z^{L_1} b, -z)q_z^{L_1} a \rangle = \langle \mathbf{1}, Y^\dagger(q_z^{L_1} b, -z)q_z^{L_1} a \rangle.$$

Using invariance this becomes

$$\langle Y(q_z^{L_1} b, -z)\mathbf{1}, q_z^{L_1} a \rangle.$$

Finally, using Exercise 2.9 and (105) this is

$$\langle q_z^{L_{-1}} q_z^{L_1} b, q_z^{L_1} a \rangle = \langle b, q_z^{L_{-1}} a \rangle = \langle b, Y(a, z)\mathbf{1} \rangle.$$

Thus we have shown

$$\langle Y(a, z)\mathbf{1}, b \rangle = \langle b, Y(a, z)\mathbf{1} \rangle.$$

In particular, considering the z^0 term, this implies that the bilinear form is symmetric:

$$\langle a, b \rangle = \langle b, a \rangle. \quad (112)$$

Consider again a, b with $\text{wt}(a) = \text{wt}(b)$. Using the creation axiom $a_{-1}\mathbf{1} = a$ we obtain

$$\langle a, b \rangle = \langle \mathbf{1}, a_{-1}^\dagger b \rangle. \quad (113)$$

with $a_{-1}^\dagger b \in V_0$. Thanks to the assumption that V is of CFT-type¹⁸ we have $a_{-1}^\dagger b = \alpha\mathbf{1}$ for some $\alpha \in \mathbb{C}$ with $\langle a, b \rangle = \alpha \langle \mathbf{1}, \mathbf{1} \rangle$. Hence either $\langle \mathbf{1}, \mathbf{1} \rangle = 0$ so that $\langle a, b \rangle = 0$ for all a, b or else $\langle a, b \rangle$ is non-trivial and is uniquely determined up to the value of $\langle \mathbf{1}, \mathbf{1} \rangle \neq 0$ in which case we choose the normalization $\langle \mathbf{1}, \mathbf{1} \rangle = 1$.

It is straightforward to show that if $\langle \mathbf{1}, \mathbf{1} \rangle \neq 0$ then $L_1 V_1 = 0$ (cf. Exercise 10.16). Li has shown [Li] that the converse is also true: for a VOA of CFT-type, then $\langle \mathbf{1}, \mathbf{1} \rangle \neq 0$ if and only if $L_1 V_1 = 0$. We say that a VOA is of *Strong CFT-type* if it is of CFT-type and $L_1 V_1 = 0$. Such a VOA therefore has a *unique* normalized invariant bilinear form.

The pairing $\langle \cdot, \cdot \rangle$ determines a standard map from V to the restricted dual space V' defined by

$$a \mapsto \langle a, \cdot \rangle. \quad (114)$$

Let \mathcal{K} denote the kernel of this map. $\langle \cdot, \cdot \rangle$ is nondegenerate with \mathcal{K} trivial if, and only if, V is isomorphic to V' (in other words V is self-dual). In this case, we may identify $\langle \cdot, \cdot \rangle$ with the canonical pairing $\langle \cdot, \cdot \rangle_d$ and the dual module (106) is isomorphic to the original VOA.

The nondegeneracy of $\langle \cdot, \cdot \rangle$ is also related to the simplicity of the VOA V in much the same way that nondegeneracy of the Killing form determines semi-simplicity in Lie theory [Li]. Let $\mathcal{J} \subset V$ denote the maximal proper ideal of V (cf. Exercise 6.11), so that

$$u_n b \in \mathcal{J}, \quad (115)$$

for all $b \in \mathcal{J}, u \in V$. V is simple if \mathcal{J} is trivial (cf. Section 6). We now show that assuming V is of strong CFT-type then $\mathcal{J} = \mathcal{K}$ and hence V is simple if, and only if, $\langle \cdot, \cdot \rangle$ is nondegenerate.

We firstly note that $\mathbf{1} \notin \mathcal{J}$ (otherwise $u = u_{-1}\mathbf{1} \in \mathcal{J}$ for all $u \in V$). Because V is of CFT-type, then for all $b \in \mathcal{J}$ it follows $b \notin V_0$ and so

$$\langle \mathbf{1}, b \rangle = 0. \quad (116)$$

Consider $u \in V$ and $b \in \mathcal{J}$. Then $u_{-1}^\dagger b \in \mathcal{J}$ from (102) and so

$$\langle u, b \rangle = \langle \mathbf{1}, u_{-1}^\dagger b \rangle = 0,$$

for all u from (116). Hence we find $\mathcal{J} \subseteq \mathcal{K}$. Conversely, suppose that $c \in \mathcal{K}$. Then

$$\langle Y^\dagger(u, z)v, c \rangle = 0,$$

¹⁸The general situation is discussed in [Li]).

for all $u, v \in V$. Invariance implies $\langle v, Y(u, z)c \rangle = 0$ and hence $u_n c \in \mathcal{K}$ for all u_n . But given V is of strong CFT-type then $\langle \cdot, \cdot \rangle$ is nontrivial so that $\mathcal{K} \neq V$ and hence $\mathcal{K} \subseteq \mathcal{J}$. Thus we conclude $\mathcal{J} = \mathcal{K}$.

Altogether we may summarize these results as follows:

THEOREM 10.15. *Let V be a VOA. An invariant bilinear form $\langle \cdot, \cdot \rangle$ on V is symmetric and diagonal with respect to the canonical L_0 -grading. Furthermore, if V is of strong CFT-type, $\langle \cdot, \cdot \rangle$ is unique up to normalization and is nondegenerate if and only if V is simple.*

The invariant bilinear form is equivalent to the chiral part of the Zamolodchikov metric in CFT ([BPZ; FMS; P]) where (abusing notation)

$$\begin{aligned} \langle a, b \rangle &= \lim_{z_1 \rightarrow 0} \lim_{z_2 \rightarrow 0} \langle Y(a, z_1)\mathbf{1}, Y(b, z_2)\mathbf{1} \rangle \\ &= \lim_{z_1 \rightarrow 0} \lim_{z_2 \rightarrow 0} \langle \mathbf{1}, Y^\dagger(a, z_1)Y(b, z_2)\mathbf{1} \rangle \\ &= \langle \mathbf{1}, Y(a, w_1 = \infty)Y(b, z_2 = 0)\mathbf{1} \rangle, \end{aligned} \tag{117}$$

for $w_1 = 1/z_1$ following (101). We thus refer to the nondegenerate bilinear form as the *Li-Zamolodchikov metric* on V or *LiZ-metric* for short¹⁹.

Consider the rank one Heisenberg VOA M_0 generated by a weight one state a with V spanned by Fock vectors

$$v = a_{-1}^{e_1} a_{-2}^{e_2} \dots a_{-p}^{e_p} \mathbf{1}, \tag{118}$$

for nonnegative integers e_i . Using (104), we find that the Fock basis consisting of vectors of the form (118) is orthogonal with respect to the LiZ-metric with

$$\langle v, v \rangle = \prod_{1 \leq i \leq p} (-i)^{e_i} e_i!. \tag{119}$$

Clearly $\langle \cdot, \cdot \rangle$ is nondegenerate, so by Theorem 10.15 it follows that M_0 is a simple VOA (as already discussed in Section 6).

Consider the Virasoro VOA Vir_c generated by the Virasoro vector ω of central charge c . Using (111) it is sufficient to consider the nondegeneracy of $\langle \cdot, \cdot \rangle$ on each homogeneous space V_n . In particular, let $M_n(c) = (\langle a, b \rangle)$ be the Gram matrix of $(\text{Vir}_c)_n$ with respect to some basis. The *Kac determinant* (see [KR]) is $\det M_n(c)$, which is conveniently considered as a polynomial in c . By Theorem 10.15, Vir_c is simple if, and only if, $\det M_n(c) \neq 0$ for all n . For $n = 2$ we have $V_2 = \mathbb{C}\omega$ with Kac determinant

$$\det M_2(c) = \langle \omega, \omega \rangle = \langle \mathbf{1}, L_2 L_{-2} \mathbf{1} \rangle = \frac{c}{2}, \tag{120}$$

¹⁹Although we use the term metric here, the bilinear form is not necessarily positive definite.

with a zero at $c = 0$. For $n = 4$ we have $V_4 = \mathbb{C}\langle L_{-2}^2 \mathbf{1}, L_{-4} \mathbf{1} \rangle$ with

$$M_4(c) = \begin{bmatrix} c(4 + \frac{1}{2}c) & 3c \\ 3c & 5c \end{bmatrix}, \quad (121)$$

and Kac determinant

$$\det M_4(c) = \frac{1}{2}c^2(5c + 22) \quad (122)$$

with zeros at $c = 0, -\frac{22}{5}$.

There is a general formula for the Kac determinant $\det M_n(c)$ which turns out to have zeros for central charge

$$c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}, \quad (123)$$

where $(p-1)(q-1) = n$ for coprime $p, q \geq 2$. Thus Vir_c is a simple VOA iff $c \neq c_{p,q}$ for some coprime $p, q \geq 2$ (cf. Theorem 7.2).

EXERCISE 10.16. Show that if $\langle \mathbf{1}, \mathbf{1} \rangle \neq 0$ then $L_1 V_1 = 0$.

EXERCISE 10.17. Suppose that $a \in V_m, b \in V_n$ and at least one of a or b is quasiprimary. Prove that the 2-point correlation function is given by

$$\langle \mathbf{1}, Y(a, z_1)Y(b, z_2)\mathbf{1} \rangle = \frac{\langle a, b \rangle}{(z_1 - z_2)^{2m}} \delta_{m,n}.$$

(The Zamolodchikov metric in CFT is often introduced in this way.)

EXERCISE 10.18. Verify (119).

11. The genus-two partition function for the Heisenberg VOA

In this section we will discuss some recent research by the authors wherein we develop a theory of partition and n -point correlation functions on a Riemann surface of genus-two [T1; MT2; MT3; MT4]. The basic idea is to construct a genus-two Riemann surface by specific sewing schemes where we either sew together two once punctured tori or self-sew a twice punctured torus (i.e., attach a handle). The partition and n -point functions on the genus-two surface are then defined in terms of correlation functions on the lower genus surfaces combined together in an appropriate way. We will not explore the full details entailed in this programme. Instead we will consider the example of the Heisenberg VOA M_0 and compute the partition function on the genus-two surface formed from two tori.

11.1. Genus-one Heisenberg 1-point functions. We first discuss the genus-one 1-point correlation function for all elements of the Heisenberg VOA M_0 generated by the weight one Heisenberg vector a [MT1]. We make heavy use of the Zhu recursion formulas (Theorems 5.7 and 5.10). In particular, we prove Theorem 4.5 by considering the 1-point function $Z_{M_0}(v, \tau)$ for a Fock vector in the square bracket formulation

$$v = a[-k_1] \dots a[-k_n] \mathbf{1}, \tag{124}$$

for $k_i \geq 1$. The Fock vector v is of square bracket weight $\text{wt}[v] = \sum_i k_i$. We want to show that

$$Z_{M_0}^{(1)}(v, \tau) = \frac{Q_v(\tau)}{\eta(q)}, \tag{125}$$

for $Q_v(\tau) \in \Omega$, the algebra of quasimodular forms. $Q_v(\tau)$ is of weight $\text{wt}[v]$ and is expressed in terms of

$$C(k, l) = C(k, l, \tau) = (-1)^{l+1} \frac{(k+l-1)!}{(k-1)!(l-1)!} E_{k+l}(\tau), \tag{126}$$

for $k, l \geq 1$. Here $E_n(\tau)$ is the Eisenstein series of (24). We recall that $E_n = 0$ for n odd, $E_2(\tau)$ is a quasimodular form of weight 2 and E_n is a modular form of weight n for even $n \geq 4$. Thus $C(k, l, \tau)$ is a quasimodular form of weight $k+l$. We also note that $C(k, l) = C(l, k)$.

Each Fock vector v is described by a label set $\Phi_\lambda = \{k_1, \dots, k_n\}$ which corresponds in a natural 1-1 manner with unrestricted partitions $\lambda = \{1^{e_1}, 2^{e_2}, \dots\}$ of $\text{wt}[v]$ (where $e_i \geq 0$). We write $v = v(\lambda)$ to indicate this correspondence, which will play a significant rôle later on. Define $F(\Phi_\lambda)$ to be the subset of all permutations on Φ_λ consisting only of *fixed-point-free involutions*. Let $\varphi = \dots (k_i k_j) \dots$, a product of disjoint cycles, denote a typical element of $F(\Phi_\lambda)$.

We can now describe the 1-point function $Z_{M_0}^{(1)}(v(\lambda), \tau)$ of (125) [MT1]:

THEOREM 11.1. *For even n*

$$Q_v(\tau) = \sum_{\phi \in F(\Phi_\lambda)} \Gamma(\phi, \tau), \tag{127}$$

$$\Gamma(\phi, \tau) = \prod_{(k_i k_j)} C(k_i, k_j, \tau). \tag{128}$$

for C of (126), where the product ranges over all the cycles of $\varphi = \dots (k_i k_j) \dots$ in $F(\Phi_\lambda)$. Moreover $Q_v(\tau)$ lies in Ω and is of weight $\text{wt}[v]$. For n odd $Q_v(\tau)$ vanishes.

PROOF. Let $v(\lambda) = a[-k_1]w$ for $w = a[-k_2] \dots a[-k_n]\mathbf{1}$ and use the second Zhu recursion formula (Theorem 5.10) to find

$$\begin{aligned} Z_{M_0}^{(1)}(a[-k_1]w, \tau) &= \delta_{k_1,1} \operatorname{Tr}_{M_0}(o(a)o(w)q^{L_0-1/24}) \\ &\quad + \sum_{m \geq 1} (-1)^{m+1} \binom{k_1+m-1}{m} E_{k_1+m}(\tau) Z_{M_0}^{(1)}(a[m]w, \tau). \end{aligned}$$

But $o(a)u = 0$ for all $u \in M$ and the Heisenberg commutation relations imply

$$\begin{aligned} Z_{M_0}^{(1)}(a[-k_1]w, \tau) &= 0 + \sum_{j=2}^n (-1)^{k_j+1} \binom{k_1+k_j-1}{k_j} E_{k_1+k_j}(\tau) k_j Z_{M_0}^{(1)}(\hat{w}, \tau) \\ &= \sum_{j=2}^n C(k_1, k_j, \tau) Z_{M_0}^{(1)}(\hat{w}, \tau), \end{aligned}$$

where \hat{w} denotes the Fock vector with label set $\{k_2, \dots, \hat{k}_j, \dots, k_n\}$ with the index k_j deleted. The result follows by repeated application of this recursive formula until we obtain $\hat{w} = \mathbf{1}$ for which $Z_{M_0}^{(1)}(\mathbf{1}, \tau) = 1/\eta(q)$. The resulting expression for $Q_v(\tau)$ is clearly a quasimodular form of weight $\operatorname{wt}[v] = \sum_i k_i$. Thus Theorem 11.1 follows. \square

Some further insight into the combinatorial structure of $Q_v(\tau)$ can be garnered by a consideration of the n -point function for n Heisenberg vectors which we denote by

$$G_n^{(1)}(z_1, \dots, z_n, \tau) \equiv F_{M_0}^{(1)}((a, z_1), \dots, (a, z), \tau). \quad (129)$$

This is a symmetric function in z_i with a pole of order two at $z_i = z_j$ for all $i \neq j$ (from locality). For $n = 1$ we immediately find

$$G_1^{(1)}(z_1, \tau) = \operatorname{Tr}_{M_0} o(a)q^{L_0-1/24} = 0.$$

The 2-point function is easily computed via the first Zhu recursion formula (Theorem 5.7):

$$\begin{aligned} G_2^{(1)}(z_1, z_2, \tau) &= \operatorname{Tr}_{M_0} o(a)o(a)q^{L_0-1/24} - \sum_{m \geq 1} \frac{(-1)^m}{m!} P_1^{(m)}(z_{12}, \tau) Z_{M_0}^{(1)}(a[m]a, \tau) \\ &= 0 + P_2(z_{12}, \tau) \frac{1}{\eta(q)}, \end{aligned} \quad (130)$$

since $a[m]a = \mathbf{1}\delta_{m,1}$ and where, from Theorem 5.1, we recall

$$P_2(z, \tau) = \frac{d}{dz} P_1(z, \tau) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (n-1) E_n(\tau) z^{n-2}.$$

(130) is the elliptic analogue of the genus zero formula (94) and reflects a deeper geometrical structure underlying the Heisenberg VOA e.g. [MT4].

Using the n -point correlation function version of the first Zhu recursion we can similarly obtain the genus-one analogue of Theorem 10.5 to find [MT1]:

THEOREM 11.2. *For n even*

$$G_n^{(1)}(z_1, \dots, z_n, \tau) = \frac{1}{\eta(q)} \sum_{\varphi \in F(\Phi)} \prod_{(ij)} P_2(z_{ij}, \tau), \quad (131)$$

where the product ranges over all the cycles of $\varphi = \dots (ij) \dots$ for $\Phi = \{1, 2, \dots, n\}$ whereas for n odd $G_n^{(1)}$ vanishes.

We may use this result to compute any genus-one n -point correlation function for M_0 by considering an appropriate analytic expansion of $G_n^{(1)}(z_1, \dots, z_n, \tau)$ [MT1]. In particular, we can rederive (127) by making use of the identity

$$\begin{aligned} G_n^{(1)}(z_1, \dots, z_n, \tau) &= Z_{M_0}^{(1)}(Y[a, z_1] \dots Y[a, z_n] \mathbf{1}, \tau) \\ &= \sum_{k_1, \dots, k_n} Z_{M_0}^{(1)}(v, \tau) z_1^{k_1-1} \dots z_n^{k_n-1}, \end{aligned} \quad (132)$$

for Fock vector $v = a[-k_1] \dots a[-k_n] \mathbf{1}$ for all k_i . We may extract the nonnegative values of k_1, \dots, k_n from the expansion

$$P_2(z_{ij}, \tau) = \frac{1}{(z_i - z_j)^2} + \sum_{k_i, k_j \geq 1} C(k_i, k_j, \tau) z_i^{k_i-1} z_j^{k_j-1}, \quad (133)$$

for C of (126). Thus (131) implies the formula (127) of Theorem 11.1 found for $Q_v(\tau)$.

It is very useful to recast Theorem 11.1 in terms of graph theory as follows. Consider a Fock vector $v(\lambda)$ with label set $\Phi_\lambda = \{k_1, \dots, k_n\}$ and let $\phi \in F(\Phi_\lambda)$ be a fixed-point free involution of Φ_λ leading to a contribution $\Gamma(\phi, \tau)$ to $Q_v(\tau)$ in (127). We may then associate to each $\phi \in F(\Phi_\lambda)$ a ϕ -graph γ_ϕ consisting of n vertices labelled by Φ_λ of unit valence with $n/2$ unoriented edges connecting the pairs of vertices (k_i, k_j) determined by $\phi = \dots (k_i k_j) \dots$. Following Exercise 10.7 there are $(n-1)!!$ such graphs for a given label set Φ_λ . Thus, in Exercise 11.4 with $v = a[-1]^3 a[-2]^2 a[-5] \mathbf{1}$ there are 15 independent ϕ -graphs (cf. Exercise 11.5). A ϕ -graph for a fixed point involution $\phi = (11)(22)(15)$ is shown in Figure 1.²⁰

Given a ϕ -graph γ_ϕ we define a *weight function*

$$\kappa : \{\gamma_\phi\} \longrightarrow \Omega,$$

²⁰Note that there are 3 distinct fixed point involutions notated by (11)(22)(15).

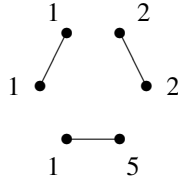


Figure 1. A ϕ -graph for (11)(22)(15).

as follows: for every edge E labeled as $\bullet^k - \bullet^l$ define

$$\kappa(E, \tau) = C(k, l, \tau), \tag{134}$$

with

$$\kappa(\gamma_\phi, \tau) = \prod \kappa(E, \tau), \tag{135}$$

where the product is taken over all edges of γ_ϕ . Thus the ϕ -graph of Figure 1 has weight $C(1, 1)C(2, 2)C(1, 5) = -30E_2(\tau)E_4(\tau)E_6(\tau)$.

Clearly Theorem 11.1 can now be restated in terms of graphs:

THEOREM 11.3. For a Fock vector $v(\lambda)$ with label set $\Phi_\lambda = \{k_1, \dots, k_n\}$ and even n

$$Q_v(\tau) = \sum_{\gamma_\phi} \kappa(\gamma_\phi, \tau), \tag{136}$$

where the sum is taken over all independent ϕ -graphs for Φ_λ .

EXERCISE 11.4. For $v = a[-1]^3 a[-2]^2 a[-5] \mathbf{1}$ of weight $\text{wt}[v] = 12$ with $\Phi_\lambda = \{1, 1, 1, 2, 2, 5\}$ and $|F(\Phi_\lambda)| = 5!! = 15$ (cf. Exercise 10.7) show that

$$\begin{aligned} Q_v(\tau) &= 6C(1, 1)C(1, 2)C(2, 5) + 3C(1, 1)C(2, 2)C(1, 5) + 6C(1, 2)^2 C(1, 5) \\ &= 0 - 90E_2(\tau)E_4(\tau)E_6(\tau) + 0. \end{aligned}$$

Thus only 3 elements of $F(\Phi_\lambda)$ make a nonzero contribution to $Q_v(\tau)$.

EXERCISE 11.5. Find all the ϕ -graphs for $v = a[-1]^3 a[-2]^2 a[-5] \mathbf{1}$.

11.2. Sewing two tori. In this section we digress from VOA theory to briefly review some aspects of Riemann surface theory and the construction of a genus-two surface by sewing together two punctured tori. A genus-two Riemann surface can also be constructed by sewing a handle to a torus but we do not consider that situation here. For more details see [MT2], [MT4].

Let $S^{(2)}$ denote a compact Riemann surface of genus-two and let a_1, a_2, b_1, b_2 be the canonical homology basis (see [FK], for example). There exists two holomorphic 1-forms $v_i, i = 1, 2$, which we may normalize by

$$\oint_{a_i} v_j = 2\pi i \delta_{ij}.$$

The genus-two period matrix Ω is defined by

$$\Omega_{ij} = \frac{1}{2\pi i} \oint_{b_i} v_j, \tag{137}$$

for $i, j = 1, 2$. Using the Riemann bilinear relations, one finds that Ω is a complex symmetric matrix with positive-definite imaginary part, i.e., $\Omega \in \mathfrak{H}_2$, the genus-two Siegel complex upper half-space.

The intersection form \mathcal{E} is a natural nondegenerate symplectic bilinear form on the first homology group $H_1(\mathcal{S}^{(2)}, \mathbb{Z}) \cong \mathbb{Z}^4$, satisfying

$$\mathcal{E}(a_i, a_j) = \mathcal{E}(b_i, b_j) = 0, \quad \mathcal{E}(a_i, b_j) = \delta_{ij}, \quad i, j = 1, 2.$$

The mapping class group is given by the symplectic group

$$\text{Sp}(4, \mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}(4, \mathbb{Z}) \mid \begin{aligned} AB^T &= BA^T, \quad CD^T = D^T C \\ AD^T - BC^T &= I_2 \end{aligned} \right\},$$

where A^T denotes the transpose of A . The group $\text{Sp}(4, \mathbb{Z})$ acts on \mathfrak{H}_2 via

$$\gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{138}$$

and naturally on $H_1(\mathcal{S}, \mathbb{Z})$, where it preserves \mathcal{E} .

We now briefly review a general method originally due to Yamada [Y] and discussed at length in [MT2] for calculating the period matrix (and other structures) on a Riemann surface formed by sewing together two other Riemann surfaces. In particular, we wish to describe Ω_{ij} on a genus-two Riemann surface formed by sewing together two tori \mathcal{S}_a for $a = 1, 2$ (Figure 11.2). Consider an oriented torus $\mathcal{S}_a = \mathbb{C}/\Lambda_a$ with lattice Λ_{τ_a} with basis $(2\pi i, 2\pi i \tau_a)$ for $\tau_a \in \mathfrak{H}$, the complex upper half plane. For local coordinate $z_a \in \mathcal{S}_a$ consider the closed disk $|z_a| \leq r_a$. This is contained in \mathcal{S}_a provided $r_a < \frac{1}{2}D(q_a)$ where

$$D(q_a) = \min_{\lambda \in \Lambda_{\tau_a}, \lambda \neq 0} |\lambda|, \tag{139}$$

is the minimal lattice distance.

Introduce a sewing parameter $\varepsilon \in \mathbb{C}$ where $|\varepsilon| \leq r_1 r_2 < \frac{1}{4}D(q_1)D(q_2)$ and excise the disk $\{z_a, |z_a| \leq |\varepsilon|/r_a\}$ centered at $z_a = 0$ to form a punctured torus

$$\hat{\mathcal{S}}_a = \mathcal{S}_a \setminus \{z_a, |z_a| \leq |\varepsilon|/r_a\},$$

where we use the convention

$$\bar{1} = 2, \quad \bar{2} = 1. \tag{140}$$

Define the annulus

$$\mathcal{A}_a = \{z_a, |\varepsilon|/r_a \leq |z_a| \leq r_a\} \subset \hat{\mathcal{S}}_a, \tag{141}$$

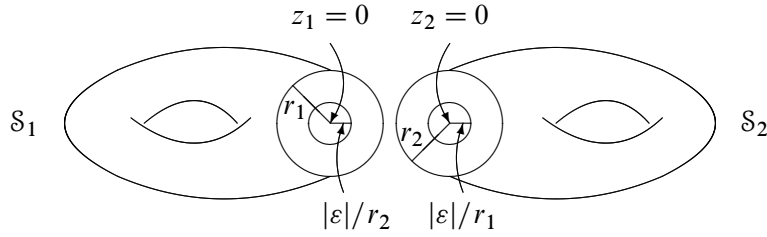


Figure 2. Sewing two tori.

We then identify \mathcal{A}_1 with \mathcal{A}_2 via the sewing relation

$$z_1 z_2 = \varepsilon, \tag{142}$$

to obtain an explicit construction of a genus-two Riemann surface

$$\mathcal{S}^{(2)} = \hat{\mathcal{S}}_1 \cup \hat{\mathcal{S}}_2 \cup (\mathcal{A}_1 \simeq \mathcal{A}_2)$$

which is parameterized by the domain

$$\mathcal{D}^\varepsilon = \{(\tau_1, \tau_2, \varepsilon) \in \mathfrak{H} \times \mathbb{H}_1 \times \mathbb{C} \mid |\varepsilon| < \frac{1}{4} D(q_1) D(q_2)\}. \tag{143}$$

In [Y], Yamada describes a general method for computing the period matrix on the sewn Riemann surface $\mathcal{S}^{(2)}$ in terms of data obtained from the two tori. This is described in detail in [MT2] where we obtain the explicit form for Ω in terms of the infinite matrix $A_a(\tau_a, \varepsilon) = (A_a(k, l, \tau_a, \varepsilon))$ for $k, l \geq 1$ where

$$A_a(k, l, \tau_a, \varepsilon) = \frac{\varepsilon^{(k+l)/2}}{\sqrt{kl}} C(k, l, \tau_a), \tag{144}$$

and where $C(k, l, \tau_a)$ is given in (126). Thus, dropping the subscript,

$$A(\tau, \varepsilon) = \begin{pmatrix} \varepsilon E_2(\tau) & 0 & \sqrt{3}\varepsilon^2 E_4(\tau) & 0 & \cdots \\ 0 & -3\varepsilon^2 E_4(\tau) & 0 & -5\sqrt{2}\varepsilon^3 E_6(\tau) \cdots \\ \sqrt{3}\varepsilon^2 E_4(\tau) & 0 & 10\varepsilon^3 E_6(\tau) & 0 & \cdots \\ 0 & -5\sqrt{2}\varepsilon^3 E_6(\tau) & 0 & -35\varepsilon^4 E_8(\tau) \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrices A_1, A_2 not only play a central rôle here but also later on in our discussion of the genus-two partition for the Heisenberg VOA M_0 . In particular, the matrix $I - A_1 A_2$ and $\det(I - A_1 A_2)$ (where I is the infinite identity matrix here) are important, where $\det(I - A_1 A_2)$ is defined by

$$\log \det(I - A_1 A_2) = \text{Tr} \log(I - A_1 A_2) = - \sum_{n \geq 1} \frac{1}{n} \text{Tr}((A_1 A_2)^n). \tag{145}$$

These expressions are power series in $\Omega[[\varepsilon]]$. One finds:

THEOREM 11.6 [MT2].

(a) *The infinite matrix*

$$(I - A_1 A_2)^{-1} = \sum_{n \geq 0} (A_1 A_2)^n, \quad (146)$$

is convergent for $(\tau_1, \tau_2, \varepsilon) \in \mathcal{D}^\varepsilon$.

(b) $\det(I - A_1 A_2)$ is nonvanishing and holomorphic on \mathcal{D}^ε .

Furthermore we may obtain an explicit formula for the genus-two period matrix on $\mathcal{S}^{(2)}$:

THEOREM 11.7. *The sewing procedure determines a holomorphic map*

$$F^\varepsilon : \mathcal{D}^\varepsilon \rightarrow \mathfrak{H}_2, \quad (\tau_1, \tau_2, \varepsilon) \mapsto \Omega(\tau_1, \tau_2, \varepsilon), \quad (147)$$

where $\Omega = \Omega(\tau_1, \tau_2, \varepsilon)$ is given by

$$\begin{aligned} 2\pi i \Omega_{11} &= 2\pi i \tau_1 + \varepsilon(A_2(I - A_1 A_2)^{-1})(1, 1), \\ 2\pi i \Omega_{22} &= 2\pi i \tau_2 + \varepsilon(A_1(I - A_2 A_1)^{-1})(1, 1), \\ 2\pi i \Omega_{12} &= -\varepsilon(I - A_1 A_2)^{-1}(1, 1). \end{aligned}$$

Here $(1, 1)$ refers to the $(1, 1)$ -entry of a matrix.

\mathcal{D}^ε is preserved under the action of

$$G \simeq (\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})) \rtimes \mathbb{Z}_2,$$

the direct product of the left and right torus modular groups, which are interchanged upon conjugation by an involution β as follows:

$$\begin{aligned} \gamma_1(\tau_1, \tau_2, \varepsilon) &= \left(\frac{a_1 \tau_1 + b_1}{c_1 \tau_1 + d_1}, \tau_2, \frac{\varepsilon}{c_1 \tau_1 + d_1} \right), \\ \gamma_2(\tau_1, \tau_2, \varepsilon) &= \left(\tau_1, \frac{a_2 \tau_2 + b_2}{c_2 \tau_2 + d_2}, \frac{\varepsilon}{c_2 \tau_2 + d_2} \right), \\ \beta.(\tau_1, \tau_2, \varepsilon) &= (\tau_2, \tau_1, \varepsilon), \end{aligned} \quad (148)$$

for $(\gamma_1, \gamma_2) \in \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$ with $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

There is a natural injection $G \rightarrow \mathrm{Sp}(4, \mathbb{Z})$ in which the two $\mathrm{SL}(2, \mathbb{Z})$ subgroups are mapped to

$$\Gamma_1 = \left\{ \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \quad \Gamma_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix} \right\}, \quad (149)$$

and the involution is mapped to

$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{150}$$

Thus as a subgroup of $\mathrm{Sp}(4, \mathbb{Z})$, G also has a natural action on the Siegel upper half plane \mathfrak{H}_2 as given in (138). This action is compatible with respect to the map (147) which is directly related to the observation that $A_a(k, l, \tau_a, \varepsilon)$ of (144) is a modular form of weight $k + l$ for $k + l > 2$, whereas $A_a(1, 1, \tau_a, \varepsilon) = \varepsilon E_2(\tau_a)$ is a quasimodular form. The exceptional modular transformation property of the latter term (29) leads via Theorem 11.7 to the following result:

THEOREM 11.8. *F^ε is equivariant with respect to the action of G ; i.e., there is a commutative diagram for $\gamma \in G$,*

$$\begin{array}{ccc} \mathcal{D}^\varepsilon & \xrightarrow{F^\varepsilon} & \mathfrak{H}_2 \\ \gamma \downarrow & & \downarrow \gamma \\ \mathcal{D}^\varepsilon & \xrightarrow{F^\varepsilon} & \mathfrak{H}_2 \end{array}$$

EXERCISE 11.9. Show that to $O(\varepsilon^4)$

$$\begin{aligned} 2\pi i \Omega_{11} &= 2\pi i \tau_1 + E_2(\tau_2) \varepsilon^2 + E_2(\tau_1) E_2(\tau_2)^2 \varepsilon^4, \\ 2\pi i \Omega_{22} &= 2\pi i \tau_2 + E_2(\tau_1) \varepsilon^2 + E_2(\tau_1)^2 E_2(\tau_1)^2 \varepsilon^4, \\ 2\pi i \Omega_{12} &= -\varepsilon + E_2(\tau_1) E_2(\tau_2) \varepsilon^3. \end{aligned}$$

11.3. The genus-two partition function for the Heisenberg VOA. In this section we define and compute the genus-two partition function for the Heisenberg VOA M_0 on the genus-two Riemann surface $\mathcal{S}^{(2)}$ described in the last section. The partition function is defined in terms of the genus-one 1-point functions $Z_{M_0}^{(1)}(v, \tau_a)$ on $\mathcal{S}_a = \mathbb{C}/\Lambda_{\tau_a}$ for all $v \in M$. The rationale behind this definition, which is strongly influenced by ideas in CFT, can be motivated by considering the following trivial sewing of a torus $\mathcal{S}_1 = \mathbb{C}/\Lambda_{\tau_1}$ to a Riemann sphere $\mathbb{C}\mathbb{P}^1$. Let $z_1 \in \mathcal{S}_1$ and $z_2 \in \mathbb{C}\mathbb{P}^1$ be local coordinates and define the sewing by identifying the annuli $r_a \geq |z_a| \geq |\varepsilon| r_a^{-1}$ via the sewing relation $z_1 z_2 = \varepsilon$ (adopting the same notation as above). The resulting surface is a torus described by the same modular parameter τ_1 .

Let V be a VOA with LiZ metric $\langle \cdot, \cdot \rangle$ and consider an n -point function²¹ $F_V^{(1)}((v^1, x_1), \dots, (v^n, x_n), \tau_1)$ for $x_i \in \mathcal{A}_1$, the torus annulus (141). This can

²¹Here and below we include a superscript (1) to indicate the genus of the Riemann torus.

be expressed in terms of a 1-point function ([MT1], Lemma 3.1) by

$$\begin{aligned} F_V^{(1)}((v^1, x_1), \dots, (v^n, x_n), \tau_1) &= Z_V^{(1)}(Y[v^1, x_1] \dots Y[v^n, x_n] \mathbf{1}, \tau_1) \\ &= Z_V^{(1)}(Y[v^1, x_1] \dots Y[v^{n-1}, x_{n-1}] v^n, \tau_1), \end{aligned} \quad (151)$$

for $x_{in} = x_i - x_n$ (see (132)). Denote the square bracket LiZ metric by $\langle \cdot, \cdot \rangle_{\text{sq}}$, and choose a basis $\{u\}$ of $V_{[r]}$ with dual basis $\{\bar{u}\}$ with respect to $\langle \cdot, \cdot \rangle_{\text{sq}}$. Expanding in this basis we find that for any $0 \leq k \leq n - 1$

$$Y[v^{k+1}, x_{k+1}] \dots Y[v^n, x_n] \mathbf{1} = \sum_{r \geq 0} \sum_{u \in V_{[r]}} \langle \bar{u}, Y[v^{k+1}, x_{k+1}] \dots Y[v^n, x_n] \mathbf{1} \rangle_{\text{sq}} u,$$

so that

$$\begin{aligned} F_V^{(1)}((v^1, x_1), \dots, (v^n, x_n), \tau_1) &= \sum_{r \geq 0} \sum_{u \in V_{[r]}} Z_V^{(1)}(Y[v^1, x_1] \dots Y[v^k, x_k] u, \tau_1) \\ &\quad \cdot \langle \bar{u}, Y[v^{k+1}, x_{k+1}] \dots Y[v^n, x_n] \mathbf{1} \rangle_{\text{sq}}. \end{aligned}$$

Using (151) we have

$$\begin{aligned} Z_V^{(1)}(Y[v^1, x_1] \dots Y[v^k, x_k] u, \tau_1) \\ = \text{Res}_{z_1} z_1^{-1} F_V^{(1)}((v^1, x_1), \dots, (v^k, x_k), (u, z_1), \tau_1). \end{aligned} \quad (152)$$

Let us now assume that each v^i is quasiprimary of $L[0]$ weight $\text{wt}[v^i]$ and let $y_i = \varepsilon/x_i \in \mathbb{C}\mathbb{P}^1$. Then (109), (112), (98) and (103) respectively imply

$$\begin{aligned} &\langle \bar{u}, Y[v^{k+1}, x_{k+1}] \dots Y[v^n, x_n] \mathbf{1} \rangle_{\text{sq}} \\ &= \langle \mathbf{1}, Y^\dagger[v^n, x_n] \dots Y^\dagger[v^{k+1}, x_{k+1}] \bar{u} \rangle_{\text{sq}} \\ &= \langle \mathbf{1}, \varepsilon^{L[0]} Y^\dagger[v^n, x_n] \varepsilon^{-L[0]} \dots \varepsilon^{L[0]} Y^\dagger[v^{k+1}, x_{k+1}] \varepsilon^{-L[0]} \varepsilon^{L[0]} \bar{u} \rangle_{\text{sq}} \\ &= \varepsilon^r \langle \mathbf{1}, Y[v^n, y_n] \dots Y[v^{k+1}, y_{k+1}] \bar{u} \rangle_{\text{sq}} \prod_{k+1 \leq j \leq n} \left(-\frac{\varepsilon}{x_j^2} \right)^{\text{wt}[v^j]} \\ &= \varepsilon^r \text{Res}_{z_2} z_2^{-1} Z_V^{(0)}((v^n, y_n), \dots, (v^{k+1}, y_{k+1}), (\bar{u}, z_2)) \prod_{k+1 \leq j \leq n} \left(\frac{dy_j}{dx_j} \right)^{\text{wt}[v^j]}. \end{aligned}$$

We are also making use here of the isomorphism between the round and square bracket formalisms in the identification of the genus zero correlation function. The result of these calculations is that, for any $0 \leq k \leq n - 1$,

$$\mathcal{F}_V^{(1)}(v^1, \dots, v^n; \tau_1) \equiv F_V^{(1)}((v^1, x_1), \dots, (v^n, x_n), \tau_1) \prod_{1 \leq i \leq n} dx_i^{\text{wt}[v^i]} =$$

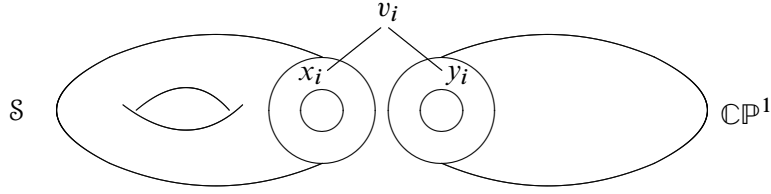


Figure 3. Equivalent insertion of v^i at x_i or $y_i = \varepsilon/x_i$.

$$= \sum_{r \geq 0} \varepsilon^r \sum_{u \in V_{[r]}} \left(\text{Res}_{z_1 z_1^{-1}} F_V^{(1)}((v^1, x_1), \dots, (v^k, x_k), (u, z_1), \tau_1) \right. \\ \times \text{Res}_{z_2 z_2^{-1}} F_V^{(0)}((v^{k+1}, y_{k+1}), \dots, (v^1, y_1), (\bar{u}, z_2)) \\ \left. \times \prod_{1 \leq i \leq k} dx_i^{\text{wt}[v^i]} \prod_{k+1 \leq j \leq n} dy_j^{\text{wt}[v^j]} \right). \quad (153)$$

Following Exercises 10.10 and 10.11 the (formal) form $\mathcal{F}_V^{(1)}(v^1, \dots, v^n; \tau_1)$ is invariant with respect to Möbius transformations. (Similarly to Remark 10.12, we note that $\mathcal{F}_V^{(1)}(v^1, \dots, v^n; \tau_1)$ is a conformally invariant global form on S_1 for primary v^1, \dots, v^n). Geometrically, (153) is telling us that we express $\mathcal{F}_V^{(1)}(v^1, \dots, v^n; \tau_1)$ via the sewing procedure in terms of data arising from $\mathcal{F}_V^{(1)}(v^1, \dots, v^k, u; \tau_1)$ and $\mathcal{F}_V^{(0)}(v^{k+1}, \dots, v^1, \bar{u})$ (cf. (108)). Furthermore, we may choose to consider the contribution from a quasiprimary vector v^i as arising from either an “insertion” at $x_i \in S_1$ or at the identified point $y_i = \varepsilon/x_i \in \mathbb{CP}^1$.

A special case of (153) is the partition (0-point) function for which we find the trivial identity

$$Z_V^{(1)}(\tau_1) = \sum_{r \geq 0} \varepsilon^r \sum_{u \in V_{[r]}} Z_V^{(1)}(u, \tau_1) \text{Res}_{z_2 z_2^{-1}} F_V^{(0)}(\bar{u}, z_2) = Z_V^{(1)}(\tau_1) + 0, \quad (154)$$

since $F_V^{(0)}(\bar{u}, z_2) = 0$ for $\bar{u} \notin V_{[0]}$.

Motivated by this example, we define the genus-two partition function where we effectively replace the Riemann sphere in Figure 3, right, by a second torus $S_2 = \mathbb{C}/\Lambda_{\tau_2}$ as described in the Section 11.2. Thus replacing the genus-zero 1-point function $F_V^{(0)}(\bar{u}, 0)$ of (154) by $Z_V^{(1)}(\bar{u}, \tau_2)$ we define the genus-two partition function for a VOA V with a LiZ metric by

$$Z_V^{(2)}(\tau_1, \tau_2, \varepsilon) = \sum_{r \geq 0} \varepsilon^r \sum_{u \in V_{[r]}} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(\bar{u}, \tau_2). \quad (155)$$

The inner sum is taken over any basis $\{u\}$ for $V_{[r]}$ with dual basis $\{\bar{u}\}$ with respect to the square bracket LiZ metric. Although the definition is associated with the specific genus-two sewing scheme, it is regarded at this stage as a purely formal

expression which can be computed to any given order in ε . One can also define genus-two correlation functions by inserting appropriate genus-one correlation functions in (155). We do not consider these here.

Let us now compute the genus-two partition function for the rank one Heisenberg VOA M_0 generated by a of weight 1. We employ the square bracket Fock basis of (124) which we alternatively notate here (cf. (118)) by

$$v = v(\lambda) = a[-1]^{e_1} \dots a[-p]^{e_p} \mathbf{1}, \tag{156}$$

for nonnegative integers e_i . We recall that $v(\lambda)$ is of square bracket weight $\text{wt}[v] = \sum_i i e_i$ and is described by a label set $\Phi_\lambda = \{1, \dots, p\}$ with $n = \sum e_i$ elements corresponding to an unrestricted partition $\lambda = \{1^{e_1}, \dots, p^{e_p}\}$ of $\text{wt}[v]$. The Fock vectors (156) form a diagonal basis for the LiZ metric $\langle \cdot, \cdot \rangle_{\text{sq}}$ with

$$\bar{v} = \frac{1}{\prod_{1 \leq i \leq p} (-i)^{e_i} e_i!} v \tag{157}$$

from (119). Following (155), we find

$$Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon) = \sum_{v \in V} \frac{\varepsilon^{\text{wt}[v]}}{\prod_i (-i)^{e_i} e_i!} Z_{M_0}^{(1)}(v, \tau_1) Z_{M_0}^{(1)}(v, \tau_2), \tag{158}$$

where the sum is taken over the basis (156). $Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon)$ is given by the following closed formula [MT4]:

THEOREM 11.10. *The genus-two partition function for the rank one Heisenberg VOA is*

$$Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon) = \frac{1}{\eta(\tau_1)\eta(\tau_2)} (\det(I - A_1 A_2))^{-1/2}, \tag{159}$$

with A_a of (144).

PROOF. The proof relies on an interesting graph-theoretic interpretation of (158). This follows the technique introduced in Theorem 11.3 for graphically interpreting the genus-one 1-point function $Z_{M_0}^{(1)}(v(\lambda), \tau_1)$ in terms the sum of weights for the ϕ -graphs. We sketch the main features of the proof leaving the interested reader to explore the details in [MT4].

Since $v(\lambda)$ is indexed by unrestricted partitions $\lambda = \{1^{e_1}, 2^{e_2}, \dots\}$ we may write (158) as

$$Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon) = \sum_{\lambda = \{i^{e_i}\}} \frac{1}{\prod_i e_i!} \prod_i \left(\frac{\varepsilon^i}{-i}\right)^{e_i} Z_{M_0}^{(1)}(v(\lambda), \tau_1) Z_{M_0}^{(1)}(v(\lambda), \tau_2). \tag{160}$$

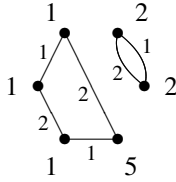


Figure 4. A chequered diagram.

Theorem 11.3 implies $Z_{M_0}^{(1)}(v(\lambda), \tau_1) = 0$ for odd $n = \sum e_i$ whereas for n even

$$Z_{M_0}^{(1)}(v(\lambda), \tau_1)Z_{M_0}^{(1)}(v(\lambda), \tau_2) = \frac{1}{\eta(\tau_1)\eta(\tau_2)} \sum_{\gamma_{\phi_1}} \sum_{\gamma_{\phi_2}} \kappa(\gamma_{\phi_1}, \tau_1)\kappa(\gamma_{\phi_2}, \tau_2),$$

where $\gamma_{\phi_1}, \gamma_{\phi_2}$ independently range over the ϕ -graphs for Φ_λ . Any pair $\gamma_{\phi_1}, \gamma_{\phi_2}$ can be naturally combined to form a *chequered diagram* D consisting of n vertices labelled by Φ_λ of valence 2 with n unoriented edges $\overset{k}{\bullet} \overset{l}{\bullet}$ consecutively labelled by $a = 1, 2$ as specified by $\phi_a = \dots (k) \dots$. Following Exercise 10.7 there are $(n!)^2$ chequered diagrams for a given $v(\lambda)$. We illustrate an example of such a diagram in Figure 4 for $v = a[-0^2 a[-2]^2 a[-5]1$ with ϕ_1 of Figure 1 and a separate choice for ϕ_2 with cycle shape $(11)(22)(15)$

For $\lambda = \{1^{e_1} \dots p^{e_p}\}$ the symmetric group $\Sigma(\Phi_\lambda)$ acts on the chequered diagrams which have Φ_λ as underlying set of labeled nodes. We define $\text{Aut}(D)$, the *automorphism group of D* , to be the subgroup of $\Sigma(\Phi_\lambda)$ which preserves node labels. $\text{Aut}(D)$ is isomorphic to $\Sigma_{e_1} \times \dots \times \Sigma_{e_p}$ of order $|\text{Aut}(D)| = \prod_i e_i!$. We may thus express (160) as a sum over the isomorphism classes of chequered diagrams D with

$$Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon) = \frac{1}{\eta(\tau_1)\eta(\tau_2)} \sum_D \frac{\zeta(D)}{|\text{Aut}(D)|},$$

and

$$\zeta(D) = \prod_i \left(\frac{\varepsilon^i}{i}\right)^{e_i} \kappa(\gamma_{\phi_1}, \tau_1)\kappa(\gamma_{\phi_2}, \tau_2), \tag{161}$$

where D is determined by $\gamma_{\phi_1}, \gamma_{\phi_2}$ and noting that $\prod_i (-1)^{e_i} = 1$ for n even. From (135) we recall that $\kappa(\gamma_{\phi_a}, \tau_1)$ is a product of the weights of the a labelled edges. Then $\zeta(D)$ can be more naturally expressed in terms of a weight function on chequered diagrams defined by

$$\zeta(D) = \Pi_E \zeta(E), \tag{162}$$

where the product is taken over the edges E of D and where for an edge E

labeled $\bullet \xrightarrow{k} a \xrightarrow{l} \bullet$ we define

$$\zeta(E) = \frac{\varepsilon^{\frac{k+l}{2}}}{\sqrt{kl}} C(k, l, \tau_a) = A_a(k, l, \tau_a, \varepsilon),$$

for A_a of (144).

Every chequered diagram can be formally represented as a product

$$D = \prod_i L_i^{m_i},$$

with D a disjoint union of unoriented chequered cycles (connected diagrams) L_i with multiplicity m_i (e.g. the chequered diagram of Figure 4 is the product of two disjoint cycles). Then $\text{Aut}(D)$ is isomorphic to the direct product of the groups $\text{Aut}(L_i^{m_i})$ of order $|\text{Aut}(L_i^{m_i})| = |\text{Aut}(L_i)|^{m_i}$ so that

$$|\text{Aut}(D)| = \prod_i |\text{Aut}(L_i)|^{m_i} m_i!.$$

But from (162) it is clear that $\zeta(D)$ is multiplicative over disjoint unions of diagrams, and we find

$$\sum_D \frac{\zeta(D)}{|\text{Aut}(D)|} = \prod_L \sum_{m \geq 0} \frac{\zeta(L)^m}{|\text{Aut}(L)|^m m!} = \exp \sum_L \frac{\zeta(L)}{|\text{Aut}(L)|},$$

where L ranges over isomorphism classes of unoriented chequered cycles. Further analysis shows that [MT4]

$$\sum_L \frac{\zeta(L)}{|\text{Aut}(L)|} = \text{Tr} \sum_{n \geq 1} \frac{1}{n} (A_1 A_2)^n = -\frac{1}{2} \text{Tr} \log(1 - A_1 A_2),$$

so that we find

$$\sum_D \frac{\zeta(D)}{|\text{Aut}(D)|} = (\det(1 - A_1 A_2))^{-1/2},$$

following (145). Thus Theorem 11.10 holds. □

The convergence and holomorphy of the determinant is the subject of Theorem 11.6 (b) so that having computed the closed formula (159) we may conclude that $Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon)$ is not just a formal function but can be evaluated on \mathcal{D}^ε to find

THEOREM 11.11. $Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon)$ is holomorphic on the domain \mathcal{D}^ε .

We next consider the automorphic properties of $Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon)$ with respect to the modular group $G \subset \text{Sp}(4, \mathbb{Z})$ of (148) which acts on \mathcal{D}^ε . We first recall a little

from the classical theory of modular forms (cf. Section 3). For a meromorphic function $f(\tau)$ on \mathfrak{H} , $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, we define the right action

$$f(\tau)|_k \gamma = f(\gamma\tau) (c\tau + d)^{-k}, \quad (163)$$

where, as usual

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

$f(\tau)$ is called a weak modular form for a subgroup $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$ of weight k if $f(\tau)|_k \gamma = f(\tau)$ for all $\gamma \in \Gamma$.

We have already discussed the (genus-one) partition function for the rank n Heisenberg VOA $V = M_0^{\otimes n}$ in Section 4.2 (cf. (41)). In particular, for $n = 2$ we have

$$Z_{M_0^2}^{(1)}(\tau) = Z_{M_0}^{(1)}(\tau)^2 = \frac{1}{\eta(\tau)^2}.$$

Then we find

$$Z_{M_0^2}^{(1)}(\tau)|_{-1}\gamma = \chi(\gamma)Z_{M_0^2}^{(1)}(\tau), \quad (164)$$

where χ is a character of $\mathrm{SL}(2, \mathbb{Z})$ of order 12 (cf. Exercise 8.5 and [Se]), and

$$Z_{M_0^{24}}^{(1)}(\tau)^{-1} = \Delta(\tau). \quad (165)$$

Similarly, we consider the genus-two partition function for the rank two Heisenberg VOA given by

$$Z_{M_0^2}^{(2)}(\tau_1, \tau_2, \varepsilon) = Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon)^2 = \frac{1}{\eta(\tau_1)^2 \eta(\tau_2)^2 \det(I - A_1 A_2)}. \quad (166)$$

Analogously to (163), we define for all $\gamma \in G$

$$f(\tau_1, \tau_2, \varepsilon)|_k \gamma = f(\gamma(\tau_1, \tau_2, \varepsilon)) \det(C\Omega + D)^{-k}, \quad (167)$$

where the action of γ on the right-hand-side is as in (148) and $\Omega(\tau_1, \tau_2, \varepsilon)$ is determined by Theorem 11.7. Then (167) defines a right action of G on functions $f(\tau_1, \tau_2, \varepsilon)$. We next obtain a natural genus-two extension of (164). Define the a character $\chi^{(2)}$ of G by

$$\chi^{(2)}(\gamma_1 \gamma_2 \beta^m) = (-1)^m \chi(\gamma_1) \chi(\gamma_2), \quad \gamma_i \in \Gamma_i, \quad i = 1, 2.$$

with Γ_i, β of (149) and (150). $\chi^{(2)}$ takes values which are twelfth roots of unity. Then, much as for Theorem 11.7, the exceptional transformation law of $A_a(1, 1, \tau_a, \varepsilon) = E_2(\tau_a)$ implies that

THEOREM 11.12. *If $\gamma \in G$ then*

$$Z_{M_0^2}^{(2)}(\tau_1, \tau_2, \varepsilon)|_{-1}\gamma = \chi^{(2)}(\gamma)Z_{M_0^2}^{(2)}(\tau_1, \tau_2, \varepsilon).$$

The definition (167) is analogous to that for a *Siegel modular form* for the symplectic group $\text{Sp}(4, \mathbb{Z})$ defined as follows (e.g. [Fr]). For a meromorphic function $F(\Omega)$ on \mathfrak{H}_2 , $k \in \mathbb{Z}$ and $\gamma \in \text{Sp}(4, \mathbb{Z})$, we define the right action

$$F(\Omega)|_k \gamma = F(\gamma.\Omega) \det(C\Omega + D)^{-k}, \tag{168}$$

with $\gamma.\Omega$ of (138). $F(\Omega)$ is called a modular form for $\Gamma \subseteq \text{Sp}(4, \mathbb{Z})$ of weight k if $F(\Omega)|_k \gamma = F(\Omega)$ for all $\gamma \in \Gamma$.

Theorem 11.12 implies that for the rank 24 Heisenberg VOA M_0^{24}

$$Z_{M_0^{24}}^{(2)}(\tau_1, \tau_2, \varepsilon)|_{-12}\gamma = Z_{M_0^{24}}^{(2)}(\tau_1, \tau_2, \varepsilon), \tag{169}$$

for all $\gamma \in G$. This might lead one to speculate that, analogously to the genus-one case in (165), $Z_{M_0^{24}}^{(2)}(\tau_1, \tau_2, \varepsilon)^{-1}$ is a holomorphic Siegel modular form of weight 12. Indeed, there does exist a unique holomorphic Siegel 12 form, $\Delta_{12}^{(2)}(\Omega)$, such that

$$\Delta_{12}^{(2)}(\Omega) \rightarrow \Delta(\tau_1)\Delta(\tau_2),$$

as $\varepsilon \rightarrow 0$, but explicit calculations show that $Z_{M_0^{24}}^{(2)}(\tau_1, \tau_2, \varepsilon)^{-1} \neq \Delta_{12}^{(2)}(\Omega)$. In any case, we cannot naturally extend the action of G on \mathcal{D}^ε to $\text{Sp}(4, \mathbb{Z})$. These observations are strongly expected to be related to the conformal anomaly [BK] in string theory and to the non-existence of a global section for the Hodge line bundle in algebraic geometry [Mu2].

Siegel modular forms do arise in the determination of the genus-two partition function for a lattice VOA V_L for even lattice L of rank l (and conjecturally for all rational theories) as follows. We recall the genus-one partition function for V_L is (cf. Section 7.3)

$$Z_{V_L}^{(1)}(\tau) = Z_{M_0^l}^{(1)}(\tau)\theta_L^{(1)}(\tau), \tag{170}$$

for $\theta_L^{(1)}(\tau) = \sum_{\alpha} q^{(\alpha, \alpha)/2}$. In the genus-two case, we may define the Siegel lattice theta function by [Fr]

$$\theta_L^{(2)}(\Omega) = \sum_{\alpha, \beta \in L} \exp(\pi i((\alpha, \alpha)\Omega_{11} + 2(\alpha, \beta)\Omega_{12} + (\beta, \beta)\Omega_{22})).$$

$\theta_L^{(2)}(\Omega)$ is a Siegel modular form of weight $l/2$ for a subgroup of $\text{Sp}(4, \mathbb{Z})$. The genus-one result (170) is naturally generalized to find [MT4]:

THEOREM 11.13. *For a lattice VOA V_L we have*

$$Z_{V_L}^{(2)}(\tau_1, \tau_2, \varepsilon) = Z_{M_0^l}^{(2)}(\tau_1, \tau_2, \varepsilon)\theta_L^{(2)}(\Omega).$$

EXERCISE 11.14. Show that $Z_{M_0}^{(2)}(\tau_1, \tau_2, \varepsilon)$ to $O(\varepsilon^4)$ is given by

$$\frac{1}{\eta(\tau_1)\eta(\tau_2)} \left(1 + \frac{1}{2} E_2(\tau_1)E_2(\tau_2)\varepsilon^2 + \left(\frac{3}{8} E_2(\tau_1)^2 E_2(\tau_2)^2 + \frac{15}{2} E_4(\tau_1)E_4(\tau_2) \right) \varepsilon^4 \right).$$

EXERCISE 11.15. Verify (159) to $O(\varepsilon^4)$ by showing that

$$\det(I - A_1 A_2) = 1 - E_2(\tau_1)E_2(\tau_2)\varepsilon^2 - 15 E_4(\tau_1)E_4(\tau_2)\varepsilon^4 + O(\varepsilon^6).$$

12. Exceptional VOAs and the Virasoro algebra

In this section we review some recent research concerning a rôle played by the Virasoro algebra in certain exceptional VOAs [T2], [T3]. We will mainly concern ourselves here with simple VOAs V of strong CFT-type for which $\dim V_1 > 0$. We construct certain quadratic Casimir vectors from the elements of V_1 and examine the constraints on V arising from the assumption that the Casimir vectors of low weight are Virasoro descendants of the vacuum. This sort of assumption is similar to that of ‘minimality’ in the holomorphic VOAs $V^{(k)}$ that we discussed in Section 9.4. In particular we discuss how a special set of simple Lie algebras: $A_1, A_2, G_2, D_4, E_4, E_6, E_7, E_8$, known as Deligne’s exceptional series [De], arises in this context. We also show that the genus-one partition function is determined by the same Virasoro condition. These constraints follow from an analysis of appropriate genus zero matrix elements and genus-one 2-point functions. In particular, we will make a relatively elementary use of rational matrix elements, the LiZ metric, the Zhu reduction formula and modular differential equations. As such, this example offers a useful and explicit application of many of the concepts reviewed in these notes.

12.1. Quadratic Casimirs and genus-zero constraints. Consider a simple VOA V of strong CFT-type of central charge c with $d = \dim V_1 > 0$. From Theorem 10.15, V possesses an LiZ metric $\langle \cdot, \cdot \rangle$, i.e., a unique (nondegenerate) normalized symmetric bilinear form. For $a, b \in V_1$ define $[a, b] \equiv a_0 b$. From Exercise 9.7 this defines a Lie algebra on V_1 with invariant bilinear form $\langle \cdot, \cdot \rangle$. We denote this Lie algebra by \mathfrak{g} . The modes of elements of V_1 satisfy the Kac–Moody algebra (cf. Exercise 12.7)

$$[a_m, b_n] = [a, b]_{m+n} - m \langle a, b \rangle \delta_{m+n,0}. \quad (171)$$

which we denote by $\hat{\mathfrak{g}}$.

Let $\{u^\alpha \mid \alpha = 1 \dots d\}$ and $\{\bar{u}^\beta \mid \beta = 1 \dots d\}$ denote a \mathfrak{g} -basis and LiZ dual basis respectively. Define the *quadratic Casimir vectors* by

$$\lambda^{(n)} = u_{1-n}^\alpha \bar{u}^\alpha \in V_n, \quad (172)$$

where α is summed. Since $u^\alpha \in V_1$ is a primary vector it follows that $[L_m, u_n^\alpha] = -nu_{m+n}^\alpha$ and hence

$$L_m \lambda^{(n)} = (n-1)\lambda^{(n-m)} \text{ for } m > 0. \tag{173}$$

Let Vir_c denote the subVOA of V generated by the Virasoro vector ω . We then find:

LEMMA 12.1. *The LiZ metric is nondegenerate on Vir_c .*

PROOF. Let $v = L_{-n_1} L_{-n_2} \dots L_{-n_k} \mathbf{1} \in \text{Vir}_c$. Then (105) gives

$$\langle v, a \rangle = \langle \mathbf{1}, L_{n_k} \dots L_{n_2} L_{n_1} a \rangle = 0,$$

for $a \in V \setminus \text{Vir}_c$. Since $\langle \cdot, \cdot \rangle$ is nondegenerate on V it must be nondegenerate on Vir_c . \square

REMARK 12.2. This implies from Theorem 10.15 that Vir_c is simple with $c \neq c_{p,q}$ of (123).

We now consider the constraints on \mathfrak{g} that follow from assuming that $\lambda^{(n)} \in \text{Vir}_c$ for small n .²² Firstly let us note [Mat]

LEMMA 12.3. *If $\lambda^{(n)} \in \text{Vir}_c$ then $\lambda^{(m)} \in \text{Vir}_c$ and is uniquely determined for all $m \leq n$.*

PROOF. If $\lambda^{(n)} \in \text{Vir}_c$ then $\lambda^{(n)} = \sum_{v \in (\text{Vir}_c)_n} \langle \bar{v}, \lambda^{(n)} \rangle v$ summing over a basis for $(\text{Vir}_c)_n$. But $\langle \bar{v}, \lambda^{(n)} \rangle$ is uniquely determined by repeated use of (173) and Exercise 12.8. Furthermore, for $m \leq n$ we have $\lambda^{(m)} = \frac{1}{n-1} L_{n-m} \lambda^{(n)} \in (\text{Vir}_c)_m$. \square

It follows that $\lambda^{(2)} \in \text{Vir}_c$ implies

$$\lambda^{(2)} = -\frac{2d}{c} \omega, \tag{174}$$

where $c \neq 0$ following Remark 12.2. Note that for \mathfrak{g} simple, this is the standard Sugawara construction for ω of (69). Similarly $\lambda^{(4)} \in \text{Vir}_c$ implies

$$\lambda^{(4)} = -\frac{3d}{c(5c+22)} (4L_{-2}^2 \mathbf{1} + (2+c)L_{-4} \mathbf{1}), \tag{175}$$

with $c \neq 0, -\frac{22}{5}$ following Remark 12.2. \square

²²The original motivation, due to Matsuo [Mat], for considering quadratic Casimirs is that both they and Vir_c are invariant under the automorphism group of V . Matsuo considered VOAs for which the automorphism invariants of V_n consist *only* of Virasoro descendents for small n . Hence for these VOAs it necessarily follows that $\lambda^{(n)} \in \text{Vir}_c$.

We next consider the constraints on \mathfrak{g} if either (174) or (175) hold. We do this by analysing the following genus zero matrix element

$$F(a, b; x, y) = \langle a, Y(u^\alpha, x)Y(\bar{u}^\alpha, y)b \rangle, \quad (176)$$

where α is summed and $a, b \in V_1$. Using associativity and (172) we find

$$\begin{aligned} F(a, b; x, y) &= \langle a, Y(Y(u^\alpha, x-y)\bar{u}^\alpha, y)b \rangle \\ &= \frac{1}{(x-y)^2} \sum_{n \geq 0} \langle a, o(\lambda^{(n)})b \rangle \left(\frac{x-y}{y}\right)^n, \end{aligned} \quad (177)$$

where $o(\lambda^{(n)}) = \lambda_{n-1}$ from (37). Thus Exercise 12.8 implies

$$F(a, b; x, y) = \frac{1}{(x-y)^2} \left(-d\langle a, b \rangle + 0 + \langle a, o(\lambda^{(2)})b \rangle \left(\frac{x-y}{y}\right)^2 + \dots \right). \quad (178)$$

Alternatively, we also have

$$\begin{aligned} F(a, b; x, y) &= \langle a, Y(u^\alpha, x)e^{yL-1}Y(b, -y)\bar{u}^\alpha \rangle \\ &= \langle a, e^{yL-1}Y(u^\alpha, x-y)Y(b, -y)\bar{u}^\alpha \rangle \\ &= \langle e^{yL}a, Y(u^\alpha, x-y)Y(b, -y)\bar{u}^\alpha \rangle \\ &= \langle a, Y(u^\alpha, x-y)Y(b, -y)\bar{u}^\alpha \rangle \\ &= \frac{1}{y^2} \sum_{m \geq 0} \langle a, u_{m-1}^\alpha b_{1-m}\bar{u}^\alpha \rangle \left(\frac{-y}{x-y}\right)^m \\ &= \frac{1}{y^2} \left(\langle a, u_{-1}^\alpha b_1\bar{u}^\alpha \rangle - \langle a, u_0^\alpha b_0\bar{u}^\alpha \rangle \frac{y}{x-y} + \dots \right), \end{aligned}$$

using skew-symmetry and translation (cf. Exercises 2.15 and 2.16), invariance of the LiZ metric and that a is primary. The leading term is

$$\langle a, u_{-1}^\alpha b_1\bar{u}^\alpha \rangle = -\langle a, u^\alpha \rangle \langle b, \bar{u}^\alpha \rangle = -\langle a, b \rangle.$$

The next to leading term is

$$-\langle a, u_0^\alpha b_0\bar{u}^\alpha \rangle = \langle u^\alpha, a_0 b_0 \bar{u}^\alpha \rangle = K(a, b),$$

the Lie algebra Killing form

$$K(a, b) = \text{Tr}_{\mathfrak{g}}(a_0 b_0). \quad (179)$$

Thus we have

$$F(a, b; x, y) = \frac{1}{y^2} \left(-\langle a, b \rangle + K(a, b) \frac{y}{x-y} + \dots \right). \quad (180)$$

From Theorem 10.1 we know that $F(a, b; x, y)$ is given by a rational function

$$F(a, b; x, y) = \frac{f(a, b; x, y)}{x^2 y^2 (x - y)^2}, \quad (181)$$

where $f(a, b; x, y)$ is a homogeneous polynomial of degree 4. Furthermore $f(a, b; x, y)$ is clearly symmetric in x, y so that it may be parameterized

$$f(a, b; x, y) = p(a, b)x^2 y^2 + q(a, b)xy(x - y)^2 + r(a, b)(x - y)^4, \quad (182)$$

for some bilinears $p(a, b)$, $q(a, b)$ and $r(a, b)$. We find:

PROPOSITION 12.4. $p(a, b)$, $q(a, b)$, $r(a, b)$ are given by

$$p(a, b) = -d\langle a, b \rangle, \quad (183)$$

$$q(a, b) = K(a, b) - 2\langle a, b \rangle, \quad (184)$$

$$r(a, b) = -\langle a, b \rangle. \quad (185)$$

PROOF. Expanding (181) in $(x - y)/y$ we have

$$F(a, b; x, y) = \frac{1}{(x - y)^2} \left[p(a, b) + q(a, b) \left(\frac{x - y}{y} \right)^2 + \dots \right], \quad (186)$$

whereas expanding (181) in $y/(x - y)$ gives

$$F(a, b; x, y) = \frac{1}{y^2} \left[r(a, b) + (-2r(a, b) + q(a, b)) \frac{y}{x - y} + \dots \right]. \quad (187)$$

Comparing to (178) and (186) gives the result. \square

We next show that if $\lambda^{(2)} \in \text{Vir}_c$ then the Killing form is proportional to the LiZ metric:

PROPOSITION 12.5. If $\lambda^{(2)} \in \text{Vir}_c$ then

$$K(a, b) = -2\langle a, b \rangle \left(\frac{d}{c} - 1 \right), \quad (188)$$

so that

$$f(a, b; x, y) = -\langle a, b \rangle \left(dx^2 y^2 + \frac{2d}{c} xy(x - y)^2 + (x - y)^4 \right). \quad (189)$$

PROOF. Equation (174) implies $o(\lambda^{(2)}) = -\frac{2d}{c} L_0$. Comparing the next to leading terms in (178) and (186) we find

$$q(a, b) = \langle a, o(\lambda^{(2)})b \rangle = -\frac{2d}{c} \langle a, b \rangle,$$

which implies the result. \square

Since the LiZ metric is nondegenerate, it follows from Cartan's criterion in Lie theory that \mathfrak{g} is solvable for $d = c$ and is semisimple for $d \neq c$, i.e.,

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^r,$$

for simple components \mathfrak{g}^i of dimension d^i . The corresponding Kac–Moody algebra $\hat{\mathfrak{g}}^i$ has level $l^i = -\frac{1}{2}\langle \alpha^i, \alpha^i \rangle$ where α^i is a long root²³ so that the dual Coxeter number is

$$h_i^\vee = l^i \left(\frac{d}{c} - 1 \right). \quad (190)$$

Furthermore, (174) implies that $\omega = \sum_{1 \leq i \leq r} \omega^i$ with ω^i the Sugawara Virasoro vector for central charge $c^i = l^i d^i / (l^i + h_i^\vee)$ for the simple component $\hat{\mathfrak{g}}^i$. It follows that for each component

$$\frac{d^i}{c^i} = \frac{d}{c}, \quad (191)$$

so that

$$\lambda^{i(2)} = -\frac{2d}{c} \omega^i, \quad (192)$$

for the quadratic Casimir on \mathfrak{g}^i .

We next show that if $\lambda^{(4)} \in \text{Vir}_c$ then \mathfrak{g} must be simple. Let L_n^i denote the modes of ω^i and $L_n = \sum_i L_n^i$ denote the modes of ω with $[L_m^i, L_n^j] = 0$ for $i \neq j$. Using $\lambda^{(4)} = \sum_i \lambda^{i(4)}$ (for quadratic Casimirs on \mathfrak{g}^i) it follows from (173) that

$$L_n^i \lambda^{i(4)} = 3\lambda^{i(2)}. \quad (193)$$

Since L_n^i satisfies the Virasoro algebra of central charge c^i we find

$$L_2^i L_{-2}^i \mathbf{1} = 6\omega^i + c^i \omega, \quad L_2^i L_{-4}^i \mathbf{1} = 6\omega^i.$$

If $\lambda^{(4)} \in \text{Vir}_c$ then (175) holds and hence

$$L_n^i \lambda^{i(4)} = -\frac{3d}{c(5c+22)} ((44+6c)\omega^i + 4c^i \omega).$$

Equating to (193) and using (192) implies that

$$\omega^i = \frac{c^i}{c} \omega.$$

But since the Virasoro vectors $\omega^1, \dots, \omega^r$ are independent it follows that $r = 1$; i.e., \mathfrak{g} is a simple Lie algebra.

If (175) holds one also finds that

$$\langle a, o(\lambda^{(4)})b \rangle = -\frac{9d(6+c)}{c(5c+22)} \langle a, b \rangle.$$

²³Then $(a, b)_i \equiv -\langle a, b \rangle / l_i$ is the unique nondegenerate form on $\hat{\mathfrak{g}}_i^{(l_i)}$ with normalization $(\alpha_i, \alpha_i)_i = 2$.

Comparing to the corresponding term in (177) this results in a further constraint on the parameters d, c in (189) given by

$$d = \frac{c(5c + 22)}{10 - c}. \tag{194}$$

Notice that the numerator vanishes for $c = 0, -22/5$, the zeros of the Kac determinant $\det M_4(c)$ (122).

For integral $d > 0$ there are only 42 rational values of c satisfying (194). This list is further restricted by the possible values of d for \mathfrak{g} simple. The level l is necessarily rational from (190). Restricting l to be integral (for example, if V is assumed to be C_2 -cofinite [DM1]) we find that $l = 1$ and \mathfrak{g} must be one of Deligne's exceptional Lie algebras:

THEOREM 12.6. *Suppose $\lambda^{(4)} \in \text{Vir}_c$.*

- (a) \mathfrak{g} is a simple Lie algebra.
- (b) If c is rational and the level l of $\hat{\mathfrak{g}}$ is integral then

$$\mathfrak{g} = A_1, A_2, G_2, D_4, F_4, E_6, E_7 \text{ or } E_8,$$

with dual Coxeter number

$$h^\vee = \frac{d}{c} - 1 = \frac{12 + 6c}{10 - c},$$

for central charge $c = 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, 8$ respectively and level $l = 1$.

The simple Lie algebras appearing in Theorem 12.6 are known as Deligne's exceptional Lie algebras [De]. These algebras are of particular interest because not only is the dimension d of the adjoint representation \mathfrak{g} described by a rational function of c in (194) but also the dimensions of the irreducible representations that arise in decomposition of up to four tensor products of \mathfrak{g} . In Deligne's original calculations, these dimensions were expressed as rational functions of a convenient parameter λ . In this VOA setting we instead employ the canonical parameter c , where

$$\lambda = \frac{c - 10}{2 + c}.$$

EXERCISE 12.7. Verify (171).

EXERCISE 12.8. Show that $\lambda^{(0)} = -d\mathbf{1}$ and $\lambda^{(1)} = 0$.

EXERCISE 12.9. Verify (174).

EXERCISE 12.10. Verify (175) using (121).

12.2. Genus-one constraints from quadratic Casimirs. We next consider the constraints on the genus-one partition function $Z_V(\tau)$ that follow if $\lambda^{(4)} \in \text{Vir}_c$. We will show that in this case, $Z_V(\tau)$ is the unique solution to a second order Modular Linear Differential Equation (MLDE) (cf. Section 8.2). As a consequence, we prove that $V = L_{\mathfrak{g}}(1, 0)$, the level 1 WZW VOA where \mathfrak{g} is an Deligne exceptional series. To prove this we apply both versions of Zhu's recursion formulas (Theorems 5.7 and 5.10). In particular, we evaluate the 1-point correlation function for a Virasoro descendent of the vacuum from where an MLDE naturally arises. This is similar in spirit to Zhu's [Z] analysis of correlation functions for the modules of C_2 -cofinite VOAs but has the advantage of being considerably less technical.

We recall the genus-one partition function

$$Z_V(\tau) = \text{Tr}_V(q^{L_0 - c/24}),$$

the 1-point correlation function for $a \in V$

$$Z_V(a, \tau) = \text{Tr}_V o(a) q^{L_0 - c/24}, \quad (195)$$

and the 2-point correlation function which can be expressed in terms of 1-point functions by

$$F_V((a, x), (b, y), \tau) = Z_V(Y[a, x]Y[b, y]\mathbf{1}, \tau) \quad (196)$$

$$= Z_V(Y[a, x - y]b, \tau), \quad (197)$$

for square bracket vertex operators $Y[a, z] = Y(q_z^{L_0} a, q_z - 1)$.

We define quadratic Casimir vectors in the square bracket VOA formalism

$$\lambda^{[n]} = u^\alpha [1 - n] \bar{u}^\alpha \in V_{[n]},$$

(for α summed) for basis $\{u^\alpha\}$ and square bracket LiZ dual basis $\{\bar{u}^\alpha\}$. Consider the genus-one analogue of (176) given by the 2-point function

$$F_V((u^\alpha, x), (\bar{u}^\alpha, y), \tau) = Z_V(Y[u^\alpha, x]Y[\bar{u}^\alpha, y]\mathbf{1}, \tau),$$

(α summed). Associativity (196) implies the genus-one analogue of (177) so that

$$F_V((u^\alpha, x), (\bar{u}^\alpha, y), \tau) = \sum_{n \geq 0} Z_V(\lambda^{[n]}, \tau) (x - y)^{n-2}. \quad (198)$$

From Zhu's first recursion formula (Theorem 5.7) we may alternatively expand $F((u^\alpha, x), (\bar{u}^\alpha, y), \tau)$ in terms of Weierstrass functions as follows:

$$\begin{aligned}
 F_V((u^\alpha, x), (\bar{u}^\alpha, y), \tau) &= \text{Tr}_V(o(u^\alpha)o(\bar{u}^\alpha)q^{L_0-c/24}) \\
 &\quad + \sum_{m \geq 1} \frac{(-1)^{m+1}}{m!} P_1^{(m)}(x-y, \tau) Z_V(u^\alpha[m]\bar{u}^\alpha, \tau) \\
 &= \text{Tr}_V(o(u^\alpha)o(\bar{u}^\alpha)q^{L_0-c/24}) - dP_2(x-y, \tau)Z_V(\tau).
 \end{aligned}$$

Recalling Theorem 5.1 we may compare the $(x-y)^2$ terms in this expression and (198) to obtain

$$Z_V(\lambda^{[4]}, \tau) = -3dE_4(\tau)Z_V(\tau). \tag{199}$$

Since $(V, Y(\cdot), \mathbf{1}, \omega)$ is isomorphic to $(V, Y[\cdot], \mathbf{1}, \tilde{\omega})$ it follows that $\lambda^{(n)} \in \text{Vir}_c$ iff $\lambda^{[n]} \in \text{Vir}_c$. Thus assuming $\lambda^{(4)} \in \text{Vir}_c$ we have

$$Z_V(\lambda^{[4]}, \tau) = \frac{-3d}{c(5c+22)} (4Z_V(L[-2]^2\mathbf{1}, \tau) + (2+c)Z_V(L[-4]\mathbf{1}, \tau)), \tag{200}$$

by (175). The Virasoro 1-point functions $Z_V(L[-2]^2\mathbf{1}, \tau)$, $Z_V(L[-4]\mathbf{1}, \tau)$ can be evaluated via Zhu's second recursion formula (Theorem 5.10). In particular taking $u = \tilde{\omega}$ and v of $L[0]$ weight k in (61) we obtain the general Virasoro recursion formula

$$\begin{aligned}
 Z_V(L[-n]v, \tau) &= \delta_{n,2} \text{Tr}_V(o(\tilde{\omega})o(v)q^{L_0-c/24}) \\
 &\quad + \sum_{0 \leq m \leq k} (-1)^m \binom{m+n-1}{m+1} E_{m+n}(\tau) Z_V(L[m]v, \tau). \tag{201}
 \end{aligned}$$

But $o(\tilde{\omega}) = L_0 - c/24$ and hence

$$\text{Tr}_V(o(\tilde{\omega})o(v)q^{L_0-c/24}) = \theta Z_V(v, \tau),$$

where $\theta = q d/dq$. It follows that

$$Z_V(L[-2]v, \tau) = D_k Z_V(v, \tau) + \sum_{2 \leq m \leq k} E_{2+m}(\tau) Z_V(L[m]v, \tau), \tag{202}$$

where $D_k = \theta + kE_2(\tau)$ is the modular derivative (30). (Zhu makes extensive use of the identities (201) and (202) in his analysis of correlation functions for C_2 -cofinite VOAs [Z]. This is the origin of MLDEs as discussed in Section 9).

We immediately find from (201) that $Z_V(L[-4]\mathbf{1}, \tau) = 0$ and

$$\begin{aligned}
 Z_V(L[-2]^2\mathbf{1}, \tau) &= D_2 Z_V(L[-2]\mathbf{1}, \tau) + E_4(\tau) Z_V(L[2]L[-2]\mathbf{1}, \tau) \\
 &= (D^2 + \frac{1}{2}cE_4(\tau)) Z_V(\tau),
 \end{aligned}$$

where $D^2 = D_2 D_0 = (q d/dq)^2 + 2E_2(\tau)q d/dq$. Substituting into (200) we find $Z_V(\tau)$ satisfies the following second order MLDE:

$$(D^2 - \frac{5}{4}c(c+4)E_4(\tau)) Z_V(\tau) = 0. \tag{203}$$

(203) has a regular singular point at $q = 0$ with indicial roots $-c/24$ and $(c + 4)/24$. Applying (194) it follows that there exists a unique solution with leading q expansion $Z_V(\tau) = q^{-c/24}(1 + O(q))$. Furthermore, since $E_4(\tau)$ is holomorphic then $Z_V(\tau)$ is also holomorphic for $0 < |q| < 1$. In summary, we find:

THEOREM 12.11. *If $\lambda^{(4)} \in \text{Vir}_c$ then $Z_V(\tau)$ is a uniquely determined holomorphic function in \mathfrak{H} .*

An immediate consequence of Theorems 12.6 and 12.11 is:

THEOREM 12.12. *$V = L_{\mathfrak{g}}(1, 0)$ the level one WZW model generated by \mathfrak{g} .*

PROOF. Clearly $L_{\mathfrak{g}}(1, 0) \subseteq V$ with $\omega, \lambda^{(2)}, \lambda^{(4)} \in L_{\mathfrak{g}}(1, 0)$. Thus $L_{\mathfrak{g}}(1, 0)$ satisfies the conditions of Theorem 12.11 for the same central charge c . Hence $Z_{L_{\mathfrak{g}}(1,0)}(\tau) = Z_V(\tau)$ and so $L_{\mathfrak{g}}(1, 0) = V$. \square

It is straightforward to substitute $Z(\tau) = q^{-c/24} \sum_n \dim V_n q^n$ into (203) and solve recursively for $\dim V_n$ as a rational function in c . In this way we recover $\dim V_1 = d$ of (194). The next two terms are

$$\dim V_2 = \frac{c(804 + 508c + 175c^2 + 25c^3)}{2(22 - c)(10 - c)},$$

$$\dim V_3 = \frac{c(33344 + 148872c + 68308c^2 + 10330c^3 + 975c^4 + 125c^5)}{6(34 - c)(22 - c)(10 - c)}.$$

These dimension formulas can be further refined as follows. Consider the Virasoro decomposition of V_2 :

$$V_2 = \mathbb{C}\omega \oplus L_{-1}\mathfrak{g} \oplus P_2, \quad (204)$$

where P_2 is the space of weight two primary vectors. Let $p_2 = \dim P_2$. Then $\dim V_2 = 1 + d + p_2$ with

$$p_2 = \frac{5(5c + 22)(c + 2)^2(c - 1)}{2(22 - c)(10 - c)}. \quad (205)$$

Comparing with Deligne's analysis of the irreducible decomposition of tensor products of \mathfrak{g} we find that

$$p_2 = \dim Y_2^*,$$

where Y_2^* denotes an irreducible representation of \mathfrak{g} in Deligne's notation [De]. This is explored further in [T3].

Similarly for V_3 we find

$$V_3 = \mathbb{C}[L_{-1}\omega] \oplus L_{-1}^2\mathfrak{g} \oplus L_{-2}\mathfrak{g} \oplus L_{-1}P_2 \oplus P_3,$$

where P_3 is the space of weight three primary vectors. Let $p_3 = \dim P_3$. Then $\dim V_3 = 1 + 2d + p_2 + p_3$ with

$$p_3 = \frac{5c(5c + 22)(c - 1)(c + 5)(5c^2 + 268)}{6(34 - c)(22 - c)(10 - c)} = \dim X_2 + \dim Y_3^*,$$

where X_2, Y_3^* denote two other irreducible representations of \mathfrak{g} in Deligne's notation of dimension

$$\dim X_2 = \frac{5c(5c + 22)(c + 6)(c - 1)}{2(10 - c)^2},$$

$$\dim Y_3^* = \frac{5c(5c + 22)(c + 2)^2(c - 8)(5c - 2)(c - 1)}{6(10 - c)^2(22 - c)(34 - c)}.$$

12.3. Higher-weight constructions. We can generalize the arguments given above to consider a VOA V with $\dim V_1 = 0$. Here we construct Casimir vectors from the weight two primary space P_2 (provided $\dim P_2 > 0$) and obtain constraints on V that follow from such Casimirs being Virasoro vacuum descendents. If $\dim P_2 = 0$ we consider primaries of weight 3 and so on. In general, let V be a VOA with primary vector space P_K of lowest weight K ; i.e., $V_n = (\text{Vir}_c)_n$ for all $n < K$, so that

$$Z_V(\tau) = q^{-c/24} \left(\sum_{n < K} \dim(\text{Vir}_c)_n q^n + O(q^K) \right). \tag{206}$$

(Recall from (42) that $\sum_{n \geq 0} \dim(\text{Vir}_c)_n q^n = \prod_{m \geq 2} (1 - q^m)^{-1} = 1 + q^2 + q^3 + 2q^3 + \dots$.) We construct Casimir vectors, as in (172), from a P_k basis $\{u^\alpha\}$ and LiZ dual basis $\{\bar{u}^\alpha\}$

$$\lambda^{(n)} = u_{2K-1-n}^\alpha \bar{u}^\alpha \in V_n.$$

We find the following natural generalization of Theorems 12.11 and 12.12:

THEOREM 12.13. *Let V be a VOA with primary vectors of lowest weight $K = 2$ or 3. If $\lambda^{(2K+2)} \in \text{Vir}_c$, then*

- (a) $Z_V(\tau)$ of (206) is a holomorphic function in \mathfrak{H} and is the unique solution to a MLDE of order $K + 1$; and
- (b) V is generated by P_K .

REMARK 12.14. We conjecture that Theorem 12.13 holds for all K .

For $K = 2$ the elements of P_2 satisfy a commutative nonassociative algebra with invariant (LiZ) form known as a Griess algebra (cf. Exercise 9.9). Theorem 12.13 implies the dimension of the Griess algebra is

$$\dim P_2 = \frac{1}{2} \frac{(5c + 22)(2c - 1)(7c + 68)}{c^2 - 55c + 748}. \tag{207}$$

This result originally appeared in [Mat] subject to stronger assumptions. Following Remark 12.2 we note that the zeros of the numerator are the zeros $c_{p,q}$ of the Kac determinant $\det M_n(c)$ for $n \leq 6$. There are 37 rational values of c for which $\dim P_2$ is a positive integer. Furthermore, we may solve $Z_V(\tau)$ iteratively for $\dim V_n$ as rational functions in c . There are 9 values of c for which $\dim V_n$ is a positive integer for $n \leq 400$ given by [T3]:

$$\begin{array}{cccccccccc} c: & -\frac{44}{5} & 8 & 16 & \frac{47}{2} & 24 & 32 & \frac{164}{5} & \frac{236}{7} & 40 \\ \dim P_2: & 1 & 155 & 2295 & 96255 & 196883 & 139503 & 90117 & 63365 & 20619 \end{array}$$

The first five cases can all be realized by explicit constructions. Of particular interest is the case $c = 24$ realized by the FLM Moonshine Module V^\natural with $Z_{V^\natural}(\tau) = j(\tau) - 744$ for which P_2 is the original Griess algebra of dimension 196883 and whose automorphism group is the Monster group (cf. Section 9.4). There are constructions for $c = 32$ and 40 with the appropriate partition function but it is not known if $\lambda^{(6)} \in \text{Vir}_c$. There are no known constructions for $c = \frac{164}{5}$ and $\frac{236}{7}$.

For $K = 3$ we find

$$\dim P_3 = \frac{(5c + 22)(2c - 1)(7c + 68)(5c + 3)(3c + 46)}{-5c^4 + 703c^3 - 32992c^2 + 517172c - 3984},$$

where the zeros of the numerator are Kac determinant zeros $c_{p,q}$ for $(p-1) \times (q-1) = n \leq 8$. Iteratively solving the appropriate MLDE for $Z_V(\tau)$ we find $\dim P_3$ and $\dim V_n$ are positive integral for only 3 rational values of c :

$$\begin{array}{cccc} c: & -\frac{114}{7} & \frac{4}{5} & 48 \\ \dim P_3: & 1 & 1 & 42987519 \end{array}$$

The first two examples can be realized by known VOAs. For $c = 48$ we find $Z_V(\tau) = J(\tau)^2 - 393767$ which, intriguingly, is the partition function of the minimal holomorphic VOA $V^{(2)}$ briefly discussed in Section 9.4.

Part IV. Appendices

13. Lie algebras and representations

An *associative algebra* is a linear space A equipped with a bilinear, associative product $A \otimes A \rightarrow A$, denoted by juxtaposition. Thus $a \otimes b \mapsto ab$ and

$$(ab)c = a(bc).$$

A *Lie algebra* is a linear space L equipped with a bilinear product (usually called bracket) $[\] : L \otimes L \rightarrow L$ such that

$$[ab] = -[ba] \quad (\text{skew-commutativity})$$

$$[a[bc]] + [b[ca]] + [c[ab]] = 0 \quad (\text{Jacobi identity})$$

An associative algebra A gives rise to a Lie algebra A^- on the *same* linear space by defining $[ab] = ab - ba$. A basic example is $\text{End}(V)$ for a linear space V , where the associative product is composition of endomorphisms. This situation can be exploited using another basic associative algebra, the *tensor algebra*

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus V \otimes V \oplus \dots$$

over V . Let $\iota : V \rightarrow T(V)$ be canonical identification of V with the degree 1 piece of $T(V)$. The *universal mapping property* (UMP) for tensor algebras says that any linear map $f : V \rightarrow A$ into an associative algebra A has a *unique* extension to a morphism of associative algebras $\alpha : T(V) \rightarrow A$:

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ & \searrow \iota & \uparrow \alpha \\ & & T(V) \end{array}$$

with $f = \alpha \circ \iota$.

A *representation* of a Lie algebra L is a linear map $\pi : L \rightarrow \text{End}(V)$ for some V such that

$$\pi([ab]) = \pi(a)\pi(b) - \pi(b)\pi(a).$$

That is, $\pi : L \rightarrow \text{End}(V)^-$ is a morphism of Lie algebras. We call V an L -module.

UMP provides an extension of π to a morphism of associative algebras $\alpha : T(L) \rightarrow \text{End}(V)$. Identifying $a \in L$ with its image in $T(L)$, we see that for $a, b \in L$

$$\begin{aligned} \alpha(a \otimes b - b \otimes a - [ab]) &= \alpha(a)\alpha(b) - \alpha(b)\alpha(a) - \alpha([ab]) \\ &= \pi(a)\pi(b) - \pi(b)\pi(a) - \pi([ab]) = 0. \end{aligned}$$

Let $J \subseteq T(L)$ be the 2-sided ideal generated by $a \otimes b - b \otimes a - [ab]$, $a, b \in L$, and set

$$\mathcal{U}(L) = T(L)/J.$$

This is the *universal enveloping algebra* of L . Thus every representation π of L extends canonically to a representation of the universal enveloping algebra:

$$\begin{array}{ccc} L & \xrightarrow{\pi} & \text{End}(V) \\ & \searrow \iota' & \uparrow \alpha \\ & & \mathcal{U}(L) \end{array}$$

where ι' is the composition $L \xrightarrow{\iota} T(L) \rightarrow \mathcal{U}(L)$.

THEOREM 13.1 (POINCARÉ–BIRKHOFF–WITT, OR PBW). *Fix an ordered basis x_1, x_2, \dots of L , with \bar{x}_i the image of x_i in $\mathcal{U}(L)$. Then*

$$\{\bar{x}_{i_1} \bar{x}_{i_2} \dots \bar{x}_{i_k} \mid i_1 \geq i_2 \geq \dots \geq i_k \geq 1\}$$

is a basis for $\mathcal{U}(L)$.

From PBW we see that ι' is *injective*. Then for a representation of $\mathcal{U}(L)$, restriction to the subspace $L = \iota(L)$ furnishes a representation of L . In this way, representations of L and $\mathcal{U}(L)$ determine each other in a canonical fashion - a statement that can be better stated using categories of modules.

The Lie algebra L has a *triangular decomposition* if it decomposes as

$$L = L^+ \oplus L^0 \oplus L^-$$

such that L^\pm, L^0 are Lie subalgebras, and the bracket satisfies

$$[L^+ L^-] \subseteq L^0, \quad [L^\pm L^0] \subseteq L^\pm.$$

Use of PBW and an appropriate choice of (ordered) basis leads to an identification

$$\mathcal{U}(L) = \mathcal{U}(L^-) \otimes \mathcal{U}(L^0) \otimes \mathcal{U}(L^+).$$

Noting that $L^0 \oplus L^+ \subseteq L$ is a Lie subalgebra, let $\pi : L^0 \oplus L^+ \rightarrow \text{End}(V)$ be a representation. The *induced module* is

$$\text{Ind}(V) = \text{Ind}_{\mathcal{U}(L^0 \oplus L^+)}^{\mathcal{U}(L)} V := \mathcal{U}(L) \otimes_{\mathcal{U}(L^0 \oplus L^+)} V = \mathcal{U}(L^-) \otimes V. \quad (208)$$

It is a $\mathcal{U}(L)$ -module, hence also an L -module upon restriction. A ubiquitous special case occurs when V is an L^0 -module, which then becomes an $L^0 \oplus L^+$ -module by letting L^+ annihilate V .

EXERCISE 13.2. Show that the following Lie algebras have natural triangular decompositions:

(a) Heisenberg algebra \hat{A} with

$$\hat{A}^+ = \bigoplus_{n>0} \mathbb{C}a \otimes t^n, \quad \hat{A}^- = \bigoplus_{n<0} \mathbb{C}a \otimes t^n, \quad \hat{A}^0 = \mathbb{C}a \otimes t^0 \oplus \mathbb{C}K.$$

(b) Virasoro algebra Vir with

$$\text{Vir}^+ = \bigoplus_{n>0} \mathbb{C}L_n, \quad \text{Vir}^- = \bigoplus_{n<0} \mathbb{C}L_n, \quad \text{Vir}^0 = \mathbb{C}L_0 \oplus \mathbb{C}K.$$

(c) Finite-dimensional simple Lie algebra (equipped with a choice of Cartan subalgebra and root system) with $L^+ = \{\text{positive root spaces}\}$, $L^- = \{\text{negative root spaces}\}$, $L^0 = \{\text{Cartan subalgebra}\}$.

14. The square bracket formalism

We prove Equations (16)–(18) of Section 27. The square bracket vertex operator (14), (15) is

$$Y[v, z] = q_z^{\text{wt}(v)} Y(v, q_z - 1).$$

Thus the square bracket modes of $Y[v, z] = \sum_{m \in \mathbb{Z}} v[m]z^{-m-1}$ are given by

$$\begin{aligned} v[m] &= \text{Res}_z Y(v, q_z - 1) z^m q_z^{\text{wt}(v)} \\ &= \text{Res}_z Y(v, q_z - 1) \frac{d}{dz} (q_z - 1) z^m q_z^{\text{wt}(v)-1}. \end{aligned}$$

We may rewrite this in terms of $w = q_z - 1 = z + O(z^2)$ by means of a (formal) chain rule [FHL], [Z] so that

$$\begin{aligned} v[m] &= \text{Res}_w Y(v, w) z(w)^m q_z^{\text{wt}(v)-1} \\ &= \text{Res}_w Y(v, w) \ln(1+w)^m (1+w)^{\text{wt}(v)-1}. \end{aligned}$$

Defining $c(\text{wt}(v), i, m)$ for $i \geq m \geq 0$ by

$$\sum_{i \geq m} c(\text{wt}(v), i, m) w^i = \frac{1}{m!} \ln(1+w)^m (1+w)^{\text{wt}(v)-1},$$

we obtain (16).

Next note that $\sum_{m \geq 0} \frac{1}{m!} \ln(1+w)^m x^m = (1+w)^x$. Hence we find

$$\sum_{i \geq 0} \sum_{m=0}^i c(\text{wt}(v), i, m) w^i x^m = (1+w)^{\text{wt}(v)-1+x},$$

from which (17) follows. Finally,

$$\begin{aligned} \sum_{m \geq 0} \frac{(k+1 - \text{wt}(v))^m}{m!} v[m] &= \sum_{m \geq 0} \sum_{i \geq m} c(\text{wt}(v), i, m) (k+1 - \text{wt}(v))^m v_i \\ &= \sum_{i \geq 0} v_i \sum_{m=0}^i c(\text{wt}(v), i, m) (k+1 - \text{wt}(v))^m, \\ &= \sum_{i \geq 0} \binom{k}{i} v_i. \end{aligned}$$

giving (18).

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