1. Introduction

A few years ago, the author took note of certain sums that appeared in the physics literature in connection with thermodynamic of the BTZ black hole — a three-dimensional solution discovered by M. Bañados, C. Teitelboim, and J. Zanelli [1] of the Einstein gravitational field equations

$$R_{ij} - \frac{1}{2} R g_{ij} - \Lambda g_{ij} = 0 \tag{1.1}$$

with *negative cosmological constant* Λ. Here $R_{ij} = R_{ij}(g), R = R(g)$ are the Ricci tensor and Ricci scalar curvature, respectively, of the solution metric $g = [g_{ij}]$. We describe the BTZ metric in equation (2.1) below. These sums were used to express, for example, the nondivergent part of the effective BTZ action, or corrections to classical Bekenstein–Hawking entropy [4; 13; 15] — sums that physicists evidently did not realize were related to the *Patterson–Selberg zeta function* $Z_\Gamma(s)$ of a hyperbolic cylinder. The paper [21], for example, was written to point out this relation and thus to establish a thermodynamics-zeta function connection. Another such connection appears in my Lecture 6 of this volume.

In [23; 25; 26], for example, we see that the Mann–Solodukhin quantum correction to black hole entropy [15] is expressed, in fact, in terms of a suitable "deformation" of $Z_\Gamma(s)$. It is also possible to keep track of a corresponding deformation of the black hole topology. We review the deformation of zeta, and of the BTZ topology, in Section 4 below where we use it to set up a one-loop determinant formula (or an effective action formula) in the presence of conical singularities.

In Section 3 we express the one-loop quantum field partition function, the one-loop gravity partition function , and the full gravity partition function all in terms of the zeta function $Z_\Gamma(s)$. Using the holomorphic sector of the one-loop gravity partition function and the classical elliptic modular function $j(\tau)$, one can build up (with the help of Hecke operators) modular invariant partition functions of proposed holomorphic conformal field theories with central charge $24k$, where k is a positive integer — theories first defined by G. Höhn [12] and proposed by E. Witten [28] as the holographic dual of pure $2+1$ gravity. That is, these partition functions exist even if the theories do not (although for $k=1$ existence has been established by I. Frenkel, J. Lepowsky, and A. Meurman in [8]), and in Section 5 we take a close look (in Theorem 5.16, for example) at their Fourier coefficients — the asymptotics of which provide for quantum corrections to holomorphic sector black hole entropy.

The lecture, after this introduction, consists of four sections and an appendix:

- The BTZ black hole
- Patterson–Selberg zeta function and a one-loop determinant formula
- Determinant formula in the presence of conical singularities
- Extremal partition functions of conformal field theories with central charge $24k$
- Appendix to Section 5: Computation of $Z_k(\tau)$ for $k=2,3$
- References

The author dedicates this lecture to the memory of Professor Kenneth Hoffman. His kind support and friendliness to me, as a young MIT postdoc, remains most highly appreciated these many years later.

2. The BTZ black hole

The BTZ metric that solves the vacuum Einstein equations (1.1) in three dimensions with $\Lambda < 0$ is given (in Euclidean form) by

$$ds^2_{\text{BTZ}} = \left(N_1(r)^2 + r^2 N_2(r)^2\right) d\tau^2 + N_1(r)^{-2} dr^2 + 2r^2 N_2(r)\, d\phi\, d\tau + r^2\, d\phi^2, \quad (2.1)$$

in coordinates (r, ϕ, τ) on a region of anti-deSitter space where for mass and angular momentum parameters $M > 0$, $J \geq 0$, respectively,

$$N_1(r)^2 \stackrel{\text{def.}}{=} -M - \Lambda r^2 - J^2/4r^2, \quad N_2(r) = -J/2r^2. \quad (2.2)$$

In equation (2.1), one has periodicity of the Schwarzschild variable ϕ; i.e. there is the identification $\phi \sim \phi + 2\pi n$ for $n \in \mathbb{Z}$, the ring of integers. We return to this important point shortly. Not all scalar curvatures are created equal. So in equation (1.1), our sign convention is such that $R = -6\Lambda > 0$; in particular ds^2_{BTZ} is a constant curvature solution. The metric ds^2_{BTZ} is also a black hole solution with outer and inner event radii r_+, r_- given by

$$r_+^2 = \frac{M\sigma^2}{2}\left[1 + \left(1 + \frac{J^2}{M^2\sigma^2}\right)^{1/2}\right], \qquad r_- = -\frac{\sigma J i}{2r_+}, \tag{2.3}$$

where $i^2 = -1$ and $\sigma \overset{\text{def.}}{=} 1/\sqrt{-\Lambda} > 0$. Here $r_+ > 0$, but $r_- \in i\,\mathbb{R}$ is pure imaginary (since we are working with the Euclidean form of BTZ). Note that

$$r_-^2 = \frac{M\sigma^2}{2}\left[1 - \left(1 + \frac{J^2}{M^2\sigma^2}\right)^{1/2}\right], \qquad |r_-| = \frac{\sigma J}{2r_+} = i\,r_-. \tag{2.4}$$

Of course $r_- = 0$ is equivalent to $J = 0$, which is the case of the nonspinning black hole, in which case there is a single event horizon.

Given the periodicity $\phi \sim \phi + 2\pi n$, $n \in \mathbb{Z}$, of the Schwarzschild variable ϕ, as mentioned earlier, one can describe the topology of the space-time (where ds^2_{BTZ} lives, with τ regarded as a time variable) as a quotient space

$$B_\Gamma = \Gamma\backslash\mathbb{H}^3 \tag{2.5}$$

where $\mathbb{H}^3 \overset{\text{def.}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ is hyperbolic 3-space, and where

$$\Gamma = \Gamma_{(a,b)} \overset{\text{def.}}{=} \{\gamma^n \mid n \in \mathbb{Z}\} \quad \text{for } \gamma = \begin{bmatrix} e^{a+ib} & 0 \\ 0 & e^{-(a+ib)} \end{bmatrix}, \tag{2.6}$$

with $a \overset{\text{def.}}{=} \pi r_+/\sigma > 0$, and $b \overset{\text{def.}}{=} \pi |r_-|/\sigma = \pi J/2r^+ \geq 0$; see equations (2.3). Thus $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ is the cyclic subgroup with generator $\gamma \in \mathrm{SL}(2, \mathbb{C})$. The action of Γ on $\mathbb{H}^3$ is given by $\gamma^n \cdot (x, y, z) = (x', y', z')$ for

$$\begin{aligned} x' &= e^{2an}(x \cos 2bn - y \sin 2bn), \\ y' &= e^{2an}(x \sin 2bn + y \cos 2bn), \\ z' &= e^{2an} z. \end{aligned} \tag{2.7}$$

A fundamental domain F for this action is given by

$$F = \left\{(x, y, z) \in \mathbb{H}^3 \mid 1 < \sqrt{x^2 + y^2 + z^2} < e^{2a}\right\}, \tag{2.8}$$

a proof of which is given in Appendix A3 of [27], for example. In particular $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ is a *Kleinian* subgroup; that is, F has *infinite* hyperbolic volume:

$$\operatorname{vol} F \overset{\text{def.}}{=} \int_F dx\, dy\, dz/z^3 = \infty. \tag{2.9}$$

The description (2.5) is derived by way of a suitable change of variables $(r, \phi, \tau) \to (x, y, z)$, $z > 0$, whereby (remarkably) the BTZ metric ds^2_{BTZ} in equation (2.1) is transformed, in fact, to a multiple ds^2 of the standard hyperbolic metric $(dx^2+dy^2+dz^2)/z^2$ on $\mathbb{H}^3$:

$$ds^2 = \sigma^2(dx^2+dy^2+dz^2)/z^2 \tag{2.10}$$

where $\sigma^2 = 1/(-\Lambda)$, by definition (2.3); see [7; 18], for example.

3. Patterson–Selberg zeta function and a one-loop determinant formula

Going back to the fact that Γ is Kleinian, we can assign to the black hole $B_\Gamma = \Gamma\backslash\mathbb{H}^3$ a natural zeta function (an Euler product)

$$Z_\Gamma(s) \overset{\text{def.}}{=} \prod_{0\le k_1,k_2\in\mathbb{Z}}^{\infty} \bigl(1-(e^{2bi})^{k_1}(e^{-2bi})^{k_2}e^{-(k_1+k_2+s)2a}\bigr), \tag{3.1}$$

which is the *Patterson–Selberg zeta function* attached to the hyperbolic cylinder $\Gamma\backslash\mathbb{H}^3$; see [16; 21]. $Z_\Gamma(s)$ is an entire function whose zeros are the numbers

$$N_{k_1,k_2,n} \overset{\text{def.}}{=} -(k_1+k_2) + (k_1-k_2)\frac{2bi}{2a} + \frac{2\pi n i}{2a}$$

for $k_1, k_2, n \in \mathbb{Z}$, $k_1, k_2 \ge 0$, that come from the zeros of its factors. In particular, $Z_\Gamma(s) \ne 0$ for $\operatorname{Re} s > 0$. In fact, for $\operatorname{Re} s > 0$, $Z_\Gamma(s) = e^{\log Z_\Gamma(s)}$, where

$$\begin{aligned}\log Z_\Gamma(s) &\overset{\text{def.}}{=} -\sum_{n=1}^{\infty} \frac{e^{-(s-1)2an}}{4n\bigl(\sinh^2(an)+\sin^2(bn)\bigr)}\\ &= -\sum_{n=1}^{\infty} \frac{e^{-(s-1)2an}}{2n\bigl(\cosh(2an)-\cos(2bn)\bigr)}.\end{aligned} \tag{3.2}$$

In [10], the authors study the one-loop partition function of a free quantum field ϕ propagating in a locally anti-de Sitter background. The results they obtain cover not only the BTZ case, but higher genus generalizations of it, as well as the case of nonscalar fields ϕ (say gauge and graviton excitations). In the special BTZ case with ϕ a scalar field, for example, the *one-loop determinant formula* (equation (4.9) of [10])

$$-\log\det\Delta = \sum_{n=1}^{\infty} \frac{e^{-2\pi n\sqrt{1+m^2}\,\operatorname{Im}\tau}}{2n|\sin \pi n\tau|^2} \tag{3.3}$$

is derived, where now $\tau \overset{\text{def.}}{=} \frac{1}{2\pi}(\theta+i\beta)$ denotes the modular parameter corresponding to the anti-de Sitter temperature β^{-1} and angular potential θ, where

$$K(t,r) = \frac{e^{-(m^2+1)t-r^2/4t}}{(4\pi t)^{3/2}}\,\frac{r}{\sinh r} \tag{3.4}$$

is the scalar heat kernel on $\mathbb{H}^3$, and where in (3.3) the divergent contribution proportional to vol F (see equation (2.9)) is disregarded. We indicate how formula (3.3) also follows quite quickly from our result in [3], and we point out that the right-hand side of (3.3) in fact coincides with the special value $-2\log Z_\Gamma(1+\sqrt{1+m^2})$ (where we identify $\beta/2$ with a and $\theta/2$ with b) — this observation being an example of our initial remarks regarding sums appearing in the physics literature that are expressible in terms of the zeta function $Z_\Gamma(s)$ — a point unnoticed by physicists.

We start with the reminder that for $z = x + iy \in \mathbb{C}$, $\sin z = \sin x \cosh y + i \cos x \sinh y$. In particular $|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x \cdot \cosh^2 y + (1-\sin^2 x)\sinh^2 y = \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y = \sin^2 x + \sinh^2 y \;\Rightarrow\; |\sin \pi n \tau|^2 \overset{\text{def.}}{=} \sin^2\left(\frac{\theta n}{2}\right) + \sinh^2\left(\frac{\beta n}{2}\right) \;\Rightarrow$ the right-hand side of formula (3.3) is (since $\operatorname{Im}\tau \overset{\text{def.}}{=} \frac{\beta}{2\pi}$)

$$2\sum_{n=1}^{\infty} \frac{e^{-\sqrt{1+m^2}\beta n}}{4n\left[\sinh^2\left(\frac{\beta n}{2}\right) + \sin^2\left(\frac{\theta n}{2}\right)\right]} \doteq -2\log Z_\Gamma(1+\sqrt{1+m^2}), \tag{3.5}$$

by definition (3.2).

On the other hand, we have considered in [3; 22] a *truncated heat kernel*

$$K_t^{*\Gamma}(\widetilde{p}_1, \widetilde{p}_2) \overset{\text{def.}}{=} \sum_{n\in\mathbb{Z}-\{0\}} K_t(p_1, \gamma^n \cdot p_2) \tag{3.6}$$

for $B_\Gamma = \Gamma\backslash\mathbb{H}^3$, $t > 0$, where $\widetilde{p}_j \in B_\Gamma$ denotes the Γ-orbit of $p_j \in \mathbb{H}^3$, $j = 1, 2$, $\gamma^n \cdot p_2$ is given by definition (2.7), and where

$$K_t(p_1, p_2) \overset{\text{def.}}{=} \frac{e^{-t-d(p_1,p_2)^2/4t}}{(4\pi t)^{3/2}} \frac{d(p_1, p_2)}{\sinh(d(p_1, p_2))} \tag{3.7}$$

(compare equation (3.4)) for $d(p_1, p_2)$ the hyperbolic distance between p_1 and p_2, given by

$$\cosh(d(p_1, p_2)) = 1 + \frac{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}{2z_1z_2} \tag{3.8}$$

for $p_j = (x_j, y_j, z_j)$. The expression $K_t^{*\Gamma}(\widetilde{p}_1, \widetilde{p}_2)$ gives rise to the theta function (or heat trace)

$$\theta_\Gamma(t) = \operatorname{trace} K_t^{*\Gamma} \overset{\text{def.}}{=} \iiint_F K_t^{*\Gamma}(\widetilde{p}, \widetilde{p})\, dv(p) \tag{3.9}$$

where $dv = dx\,dy\,dz/z^3$ is the hyperbolic volume element; see (2.8) and (2.9). We regard the integral

$$I(m) \overset{\text{def.}}{=} \int_0^\infty e^{-m^2 t} \operatorname{trace} K_t^{*\Gamma} \frac{dt}{t} \tag{3.10}$$

as the meaning of the expression $-\log\det \Delta$ in the left-hand side of (3.3), with the understanding that by restricting the summation to $n \neq 0$ in (3.6), we disregard the divergent term

$$\int_0^\infty e^{-m^2 t}\left(\iiint_F \frac{e^{-t}}{(4\pi t)^{3/2}} dv\right)\frac{dt}{t} = \frac{\operatorname{vol} F}{(4\pi)^{3/2}} \int_0^\infty e^{-(1+m^2)t} e^{-3/2-1} dt. \tag{3.11}$$

Namely, $n = 0$ implies

$$K_t(p, \gamma^n \cdot p) = K_t(p, p) = e^{-t}(4\pi t)^{-3/2},$$

by (3.7) and (3.8), and this expression is independent of $p \in \mathbb{H}^3$. Thus if we were to include the term for $n = 0$ in (3.6), there would be a manifest contribution $\iiint_F e^{-t}(4\pi t)^{-3/2} dv$ to (3.9), which in turn would lead to the contribution $\int_0^\infty e^{-m^2 t}\left(\iiint_F \frac{e^{-t}}{(4\pi t)^{-3/2}} dv\right) dt/t$ to (3.10). This explains the divergent term mentioned in (3.11), where one notes not only the "infrared" divergence vol F (by (2.9)), but also the "ultraviolet" divergence reflected by the *negative* $-3/2$ in the integral $J(m) \overset{\text{def.}}{=} \int_0^\infty e^{-(1+m^2)t} t^{-3/2-1} dt$. Given the formula

$$\int_0^\infty e^{-(1+m^2)t} t^{v-1} dt = \frac{\Gamma(v)}{(1+m^2)^v} \tag{3.12}$$

for *positive* v, the authors in [10] (also compare [5]) remove the ultraviolet divergence by assigning to $J(m)$ the value

$$\frac{\Gamma(-3/2)}{(1+m^2)^{-3/2}} = \frac{4\sqrt{\pi}}{3}(1+m^2)^{3/2}.$$

Thus, in summary, the divergent term being disregarded is (by (3.11)) equal to

$$\frac{\operatorname{vol} F}{(4\pi)^{3/2}} \frac{4\sqrt{\pi}}{3}(1+m^2)^{3/2} = \frac{(1+m^2)^{3/2}}{6\pi} \operatorname{vol} F,$$

and we regard the one-loop determinant formula (3.3) as the statement that

$$I(m) \overset{(3.10)}{=} \int_0^\infty e^{-m^2 t} \operatorname{trace} K_t^{*\Gamma} \frac{dt}{t} = -2\log Z_\Gamma\left(1+\sqrt{1+m^2}\right), \tag{3.13}$$

since we have noted that the right-hand side of (3.3) is the right-hand side of equation (3.5).

Now formula (3.13) is easy to prove since the theta function $\theta_\Gamma(t) = \text{trace } K_t^{*\Gamma}$ was computed in [3]; also compare [4; 15; 17]. Namely

$$\theta_\Gamma(t) = \frac{a}{\sqrt{4\pi t}} \sum_{n=1}^{\infty} \frac{e^{-(t+n^2a^2/t)}}{\sinh^2(na)+\sin^2(nb)}. \tag{3.14}$$

Also by formula 32. on page 1145 of [11]

$$\int_0^\infty t^{-3/2} e^{-A/4t} e^{-Bt}\, dt = 2\sqrt{\frac{\pi}{A}} e^{-(AB)^{1/2}} \tag{3.15}$$

for $A > 0$, $B \geq 0$. Commutation of the integration in (3.13) with the summation in (3.14) is okay:

$$\begin{aligned} I(m) &= \frac{a}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(na)+\sin^2(nb)} \int_0^\infty t^{-3/2} e^{-4n^2a^2/(4t)} e^{-(1+m^2)}\, dt \\ &\overset{(3.15)}{=} \frac{a}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(na)+\sin^2(nb)} 2\sqrt{\frac{\pi}{4n^2a^2}} e^{-[4n^2a^2(1+m^2)]^{1/2}} \\ &= 2\sum_{n=1}^{\infty} \frac{e^{-\sqrt{1+m^2}\,2an}}{4n\big(\sinh^2(na)+\sin^2(nb)\big)} \\ &= -2\log Z_\Gamma(1+\sqrt{1+m^2}), \end{aligned} \tag{3.16}$$

as desired (again by (3.2)).

Given formula (3.13), we can go a step further and obtain in terms of $Z_\Gamma(s)$ the *one-loop partition function* denoted by $Z^{\text{1-loop}}_{\text{scalar}}(\tau, \bar{\tau})$ in [10]. By definition, it equals $(\det \Delta)^{-1/2}$, which we take to mean $(e^{-I(m)})^{-1/2}$. Thus, by (3.13),

$$Z^{\text{1-loop}}_{\text{scalar}}(\tau, \bar{\tau}) = \frac{1}{Z_\Gamma(1+\sqrt{1+m^2})}. \tag{3.17}$$

Let $q \overset{\text{def.}}{=} e^{2\pi i\tau} = e^{2bi-2a}$, $\bar{q} = e^{-2bi-2a}$, $h \overset{\text{def.}}{=} (1+\sqrt{1+m^2})/2$, and note that for $0 \leq k_1, k_2 \in \mathbb{Z}$ one has

$$\begin{aligned} q^{k_1+h}(\bar{q})^{k_2+h} &= e^{(2bi-2a)(k_1+h)} e^{(-2bi-2a)(k_2+h)} \\ &= (e^{2bi})^{k_1}(e^{-2bi})^{k_2} e^{-(k_1+k_2+2h)2a}. \end{aligned}$$

Therefore by definition (3.1) we can also write

$$\frac{1}{Z_\Gamma(1+\sqrt{1+m^2})} = \prod_{0\leq k_1,k_2\in\mathbb{Z}} \frac{1}{1-q^{k_1+h}(\bar{q})^{k_2+h}} \tag{3.18}$$

where, as noted in [10], the right-hand side has the form trace $q^{L_0}\bar{q}^{\bar{L}_0}$ for Virasoro operators $L_0, \bar{L}_0$ that generate scale transformations (in the language of boundary conformal field theory).

The one-loop gravity partition function $Z^{\text{1-loop}}_{\text{gravity}}(\tau)$ is also computed in [10]. The result is

$$Z^{\text{1-loop}}_{\text{gravity}}(\tau) = \prod_{m=2}^{\infty} \frac{1}{|1-q^m|^2} = \prod_{m=2}^{\infty} \frac{1}{|1-e^{2mbi}e^{-2am}|^2}, \tag{3.19}$$

from which one obtains the full gravity partition function

$$Z_{\text{gravity}}(\tau) = |q|^{-2k} Z^{\text{1-loop}}_{\text{gravity}}(\tau). \tag{3.20}$$

for the Chern–Simon coupling constant $k = \sigma/16G$ (see (2.3)), G being the Newton constant; see [10; 14; 28]. We claim that $Z^{\text{1-loop}}_{\text{gravity}}(\tau)$ can also be expressed in terms of the zeta function $Z_\Gamma(s)$. We have a factorization

$$Z^{\text{1-loop}}_{\text{gravity}}(\tau) = Z_{\text{hol}}(\tau)\overline{Z}_{\text{hol}}(\tau) \tag{3.21}$$

for

$$Z_{\text{hol}}(\tau) \overset{\text{def.}}{=} \prod_{m=2}^{\infty} \frac{1}{1-q^m} \tag{3.22}$$

its holomorphic sector. Since $a > 0$, we have $|q^m| = e^{-2am} < 1$ for $m > 0$; hence $\log(1-q^m) = -\sum_{n=1}^{\infty} q^{mn}/n$. That is,

$$\begin{aligned}
\log Z_{\text{hol}}(\tau) &= -\sum_{m=2}^{\infty} \log(1-q^m) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=2}^{\infty} (q^n)^m \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{2n}}{(1-q^n)} \frac{1-\bar{q}^n}{1-\bar{q}^n} = \sum_{n=1}^{\infty} \frac{e^{4bni}e^{-4an}(1-e^{-2bni}e^{-2an})}{n|1-q^n|^2} \\
&= \sum_{n=1}^{\infty} \frac{e^{4bni}e^{-4an} - e^{2bni}e^{-4an}e^{-2an}}{n|1-q^n|^2}.
\end{aligned} \tag{3.23}$$

On the other hand, $\sin^2(bn) + \sinh^2(an) = \frac{1}{2}\big(\cosh(2an) - \cos(2bn)\big)$ — this identity was used in (3.2) and will be used later in (4.8). Thus

$$\begin{aligned}
\frac{|1-q^n|^2}{4|q|^n} &= \frac{1-q^n-\bar{q}^n+|q|^{2n}}{4|q|^n} = \frac{1-e^{2bni}e^{-2an}-e^{-2bni}e^{-2an}+e^{-4an}}{4e^{-2an}} \\
&= \tfrac{1}{4}(e^{2an} - 2\cos(2bn) + e^{-2an}) \\
&= \tfrac{1}{2}\big(\cosh(2an) - \cos(2bn)\big) = \sin^2(bn) + \sinh^2(an).
\end{aligned} \tag{3.24}$$

That is, $\dfrac{1}{|1-q^n|^2} = \dfrac{1}{e^{-2an}4\big(\sinh^2(an)+\sin^2(bn)\big)}$, which by (3.23) lets us write

$$
\begin{aligned}
\log Z_{\text{hol}}(\tau) &= \sum_{n=1}^{\infty} \frac{e^{4bni}e^{-2an} - e^{2bni}e^{-4an}}{4n\big(\sinh^2(an)+\sin^2(bn)\big)} \\
&= \sum_{n=1}^{\infty} \frac{e^{-(-\frac{2b}{a}i+1)2an} - e^{-(-\frac{b}{a}i+2)2an}}{4n\big(\sinh^2(an)+\sin^2(bn)\big)} \\
&= -\log Z_\Gamma\Big(2-\frac{2b}{a}i\Big) + \log Z_\Gamma\Big(3-\frac{b}{a}i\Big)
\end{aligned}
\tag{3.25}
$$

by definition (3.2). By definition (3.1), $\overline{Z_\Gamma(s)} = Z_\Gamma(\bar{s})$. By equation (3.25) we have therefore established:

THEOREM 3.26. *For* $\tau = \dfrac{b}{\pi} + \dfrac{ai}{\pi}$, $Z_{\text{hol}}(\tau) = Z_\Gamma\Big(3-\dfrac{b}{a}i\Big)\Big/Z_\Gamma\Big(2-\dfrac{2b}{a}i\Big)$. *In particular*, *by* (3.21),

$$
Z^{\text{1-loop}}_{\text{gravity}}(\tau) = \frac{Z_\Gamma\Big(3-\frac{b}{a}i\Big)Z_\Gamma\Big(3+\frac{b}{a}i\Big)}{Z_\Gamma\Big(2-\frac{2b}{a}i\Big)Z_\Gamma\Big(2+\frac{2b}{a}i\Big)}, \tag{3.27}
$$

and thus by equation (3.20), $Z_{\text{gravity}}(\tau)$ *also has the explicit expression* e^{4ak} (the right-hand side of equation (3.27)) *in terms of the Patterson–Selberg zeta function* $Z_\Gamma(s)$.

4. Determinant formula in the presence of conical singularities

We will now extend the one-loop determinant formula (3.13) to the BTZ black holes $B_{\Gamma^{(\alpha)}}$ with *conical singularities*. Here we fix $0 < \alpha \leq 1$ and for

$$
\gamma_\alpha \overset{\text{def.}}{=} \begin{bmatrix} e^{i\pi\alpha} & 0 \\ 0 & e^{-i\pi\alpha} \end{bmatrix}
$$

we define $\Gamma^{(\alpha)}$ to be the subgroup of $\mathrm{SL}(2,\mathbb{C})$ generated by γ and γ_α, for γ as in (2.6):

$$
\Gamma^{(\alpha)} \overset{\text{def.}}{=} \{\gamma^n\gamma_\alpha^m \mid n, m \in \mathbb{Z}\}. \tag{4.1}
$$

$\Gamma^{(\alpha)}$ acts on $\mathbb{H}^3$ by $(\gamma^n\gamma_\alpha^m)\cdot(x,y,z) = (x',y',z')$, where

$$
\begin{aligned}
x' &= e^{2an}\big(x\cos 2(bn+\pi\alpha m) - y\sin 2(bn+\pi\alpha m)\big), \\
y' &= e^{2an}\big(x\sin 2(bn+\pi\alpha m) + y\cos 2(bn+\pi\alpha m)\big), \\
z' &= e^{2an}z.
\end{aligned}
\tag{4.2}
$$

This action, like that defined in equation (2.7), is the restriction of the standard action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{H}^3$. We take

$$B_{\Gamma^{(\alpha)}} \overset{\text{def.}}{=} \Gamma^{(\alpha)} \backslash \mathbb{H}^3.$$

If $\alpha = 1$, then $\Gamma^{(\alpha)} = \Gamma$ and $B_{\Gamma^{(\alpha)}} = B_\Gamma$. However, in general, $B_{\Gamma^{(\alpha)}}$ is *not* a smooth manifold since the action of $\Gamma^{(\alpha)}$ is not free. For example each point $(0, 0, z)$, $z > 0$, on the positive z-axis is a fixed point of γ_α^m, by definition (4.2).

To understand the topology of $B_{\Gamma^{(\alpha)}}$ a little better, consider the action of $\mathbb{Z}$ on $\mathbb{R}^2$ given by

$$\begin{aligned} m \cdot \begin{bmatrix} x \\ y \end{bmatrix} &\overset{\text{def.}}{=} \begin{bmatrix} x\cos(2\pi m\alpha) - y\sin(2\pi m\alpha) \\ x\sin(2\pi m\alpha) + y\cos(2\pi m\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\pi m\alpha) & -\sin(2\pi m\alpha) \\ \sin(2\pi m\alpha) & \cos(2\pi m\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned} \tag{4.3}$$

for $m \in \mathbb{Z}$ and $x, y \in \mathbb{R}$.

Thus the action is a rotation, the angle of rotation being $2\pi m\alpha$. Let $(\mathbb{Z}\backslash\mathbb{R}^2)^{(\alpha)}$ denote the corresponding quotient space, and let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the unit circle. In [24] we construct a well-defined surjective homeomorphism $\psi_\alpha : B_{\Gamma^{(\alpha)}} \to (\mathbb{Z}\backslash\mathbb{R}^2)^{(\alpha)} \times S^1$. In fact, given $(x, y, z) \in \mathbb{H}^3$ define

$$\begin{aligned} r &= r(x, y, z) \overset{\text{def.}}{=} \frac{\pi}{\alpha} \log z \quad (\text{since } z > 0), \\ u &= u(x, y, z) \overset{\text{def.}}{=} \frac{x}{z} \cos\frac{rb}{\pi} + \frac{y}{z} \sin\frac{rb}{\pi}, \\ v &= v(x, y, z) \overset{\text{def.}}{=} -\frac{x}{z} \sin\frac{rb}{\pi} + \frac{y}{z} \cos\frac{rb}{\pi}. \end{aligned} \tag{4.4}$$

If $\widetilde{(x, y, z)} \in B_{\Gamma^{(\alpha)}}$ denotes the $\Gamma^{(\alpha)}$-orbit of $(x, y, z) \in \mathbb{H}^3$, and $\widetilde{(u, v)} \in (\mathbb{Z}\backslash\mathbb{R}^2)^{(\alpha)}$ denotes the $\mathbb{Z}$-orbit of $(u, v) \in \mathbb{R}^2$, then

$$\psi_\alpha(\widetilde{(x, y, z)}) \overset{\text{def.}}{=} (\widetilde{(u, v)}, \quad e^{ir} = e^{i\frac{\pi}{\alpha}\log z}). \tag{4.5}$$

Similarly, the inverse function $\psi_\alpha^{-1} : (\mathbb{Z}\backslash\mathbb{R}^2)^{(\alpha)} \times S^1 \to B_{\Gamma^{(\alpha)}}$ is explicated in [24]. If $\alpha = 1/l$ with $2 \le l \in \mathbb{Z}$, for example, then one computes that a fundamental domain for the $\mathbb{Z}$ action in (4.3) is given by a cone in $\mathbb{R}^2$ with vertex at $(0, 0)$, and with opening angle $2\pi/l = 2\pi\alpha$. Given that the black holes $B_{\Gamma^{(\alpha)}}$ have the topology $(\mathbb{Z}\backslash\mathbb{R}^2)^{(\alpha)} \times S^1$, as just indicated, we see that they have conical singularities. In particular B_Γ has the topology $\mathbb{R}^2 \times S^1$, as is well-known.

The family $\{B_{\Gamma^{(\alpha)}}\}_{0<\alpha\le 1}$ of topological spaces is a "deformation" of B_Γ in the sense that $B_{\Gamma^{(1)}} = B_\Gamma$, as we have noted. Similarly, as indicated in the introductory remarks, we have constructed a family $\{Z_{\Gamma^{(\alpha)}}\}_{0<\alpha\le 1}$ of zeta functions such that $Z_{\Gamma^{(1)}} = Z_\Gamma$ in (3.1). We review this construction as it is the key to the extension of formula (3.13). For convenience we will also write $Z(s;\alpha)$ for $Z_{\Gamma^{(\alpha)}}(s)$.

$$Z_{\Gamma^{(\alpha)}}(s) = Z(s;\alpha) \overset{\text{def.}}{=} \tag{4.6}$$
$$\prod_{0\le k_1,k_2\in\mathbb{Z}} \bigl(1-(e^{2bi/\alpha})^{k_1}(e^{-2bi/\alpha})^{k_2}e^{-(k_1+k_2+\alpha s)2a/\alpha}\bigr)$$
$$\times \prod_{0\le k_1,k_2,k_3\in\mathbb{Z}} \frac{1-e^{-2(k_3+1)2a}(e^{2bi/\alpha})^{k_1}(e^{-2bi/\alpha})^{k_2}e^{-(k_1+k_2+\alpha s)2a/\alpha}}{1-e^{-2(k_3+1/\alpha)2a}(e^{2bi/\alpha})^{k_1}(e^{-2bi/\alpha})^{k_2}e^{-(k_1+k_2+\alpha s)2a/\alpha}}.$$

For $\alpha = 1$, the product over $k_1, k_2, k_3 \ge 0$ here is 1. So clearly $Z_{\Gamma^{(1)}} = Z_\Gamma(s)$ by definition (3.1). On the other hand, it is proved in [27] that we can also write

$$Z(s;\alpha) = \prod_{0\le k_1,k_2\in\mathbb{Z}} \bigl(1-e^{(ib-a)2k_1/\alpha}e^{-4ak_2}e^{-2as}\bigr)\cdot \prod_{0\le k_1,k_2\in\mathbb{Z}} \bigl(1-e^{-(ib+a)(2/\alpha)(k_1+1)}e^{-4ak_2}e^{-2as}\bigr), \tag{4.7}$$

which shows that $Z(s;\alpha)$ is an entire function. A third expression for $Z(s;\alpha)$ is also proved in [27]. The definition (4.6) might appear to be a bit opaque, but there are physical motivations for it. Namely, the author's interest was to find a zeta function meaning of results in [15]. By deforming $Z_\Gamma(s)$, we wished to construct a statistical mechanics type function $\log Z(s;\alpha)$ such that the evaluation $(\alpha\,\partial/\partial\alpha - 1)\log Z(s;\alpha)\big|_{\alpha=1}$, for a special value of s, would capture the quantum correction of R. Mann and S. Solodukhin to BTZ black hole entropy; see [23; 25; 26]. From definition (4.6) one can obtain for $\operatorname{Re} s > 0$ the equality $Z(s;\alpha) = e^{\log Z(s;\alpha)}$, where

$$\begin{aligned}\log Z(s;\alpha) &\overset{\text{def.}}{=} -\sum_{n=1}^{\infty} \frac{\sinh(2an/\alpha)\,e^{-(s-1)2an}}{4n\sinh(2an)\bigl(\sinh^2(an/\alpha)+\sin^2(bn/\alpha)\bigr)}\\ &= -\sum_{n=1}^{\infty} \frac{\sinh(2an/\alpha)\,e^{-(s-1)2an}}{2n\sinh(2an)\bigl(\cosh(2an/\alpha)-\cos(2bn/\alpha)\bigr)}.\end{aligned} \tag{4.8}$$

Of course the formulas in (4.8) reduce to those in (3.2) in case $\alpha = 1$.

A final ingredient needed is an extension of the trace formula (3.14). Fortunately, this is available from [27] in case $\alpha = 1/l$ (again) for $1 \le l \in \mathbb{Z}$, which we therefore assume. By averaging the heat kernel $K_t^{*\Gamma}(\tilde{p}_1, \tilde{p}_2)$ in (3.6) over the finite group $l\mathbb{Z}\backslash\mathbb{Z}$ we obtain the truncated heat kernel (for $t > 0$)

$$K_t^{*\Gamma^{(\alpha)}}(\tilde{p}_1,\tilde{p}_2) \overset{\text{def.}}{=} \sum_{m=0}^{l-1} K_t^{*\Gamma}\bigl(\tilde{p}_1, \widetilde{\gamma_\alpha^m\cdot p_2}\bigr), \tag{4.9}$$

which equals

$$\frac{1}{(4\pi t)^{3/2}} \sum_{m=0}^{l-1} \sum_{n \in \mathbb{Z}-\{0\}} e^{-t-d(p_1,(\gamma^n\gamma_\alpha^m)\cdot p_2)^2/4t} \frac{d(p_1,(\gamma^n\gamma_\alpha^m)\cdot p_2)}{\sinh d(p_1,(\gamma^n\gamma_\alpha^m)\cdot p_2)}$$

for $B_{\Gamma^{(\alpha)}}$, from which we can define (compare definition (3.9)) the theta function (for $t > 0$)

$$\theta_{\Gamma^{(\alpha)}}(t) = \operatorname{trace} K_t^{*\Gamma^{(\alpha)}} \overset{\text{def.}}{=} \iiint_{F^{(\alpha)}} K_t^{*(\alpha)}(\tilde{p},\tilde{p})\, dv(p), \tag{4.10}$$

where $F^{(\alpha)} \subset \mathbb{H}^3$ is defined in terms of spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta,\ z = \rho\cos\phi$, with $\rho \geq 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \phi < \pi/2$:

$$F^{(\alpha)} \overset{\text{def.}}{=} \{(x,y,z) \in \mathbb{H}^3 \mid \substack{1<\rho<e^{2a} \\ 2\pi(1-\alpha) \leq \theta \leq 2\pi}\}. \tag{4.11}$$

Thus $F^{(\alpha)}$ is the upper hemispherical region in $\mathbb{R}^3$ between the spheres of radii 1 and e^{2a}, but with θ at least $2\pi(1-\alpha)$, called *the defect angle*. If we choose $\theta_1 = 2\pi(1-\alpha)$ and $\theta_2 = 2\pi$ in formula (4.8) of [27] we obtain, for $t > 0$:

THEOREM 4.12. *For $\alpha = 1/l$, with $1 \leq l \in \mathbb{Z}$, one has*

$$\theta_{\Gamma^{(\alpha)}}(t) = \frac{\alpha a}{2\sqrt{4\pi t}} \sum_{\substack{n \in \mathbb{Z}-\{0\} \\ m \in \mathbb{Z} \\ 0 \leq m \leq l-1}} \frac{e^{-t-a^2n^2/t}}{\sinh^2(an) + \sin^2(bn + \pi\alpha m)}.$$

This theorem generalizes the trace formula (3.14). Similarly the following theorem generalizes the one-loop determinant formula (3.13):

THEOREM 4.13. *For $\alpha = 1/l$, with $1 \leq l \in \mathbb{Z}$, one has*

$$\int_0^\infty e^{-m^2 t} \operatorname{trace} K_t^{*\Gamma^{(\alpha)}} \frac{dt}{t} = -2 \log Z_{\Gamma^{(\alpha)}}(1 + \sqrt{1+m^2}). \tag{4.14}$$

PROOF. We follow the argument above in the proof of (3.13), given Theorem 4.12.

$$\int_0^\infty e^{-m^2 t} \operatorname{trace} K_t^{*\Gamma^{(\alpha)}} \frac{dt}{t} =$$

$$\frac{\alpha a}{2\sqrt{4\pi}} \sum_{\substack{n \neq 0 \\ 0 \leq m \leq l-1}} \frac{1}{\sinh^2(an) + \sin^2(bn+\pi\alpha m)} \int_0^\infty t^{-3/2} e^{-4n^2a^2/4t} e^{-(1+m^2)t}\, dt$$

$$\overset{(3.15)}{=} \frac{\alpha a}{2\sqrt{4\pi}} \sum_{\substack{n \neq 0 \\ 0 \leq m \leq l-1}} \frac{1}{\sinh^2(an) + \sin^2(bn+\pi\alpha m)} 2\sqrt{\frac{\pi}{4n^2a^2}} e^{-\sqrt{4n^2a^2(1+m^2)}}$$

$$= \frac{\alpha}{2} \sum_{\substack{n\neq 0\\ 0\leq m\leq l-1}} \frac{e^{-2|n|a\sqrt{1+m^2}}}{|n|\, 2\left(\sinh^2(an)+\sin^2(bn+\pi\alpha m)\right)}$$

$$= \frac{\alpha}{2} \sum_{\substack{n\neq 0\\ 0\leq m\leq l-1}} \frac{e^{-2|n|a\sqrt{1+m^2}}}{|n|\left(\cosh(2an)-\cos 2(bn+\pi\alpha m)\right)}. \tag{4.15}$$

To proceed further we employ the identity[1]

$$\sum_{m=0}^{l-1} \frac{1}{\cosh u - \cos(v - 2\pi m/l)} = \frac{l \sinh(lu)}{\sinh u\left(\cosh(lu)-\cos(lv)\right)}. \tag{4.16}$$

We apply it with $u \overset{\text{def.}}{=} 2an$ and $v \overset{\text{def.}}{=} -2bn$ to rewrite the last sum in (4.15) as

$$\frac{\alpha}{2} \sum_{n\in\mathbb{Z}-\{0\}} \frac{e^{-2|n|a\sqrt{1+m^2}}\, l \sinh(l2an)}{|n| \sinh(2an)\left(\cosh(l2an)-\cos(l2bn)\right)}$$

$$= 2\sum_{n=1}^{\infty} \frac{\sinh(2an/\alpha)\, e^{-2an\sqrt{1+m^2}}}{2n \sinh(2an)\left(\cosh(2an/\alpha)-\cos(2bn/\alpha)\right)} \tag{4.17}$$

$$= -2 \log Z_{\Gamma^{(\alpha)}}(1+\sqrt{1+m^2}),$$

by definition (4.8), which concludes the proof of Theorem 4.13. □

In the effective action formula (4.14) we have assumed that $\alpha^{-1} \in \mathbb{Z}$. This assumption can be removed and thus a more general formula can be presented if we appeal to an old contour integral formula that goes back to A. Sommerfeld in 1897, in his amazing diffraction studies. The reader can consult the references [13; 15], for example, on this point — references which of course do not employ the zeta function $Z_{\Gamma^{(\alpha)}}(s)$.

5. Extremal partition functions of conformal field theories with central charge $24k$

In this section we consider the modular invariant partition function $Z_k(\tau)$ of a holomorphic conformal field theory (CFT) with central charge $c = 24k$, $k = 1, 2, 3, 4, \ldots$. Such a theory was introduced by G. Höhn [12], and is called an *extremal* CFT (ECFT) — which according to a bold proposal of E. Witten [28] is the dual to 3-dimensional pure gravity with a negative cosmological constant;

[1] Formula (4.16) corrects a misprint in [27]. Namely, the expression $\sin(lu)$ in formula (4.10) of [27] should read $\sinh(lu)$. Also in equations (4.5) and (4.6) of [27] the often occurring expression $\gamma_m^{(\alpha)}$ should read γ_α^m.

also compare [14]. Apart from the case $k = 1$, however, there is uncertainty regarding the existence of ECFT's. I. Frenkel, J. Lepowsky, and A. Meurman (FLM) [8] have indeed constructed a holomorphic CFT with central charge $c = 24$ (i.e., with $k = 1$) and with $Z_1(\tau) = j(\tau) - 744$, where $j(\tau)$ is the classical elliptic modular invariant (see (5.6) below).

An important point regarding the FLM construction is *monster symmetry*: the states of the theory transform as a representation of the finite, simple, Fischer–Griess group M, of order $|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 10^{54}$, called the *monster* (or the *friendly giant*). However for $k = 2$, D. Gaiotto [9] has slain the "two-headed monster": there exists no self-dual ECFT for $c = 48$ with monster symmetry.

We begin by indicating how $Z_k(\tau)$ (defined for $\operatorname{Im} \tau > 0$) can be explicitly constructed from the FLM $Z_1(\tau)$ and the one-loop partition function $Z_{\text{hol}}(\tau)$ of definition (3.22), with help of Hecke operators.

Fix $k = 1, 2, 3, 4, \ldots$, and for $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$ set $q = q(\tau) \overset{\text{def.}}{=} e^{2\pi i \tau}$, so $|q| < 1$. Define

$$Z_0(\tau) \overset{\text{def.}}{=} q^{-k} Z_{\text{hol}}(\tau) \overset{\text{def.}}{=} q^{-k} \prod_{n=2}^{\infty} \frac{1}{1 - q^n}; \tag{5.1}$$

compare definition (3.22). The full gravity partition function of definition (3.20) therefore admits the factorization

$$Z_{\text{gravity}}(\tau) = Z_0(\tau) \overline{Z}_0(\tau), \tag{5.2}$$

by equation (3.21).

Let p denote the partition function on $\mathbb{Z}^+$; that is, $p(n)$ is the number of ways of writing a positive integer n as a sum of positive integers, without regard to order. Euler's formula (equation (9.10) of my introductory lectures, page 72) says that

$$\frac{1}{\prod_{n=1}^{\infty} (1 - z^n)} = \sum_{n=0}^{\infty} p(n) z^n \tag{5.3}$$

for $|z| < 1$, where $p(0) \overset{\text{def.}}{=} 1$. Therefore we can write

$$\begin{aligned} Z_0(\tau) &= q^{-k}(1-q) \prod_{n=1}^{\infty} \frac{1}{1-q^n} = q^{-k}(1-q) \sum_{n=0}^{\infty} p(n) q^n \\ &= \sum_{n=0}^{\infty} p(n) q^{n-k} - \sum_{n=0}^{\infty} p(n) q^{n+1-k}. \end{aligned}$$

Collecting coefficients here we see that

$$Z_0(\tau) = \sum_{r=-k}^{\infty} a_r(k) q^r \tag{5.4}$$

for

$$a_r(k) \overset{\text{def.}}{=} p(r+k) - p(r+k-1), \quad r \geq -k, \tag{5.5}$$

where we set $p(-1) \overset{\text{def.}}{=} 0$.

As mentioned in my introductory lectures (equations (4.44) and (4.45) on page 42), the modular j-invariant has a q-expansion with all Fourier coefficients $c_n \in \mathbb{Z}$. That is, defining

$$j(\tau) \overset{\text{def.}}{=} \frac{1728(60\,G_4(\tau))^3}{(60\,G_4(\tau))^3 - 27(140\,G_6(\tau))^2}, \tag{5.6}$$

we have

$$\begin{aligned} j(\tau) &= \frac{1}{q} + \sum_{n=0}^{\infty} c_n q^n \quad \text{with} \\ c_0 &= 744, \\ c_1 &= 196{,}884, \\ c_2 &= 21{,}493{,}760, \\ c_3 &= 864{,}299{,}970, \\ c_4 &= 20{,}245{,}856{,}256, \\ c_5 &= 333{,}202{,}640{,}600, \\ c_6 &= 4{,}252{,}023{,}300{,}096. \end{aligned} \tag{5.7}$$

The denominator in (5.6) is the Dedekind–Klein discriminant form $\Delta(\tau)$, and $G_l(z)$ is the holomorphic Eisenstein series of weight l given in definition (4.4) of the introductory lectures (page 31). *Here we depart from convention in using $J(\tau)$ not in the classical sense but to denote the function $j(\tau) - c_0$*:

$$J(\tau) \overset{\text{def.}}{=} j(\tau) - 744 = \frac{1}{q} + 196{,}884\,q + 21{,}493{,}760\,q^2 + 864{,}299{,}970\,q^3 + 20{,}245{,}856{,}256\,q^4 + \cdots. \tag{5.8}$$

Now recall the n-th Hecke operator $T(n)$ of weight w acting on a function $f(\tau)$, $\operatorname{Im}\tau > 0$, where $n, w \in \mathbb{Z}$, $n \geq 1$, $w \geq 0$. As seen in (3.22) of the introductory lectures (page 28), it is given by

$$(T(n)f)(\tau) \overset{\text{def.}}{=} n^{w-1} \sum_{\substack{d>0 \\ d|n}} \sum_{a=0}^{d-1} d^{-w} f\left(\frac{n\tau + da}{d^2}\right). \tag{5.9}$$

In particular

$$n(T(n)f)(\tau) \overset{\text{def.}}{=} \sum_{\substack{d>0 \\ d|n}} \sum_{a=0}^{d-1} f\left(\frac{n\tau + da}{d^2}\right) \tag{5.10}$$

for $w = 0$, which is the only case we will need, since $J(\tau)$ in (5.8) has weight zero. Of course $T(1)f = f$ for any weight w. We can now define the main object, where we have fixed an integer $k > 0$:

$$\begin{aligned} Z_k(\tau) &\overset{\text{def.}}{=} a_0(k) + \sum_{r=1}^{k} a_{-r}(k) r (T(r)J)(\tau) \\ &= p(k) - p(k-1) + \sum_{r=1}^{k} \big(p(k-r) - p(k-r-1)\big) r (T(r)J)(\tau) \end{aligned} \tag{5.11}$$

by definition (5.5), where $r(T(r)J)(\tau)$ is given by equation (5.10) applied to $f = J$. Since $p(0) = p(1) = 1$, $p(-1) = 0$, and $T(1)f = f$ we see that $Z_1(\tau) \equiv J(\tau)$, which (as remarked on earlier) is the partition function of the FLM holomorphic CFT of central charge $c = 24$, with monster symmetry. Frenkel, Lepowsky, and Meurman also conjecture that this ECFT is unique — a result that remains unproved at the present time.

To be a bit more precise, these authors construct a graded, infinite-dimensional M-module $V^\natural = V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus \cdots$ (the *moonshine module*), where V_0 is the trivial representation π of M, $V_1 = \{0\}$, $V_2 = \pi_1 \oplus \pi_{196,833}$, $V_3 = \pi_1 \oplus \pi_{196,883} \oplus \pi_{21,296,876}$, and so on; π_d is the irreducible representation of M of degree d, for $d \geq 1$. A remarkable observation, first made by John McKay in 1978, is that the early Fourier coefficients c_n in (5.7) are integral linear combinations of the degrees d; thus $c_1 = 196{,}884 = 1 + 196,883$, $c_2 = 21{,}493{,}760 = 1 + 196{,}883 + 21{,}296{,}876$, and

$$c_3 = 864{,}299{,}970 = 2 \times 1 + 2 \times (196{,}883) + 21{,}296{,}876 + 842{,}609{,}326.$$

$V^\natural$ has the structure, in fact, of a *vertex operator algebra* (VOA), a subject thoroughly discussed by G. Mason and M. Tuite in their lectures in this book. The submodule V_2 is actually an algebra (which is commutative but not associative), the *Griess algebra*, which has the monster M as its full symmetry group (i.e., as its automorphism group).

By equations (5.7), (5.8) we have the Fourier expansion $J(\tau) = \sum_{n \geq -1} c_n q^n$, where $c_{-1} = 1, c_0 = 0$. Accordingly, $(T(n)J)(\tau)$ has Fourier expansion

$$(T(n)J)(\tau) = \sum_{m \geq -n} c_m^{(n)} q^m,$$

where

$$c_m^{(n)} = \sum_{\substack{d>0 \\ d|n,\, d|m}} \frac{c_{mn/d^2}}{d}, \;\; m \geq 1; \quad c_0^{(n)} = c_0 \sum_{\substack{d>0 \\ d|n}} \frac{1}{d} \doteq 0 = c_{-m}^{(n)}, \;\; 1 \leq m < n; \quad c_{-n}^{(n)} = \frac{1}{n}. \tag{5.12}$$

That is,

$$(T(n)J)(\tau) = \frac{q^{-n}}{n} + \sum_{m=1}^{\infty} c_m^{(n)} q^m \tag{5.13}$$

which we use in definition (5.11):

$$Z_k(\tau) = a_0(k) + \sum_{r=1}^{k} a_{-r}(k)\Big(q^{-r} + r\sum_{n=1}^{\infty} c_n^{(r)} q^n\Big).$$

Collecting coefficients here, we see that

$$Z_k(\tau) = a_{-k}(k)q^{-k} + \cdots + a_{-2}(k)q^{-2} + a_{-1}(k)q^{-1} + a_0(k) + \sum_{n=1}^{\infty} b_{k,n} q^n, \tag{5.14}$$

where

$$a_{-k}(k) = 1, \qquad b_{k,n} \overset{\text{def.}}{=} \sum_{r=1}^{k} r a_{-r}(k) c_n^{(r)}, \quad n \geq 1, \tag{5.15}$$

with $a_{-r}(k) = p(k-r) - p(k-r-1)$ for $1 \leq r \leq k$ given by (5.5), and $c_n^{(r)} = \sum_d c_{rn/d^2}/d$ given by (5.12).

Before commenting on the important physical significance of the coefficients $b_{k,n}$ in (5.15), we state the following result:

THEOREM 5.16. *For $k, n \in \mathbb{Z}$, $k, n \geq 1$, let*

$$b_{k,n}^{\infty} \overset{\text{def.}}{=} k e^{4\pi\sqrt{kn}}/\sqrt{2}(kn)^{3/4}. \tag{5.17}$$

Then $b_{k,n}$ equals

$$b_{k,n}^{\infty}\Bigg(1 - \frac{3}{32\pi\sqrt{kn}} + \varepsilon_{kn} + T(k,n) + \frac{1}{k^{1/4}}\sum_{r=1}^{k-1}\frac{r^{1/4}a_{-r}(k)}{e^{4\pi\sqrt{n}(\sqrt{k}-\sqrt{r})}}\Big(1 - \frac{3}{32\sqrt{rn}} + \varepsilon_{rn} + T(r,n)\Big)\Bigg), \tag{5.18}$$

where $|\varepsilon_m| \leq .055/m$ for integer $m \geq 1$, and $0 \leq T(r,n)$ is bounded above by both $r^{3/2}\zeta(\frac{3}{2})/(2e^{2\pi\sqrt{rn}})$ and $n^{3/2}\zeta(\frac{3}{2})/(2e^{2\pi\sqrt{rn}})$ for $1 \leq r \leq k$, where $\zeta(s)$ is the Riemann zeta function.

In setting up the proof of Theorem 5.16, the author relied heavily on the following result of N. Brisebarre and G. Philibert [2] (as mentioned in equation (9.32) on page 76 of my introductory lectures): For $m \geq 1$

$$c_m = \frac{e^{4\pi\sqrt{m}}}{\sqrt{2}m^{3/4}}\left(1 - \frac{3}{32\pi\sqrt{m}} + \varepsilon_m\right), \tag{5.19}$$

where again $|\varepsilon_m| \leq .055/m$. Equation (5.19) immediately implies the weaker asymptotic result (see equation (9.31) on page 76)

$$c_m \sim e^{4\pi\sqrt{m}}/\sqrt{2}m^{3/4} \text{ as } m \to \infty, \tag{5.20}$$

due to H. Petersson in 1932 and H. Rademacher in 1938, who was unaware of Petersson's proof. Similarly, Theorem 5.16 immediately implies the weaker asymptotic result

$$b_{k,n} \sim b_{k,n}^{\infty} \overset{\text{def.}}{=} k e^{4\pi\sqrt{kn}}/\sqrt{2}(kn)^{3/4} \text{ as } n \to \infty \tag{5.21}$$

for every fixed k, as observed by E. Witten in Section 3 of [28]. Actually Witten assumes that k is large with n/k fixed, but we see that this assumption is unnecessary for the statement (5.21).

Now from $\log b_{k,n}$, say for n sufficiently large, one obtains both the classical, holomorphic sector Bekenstein–Hawking black hole entropy $S_{\text{hol}} = 4\pi\sqrt{kn}$ (the leading asymptotic term) and corrections (subleading asymptotic terms) to that entropy:

$$\log b_{k,n}^{\infty} = 4\pi\sqrt{kn} + \left(\tfrac{1}{4}\log k - \tfrac{3}{4}\log n - \tfrac{1}{2}\log 2\right), \tag{5.22}$$

by (5.21).

We offer further explanation regarding equation (5.22). In particular we explain why the leading term $S_{\text{hol}} = 4\pi\sqrt{kn}$ there was referred to as the holomorphic sector entropy. In formulas (2.3), (2.4), the outer and inner black hole radii for the BTZ metric in *Euclidean* form (2.1) are given by

$$r_{\pm}^2 = \frac{M\sigma^2}{2}\left(1 \pm \sqrt{1 + \left(\frac{J}{M\sigma}\right)^2}\right).$$

For convenience, we also consider the *Lorentzian form* of the metric

$$ds_L^2 = \left(-N_1(r)^2 + r^2 N_2(r)^2\right) dt^2 + N_1(r)^{-2} dr^2 + 2r^2 N_2(r)\, d\phi\, dt + r^2\, d\phi^2, \tag{5.23}$$

where now

$$N_1(r) = \left(-8GM + \frac{r^2}{\sigma^2} + \frac{16G^2}{r^2} J^2\right)^{1/2}, \tag{5.24}$$

with the gravitational constant G also included for generality. We omit the definition of $N_2(r)$, which will not be needed. The corresponding radii, which we again denote by $r_{\pm}$, are (by definition) solutions of the quartic equation $N_1(r) = 0$:

$$r_{\pm}^2 = 4GM\sigma^2\left(1 \pm \sqrt{1 - \left(\frac{J}{M\sigma}\right)^2}\right). \tag{5.25}$$

Here we assume (so ds_L^2 can have a black hole structure) that $1-(J/\sigma M)^2 \geq 0$, which is to say $|J| \leq \sigma M$; then $r_\pm \geq 0$. The key point here is that the classical Bekenstein–Hawking entropy, given by

$$S_{\mathrm{BH}} = \frac{\pi r^+}{2G}, \tag{5.26}$$

is also given by the formula of J. Cardy [6] (see equation (9.3) on page 70)

$$S_{\mathrm{BH}} = 2\pi \sqrt{\frac{cL_0}{6}} + 2\pi \sqrt{\frac{c\bar{L}_0}{6}}, \tag{5.27}$$

where L_0 and $\bar{L}_0$ are eigenvalues of the holomorphic and antiholomorphic Virasoro generators, respectively, and where the central charge c equals $3\sigma/2G$. To check equation (5.27) we use the equalities

$$L_0 = (\sigma M + J)/2, \qquad \bar{L}_0 = (\sigma M - J)/2. \tag{5.28}$$

By definition (5.25) we can write $r_+^2 + r_-^2 = 8GM\sigma^2$ and $r_+ r_- = 4G\sigma J$; therefore $(r_+ \pm r_-)^2 = 8GM\sigma^2 \pm 8G\sigma J = 16G\sigma(\sigma M \pm J)/2$, which yields

$$L_0 = (r_+ + r_-)^2/16G\sigma, \quad \bar{L}_0 = (r_+ - r_-)^2/16G\sigma. \tag{5.29}$$

By (5.26), the right-hand side of equation (5.27) is then

$$2\pi \sqrt{\frac{c}{6}} \left(\frac{r_+ + r_- + r_+ - r_-}{4\sqrt{G\sigma}} \right) = \pi r^+/2G = S_{\mathrm{BH}}$$

for $c = 3\sigma/2G$, which verifies equation (5.27).

Recall the Chern–Simon coupling constant $k = \sigma/16G$ following equation (3.20). Since $c = 3\sigma/2G$, we see that $c = 24k$. Thus our ongoing assumption $c \in 24\mathbb{Z}^+$ amounts now to the "quantization" of k; that is, $k = \sigma/16G$ is a positive integer. If we call the first term $2\pi\sqrt{cL_0/6}$ in equation (5.27) the *holomorphic sector entropy* S_{hol} (for obvious reasons), then for $c = 24k$ we have $S_{\mathrm{hol}} = 2\pi\sqrt{24kL_0/6} = 4\pi\sqrt{kL_0}$, which is the leading asymptotic term in equation (5.22), where n there is identified with the Virasoro eigenvalue L_0. This is a justification for referring to that leading term as holomorphic sector entropy.

Appendix to Section 5: Computation of $Z_k(\tau)$ for $k = 2, 3$

The explicit formulas (5.14) and (5.15) are sufficient for the direct computation of the initial terms of $Z_k(\tau)$, say for small values of k. One could employ a computer program to deal with larger values of k. For example, take $k = 2$. Then, by (5.5) and (5.15),

$$\begin{aligned} a_{-1}(2) &= p(2-1) - p(2-2) = 1 - 1 = 0, \\ a_{-2}(2) &= 1, \\ a_0(2) &= p(2) - p(1) = 1. \end{aligned}$$

Also (5.15) gives, for $n \geq 1$,

$$b_{2,n} = \sum_{r=1}^{2} r a_{-r}(2) \sum_{\substack{d>0 \\ d|r,\, d|n}} \frac{1}{d} c_{rn/d^2} \stackrel{\therefore}{=} 2 \sum_{\substack{d>0 \\ d|2,\, d|n}} \frac{1}{d} c_{2n/d^2} = \begin{cases} 2c_{2n} & \text{if } 2 \nmid n, \\ 2c_{2n} + c_{n/2} & \text{if } 2 \mid n, \end{cases}$$

leading to

$$\begin{aligned} b_{2,1} &= 2c_2 = 42{,}987{,}520, \\ b_{2,2} &= 2c_4 + c_1 = 40{,}491{,}909{,}396, \\ b_{2,3} &= 2c_6 = 8{,}540{,}046{,}600{,}192, \end{aligned}$$

by (5.7). Therefore by (5.14)

$$Z_2(\tau) = \\ q^{-2} + 1 + 42{,}987{,}520q + 40{,}491{,}909{,}396q^2 + 8{,}504{,}046{,}600{,}192q^3 + \cdots.$$

Of course,

$$Z_1(\tau) \stackrel{\text{def.}}{=} J(\tau) \stackrel{\text{def.}}{=} j(\tau) - 744 = \\ q^{-1} + 196{,}884q + 21{,}493{,}760q^2 + 864{,}299{,}970q^3 + 20{,}245{,}856{,}256q^4 + \cdots,$$

by equation (5.8).

Similarly for $k = 3$ we have $a_0(3) = 1,\ a_{-1}(3) = 1,\ a_{-2}(3) = 0,\ a_{-3}(3) = 1$, so that

$$b_{3,n} = \sum_{r=1}^{3} r a_{-r}(3) \sum_{\substack{d>0 \\ d|r,\, d|n}} \frac{1}{d} c_{rn/d^2} = c_n + 3 \sum_{\substack{d>0 \\ d|3,\, d|n}} \frac{1}{d} c_{3n/d^2},$$

which leads to

$$\begin{aligned} b_{3,1} &= c_1 + 3c_3 = 2{,}593{,}096{,}794, \\ b_{3,2} &= c_2 + 3c_6 = 12{,}756{,}091{,}394{,}048, \end{aligned}$$

and hence

$$Z_3(\tau) = q^{-3} + q^{-1} + 1 + 2{,}593{,}096{,}794q + 12{,}756{,}091{,}394{,}048q^2 + \cdots.$$

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Added in proof

The author has discovered that a (slightly incorrect) version of formula (3.27), page 337, has been obtained, independently, by A. Bytsenko and M. Guimarães, (see formula (4.13) in their *Truncated heat kernel and one-loop determinants for the BTZ geometry*, Eur. Phys. J. C **58** (2008), pp. 511–516).

The following reference provides for further connections of the Patterson–Selberg zeta function to BTZ physics: D. Diaz, *Holographic formula for the determinant of the scattering operator in thermal AdS*, preprint, arXiv:0812.2158v3 (2009).