Assignment 3

1.
   - $x(t) = \cos t$. $\omega_N = 1$rad/sec. $\omega_s = \frac{2\pi}{T} = 2$rad/sec. Thus, $T = \pi$sec. $x(nT) = (-1)^n$.
   - $x(t) = \sin t$. $\omega_N = 1$rad/sec. $\omega_s = \frac{2\pi}{T} = 2$rad/sec. Thus, $T = \pi$sec. $x(nT) = 0$.

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2. $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$. Thus, $y(0) = 0.5$, $y(1) = 2$, $y(2) = 1.5$. $Y(\omega) = 0.5 + 2e^{-i\omega} + 1.5e^{-2i\omega}$. $H(\omega)X(\omega) = (1+3e^{-i\omega})(0.5+0.5e^{-i\omega}) = 0.5+2e^{-i\omega}+1.5e^{-2i\omega}$.

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3. $K(\omega) = \left(\frac{1}{2} + \frac{1}{2}e^{-i\omega}\right)^4 = \frac{1}{16}\left(1 + 4e^{-i\omega} + 6e^{-2i\omega} + 4e^{-3i\omega} + e^{-4i\omega}\right)$. Thus, $h(0) = h(4) = \frac{1}{16}$, $h(1) = h(3) = \frac{1}{4}$, and $h(2) = \frac{3}{8}$.

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4. The exponent $-n$ appears in H times X when $k + l$ equals n. Thus, it equals $\cdots + h(0)x(n) + h(1)x(n-1) + \cdots + h(n+1)x(-1) + \cdots = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$.

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5. The answer is in our textbook equations (3.3) to (3.9), pages 89-90.

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6. The answer is no. The downsampling and upsampling matrices is a counterexample. See problem 5.

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7. Since $a_\ell = A_{\ell+1,1}$, we can deduce that $A_{\ell,k} = a_{\ell-k \mod n}$.

   METHOD 1.

   Let $Y_{\ell+1,1} = y_\ell$, $X_{k+1,1} = x_k$. If $Y = AX$ then $Y_{\ell+1,1} = \sum_k A_{\ell+1,k+1}X_{k+1,1}$ or $y_\ell = \sum_k a_{\ell-k}x_k$, which means $y = a * x$. Using the rule for the DFT of a convolution we have

   $$\mathcal{F}Y[j] = \mathcal{F}AX[j] = \mathcal{F}(a * x)[j] = \mathcal{F}a[j] \cdot \mathcal{F}x[j] = \mathcal{F}a[j]\mathcal{F}X[j]$$

   Define the diagonal matrix D by $D_{i,j} = \delta_{i,j}\mathcal{F}a[j]$. Then we have

   $$\mathcal{F}AX = D\mathcal{F}X$$

   for all X, so $\mathcal{F}A = D\mathcal{F}$, or $D = \mathcal{F}A\mathcal{F}^{-1}$

   METHOD 2.

   - $$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix} \quad (1)$$

     $y_\ell = Y_{\ell+1,1} = \sum_{k=1}^{n} A_{\ell+1,k}X_{k,1}$. Since, $A_{\ell+1,k} = A_{(\ell+1-k) \mod n,1}$. Thus, $y_\ell = \sum_{k=0}^{n-1} a_{(\ell-k) \mod n}x_k$.

- $$AF = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix} \tag{2}$$

  Let $\hat{a}(\omega) = \sum_{\ell=0}^{n-1} \omega^\ell a_\ell$, where $\omega = e^{-\frac{-2\pi i}{n}}$.

  $$\frac{1}{n}\mathcal{F}^* A\mathcal{F} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^* & (\omega^2)^* & \cdots & (\omega^{n-1})^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (\omega^{n-1})^* & (\omega^{2(n-1)})^* & \cdots & (\omega^{(n-1)^2})^* \end{pmatrix} \begin{pmatrix} \sum a_\ell & \hat{a}(\omega) & \cdots & \hat{a}(\omega^{n-1}) \\ \sum a_\ell & \omega\hat{a}_(\omega) & \cdots & \omega^{n-1}\hat{a}(\omega^{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \sum a_\ell & \omega^{n-1}\hat{a}(\omega) & \cdots & \omega^{(n-1)^2}\hat{a}(\omega^{n-1}) \end{pmatrix} \tag{3}$$

  From the orthogonality of the columns of $\mathcal{F}$, we can deduce that

  $$\frac{1}{n}\mathcal{F}^* A\mathcal{F} = \begin{pmatrix} \hat{a}(0) & 0 & \cdots & 0 \\ 0 & \hat{a}_(\omega) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \hat{a}(\omega^{n-1}) \end{pmatrix} \tag{4}$$

- The entries of the diagonal matrix are the DFT of a_ℓ.
- Read pages 265-269 of our textbook, for discussion of the properties of circular shift and discrete transform of circulants.

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8. If two low pass filters **C** and **H** satisfy condition **O**, they are polynomials of even length (odd degree). The filters that result from multiplying **C** and **H** is a polynomial odd length (even degree) and therefore not double-shift orthogonal.