1. Fields

The simple dynamical systems we have studied earlier have a finite number of degrees of freedom. A point particle for example has 3 degrees of freedom, x, y, z. A set of N point particles has $3N$ degrees of freedom (constraints could reduce this number). We will now look at systems that have an *infinite* number of degrees of freedom. Such systems are easy to find. Consider for example a string lying along the $x - axis$. The string can have transverse vibrations; let us restrict to vibrations along the y axis. Then the configuration of the string at any time t is given by a *function* $y(x)$. Thus in contrast to the point particle where we had to just specify 3 numbers x, y, z to describe the configuration, now we have to specify *one number* y *per point* x, so that we need an infinity of numbers to specify the configuration. This infinity arises because x is a continuous variable, and so such systems are also called 'continuous' systems.

Since the configuration will change with time we have $y = y(x,t)$. More generally we can have any function f which is a function of x, y, z, t. We call such a variable a *scalar field*, where the word 'scalar' tells us that at each point f is just a scalar number, and the word 'field' says that we have one such number for every point. We can also have a 'vector field', where we have a vector at every point; an example where vector fields appear is electromagnetism which is described by fields $\vec{E}, \vec{B}$ or by the potentials $\Phi, \vec{A}$.

2. Notation

There are several conventions that help us to write fields and their actions in a more compact form. Let us call the three spatial coordinates as x^1, x^2, x^3 instead of x, y, z, to avoid later confusion. Noting that in relativity time will be on the same footing as the spatial directions, we write $t = x^0$. Then all four variables are written in condensed form

$$x \equiv \{x^0, x^1, x^2, x^3\} \tag{2.1}$$

The four different components of x are thus

$$x^a, \quad a = 0, 1, 2, 3 \tag{2.2}$$

Note that we have pit the index a at the top; this will be relevant later. If we are integrating over all variables we would just write

$$\int d^4x \equiv \int dx^0 dx^1 dx^2 dx^3 \tag{2.3}$$

For a small change in the positions and time we write dx^a. The proper length of such an interval is

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{2.4}$$

We need to have a good bookkeeping device to keep track of the negative signs in (2.4). Define the matrix

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.5}$$

where note that we have put the indices a, b at the *bottom*. Then we have

$$ds^2 = \sum_{a=0}^{3}\sum_{b=0}^{3} dx^a \eta_{ab} dx^b \tag{2.6}$$

We will often get such sums over indices. We will adopt the Einstein summation convention which says that if an index appears *twice* in an expression then it is assumed to be summed (over the values $0, 1, 2, 3$) without the summation being shown explicitly. Thus we would just write

$$ds^2 = dx^a \eta_{ab} dx^b \tag{2.7}$$

If we encounter a situation where we have an index coming twice and we do *not* want it summed then we will have to say that explicitly.

Note that we have put indices up and down in different places in such a way that a summed index always appears up in one place and down in another. To see the significance of this notation consider a part of the expression in (2.7)

$$\eta_{ab} dx^b \tag{2.8}$$

Note that the index b is summed, the index a appears once so it is *not* summed. We will give this expression a simple name; we have just the vector dx^a multiplied by our standard matrix η. So we will call it

$$dx_a = \eta_{ab} dx^b \tag{2.9}$$

Now the index a is at the bottom, and this signifies that dx^a has been multiplied by η. Note the components of dx_a

$$dx_a \equiv \{dx_0, dx_1, dx_2, dx_3\} = \{dx^0, -dx^1, -dx^2, -dx^3\} \tag{2.10}$$

We have

$$ds^2 = dx^a dx_a = dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3 \tag{2.11}$$

so the signs we needed for special relativity are now encoded automatically, by the fact that we have both dx^a and dx_a in the expression.

3. Action for the scalar field

Consider a scalar field $f(x) = f(x^0, x^1, x^2, x^3)$. We wish to write an action for this field. For a point particle we had

$$S = \int dt \ L \tag{3.1}$$

Now because of symmetry between all the four directions we will get

$$S = \int d^4x \ \mathcal{L} \tag{3.2}$$

where $\mathcal{L}$ is called the *Lagrangian density*. In analogy to $\frac{1}{2}m(\frac{dx}{dt})^2$ for the point particle we write

$$\mathcal{L} \to \frac{1}{2}(\frac{\partial f}{\partial x^0})^2 \tag{3.3}$$

But by symmetry in all four directions we should actually write

$$\mathcal{L} = \frac{1}{2}(\frac{\partial f}{\partial x^0})^2 - \frac{1}{2}(\frac{\partial f}{\partial x^1})^2 - \frac{1}{2}(\frac{\partial f}{\partial x^2})^2 - \frac{1}{2}(\frac{\partial f}{\partial x^3})^2 \tag{3.4}$$

where we have put in the negative signs to again accord with special relativistic invariance. We can rewrite this in several different notations. First note that though x^a has the index up, in $\frac{\partial f}{\partial x^a}$ we have x^a in the denominator, and so effectively the index is *down*. Define another matrix with up indices

$$\eta^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.5}$$

Then we can write

$$\mathcal{L} = \frac{\partial f}{\partial x^a}\eta^{ab}\frac{\partial f}{\partial x^b} \tag{3.6}$$

In more condensed form

$$\mathcal{L} = \frac{1}{2}f_{,a}f_{,b}\eta^{ab} \tag{3.7}$$

where we have used the summation convention. Finally we can also write

$$\mathcal{L} = \frac{1}{2}f_{,a}f^{,a} \tag{3.8}$$

and the action is

$$S = \int d^4x \frac{1}{2}f_{,a}f^{,a} \tag{3.9}$$

This term is just the 'kinetic term' of the action, since it depends on the *changes* in f rather than just the *value* of f. We can also have a potential term, and then the action becomes

$$S = \int d^4x [\frac{1}{2} f_{,a} f^{,a} - V(f)] \tag{3.10}$$

More generally, the Lagrangian density can be any function of f, the derivatives $f_{,a}$ of f, and the coordinates x

$$S = \int d^4x \ \mathcal{L}[f, f_{,a}, x] \tag{3.11}$$

just like $S = \int dt L[q, \dot{q}, t]$ for the point particle.

4. The Euler-Lagrange equations

In the point particle case we held the dynamical variables q fixed at the initial and final times t_{min}, t_{max} and demanded that the correct path extremise the action. Now we will hold fixed the value of f at the sides of a *box* in spacetime – the box extends from $t = t_{min}$ to $t = t_{max}$ in time, from x^1_{min} to x^1_{max} in the direction x^1 etc. We have

$$\delta S = \int d^4x [\frac{\partial \mathcal{L}}{\partial f_{,a}} \delta(f_{,a}) + \frac{\partial \mathcal{L}}{\partial f} \delta f] \tag{4.1}$$

We have by the usual argument

$$\delta(f_{,a}) = (\delta f)_{,a} \tag{4.2}$$

We can then write

$$\delta S = \int d^4x [\frac{\partial \mathcal{L}}{\partial f_{,a}} (\delta f)_{,a} + \frac{\partial \mathcal{L}}{\partial f} \delta f] = \int d^4x [[\frac{\partial \mathcal{L}}{\partial f_{,a}} \delta f]_{,a} - [\frac{\partial \mathcal{L}}{\partial f_{,a}}]_{,a} \delta f + \frac{\partial \mathcal{L}}{\partial f} \delta f] \tag{4.3}$$

The first term is

$$\int d^4x [\frac{\partial \mathcal{L}}{\partial f_{,a}} \delta f]_{,a} = \int d^4x [\frac{\partial \mathcal{L}}{\partial f_{,0}} \delta f]_{,0} + \int d^4x [\frac{\partial \mathcal{L}}{\partial f_{,1}} \delta f]_{,1} + \int d^4x [\frac{\partial \mathcal{L}}{\partial f_{,2}} \delta f]_{,2} + \int d^4x [\frac{\partial \mathcal{L}}{\partial f_{,3}} \delta f]_{,3} \tag{4.4}$$

In the first of these terms, do the x^0 integral first, holding x^1, x^2, x^3 fixed. We get

$$\int dx^1 dx^2 dx^3 [\frac{\partial \mathcal{L}}{\partial f_{,0}} \delta f(t_{max}, x^1, x^2, x^3) - \frac{\partial \mathcal{L}}{\partial f_{,0}} \delta f(t_{min}, x^1, x^2, x^3)] = 0 \tag{4.5}$$

where we get the vanishing because δf is zero at the boundary of our 'box'. We thus get from (4.3)

$$\delta S = \int d^4x [-[\frac{\partial \mathcal{L}}{\partial f_{,a}}]_{,a} + \frac{\partial \mathcal{L}}{\partial f}] \delta f \tag{4.6}$$

Since we want $\delta S = 0$ for arbitrary δf we have at each point

$$[\frac{\partial \mathcal{L}}{\partial f_{,a}}]_{,a} - \frac{\partial \mathcal{L}}{\partial f} = 0 \tag{4.7}$$

or more explicitly

$$\frac{\partial}{\partial x^0}[\frac{\partial \mathcal{L}}{\partial(\partial_0 f)}] + \frac{\partial}{\partial x^1}[\frac{\partial \mathcal{L}}{\partial(\partial_1 f)}] + \frac{\partial}{\partial x^2}[\frac{\partial \mathcal{L}}{\partial(\partial_2 f)}] + \frac{\partial}{\partial x^3}[\frac{\partial \mathcal{L}}{\partial(\partial_3 f)}] - \frac{\partial \mathcal{L}}{\partial f} = 0 \tag{4.8}$$

This is the Euler-Lagrange equation.

5. Symmetries and conserved quantities

Suppose that the change

$$f \to f + \delta f \tag{5.1}$$

leaves the Lagrangian density unchanged

$$\mathcal{L} \to \mathcal{L} \tag{5.2}$$

Then we have

$$0 = \delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial f_{,a}}\delta(f_{,a}) + \frac{\partial \mathcal{L}}{\partial f}\delta f = \frac{\partial \mathcal{L}}{\partial f_{,a}}(\delta f)_{,a} + \frac{\partial \mathcal{L}}{\partial f}\delta f = [\frac{\partial \mathcal{L}}{\partial f_{,a}}\delta f]_{,a} - [[\frac{\partial \mathcal{L}}{\partial f_{,a}}]_{,a} - \frac{\partial \mathcal{L}}{\partial f}]\delta f \tag{5.3}$$

If we have a solution of the equations of motion then the last two terms vanish by the Euler-Lagrange equations, and we have

$$[\frac{\partial \mathcal{L}}{\partial f_{,a}}\delta f]_{,a} = 0 \tag{5.4}$$

Expanded out, we have

$$\partial_0[\frac{\partial \mathcal{L}}{\partial f_{,0}}\delta f] + \partial_1[\frac{\partial \mathcal{L}}{\partial f_{,1}}\delta f] + \partial_2[\frac{\partial \mathcal{L}}{\partial f_{,2}}\delta f] + \partial_3[\frac{\partial \mathcal{L}}{\partial f_{,3}}\delta f] = 0 \tag{5.5}$$

Writing

$$J^a = \frac{\partial \mathcal{L}}{\partial f_{,a}}\delta f \tag{5.6}$$

we get

$$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0 \tag{5.7}$$

In the point particle case we had obtained a conserved quantity from a symmetry of the Lagrangian, but (5.5) does not immediately seem to give a conserved quantity. (5.7) is like the continuity equation in electromagnetism, which is indeed a special case of (5.7). In fact the notation (5.6) is adopted to the general case from the special example of electromagnetism where the current is called J.

So how do we get a conserved quantity? A conserved quantity is something that does not change with time. Let us integrate (5.7) over all *space*, but not over time

$$\int dx^1 dx^2 dx^3 [\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3}] = 0 \tag{5.8}$$

We will assume that f dies off at spatial inifinity sufficiently rapidly; if this does not happen we do not get a conserved quantity since 'stuff can flow off to infinity'. Then we get for instance

$$\int dx^1 dx^2 dx^3 [\frac{\partial J^1}{\partial x^1}] = \int dx^2 dx^3 \int dx^1 [\frac{\partial J^1}{\partial x^1}] = \int dx^2 dx^3 [J^1(x^1_{max}, x^2, x^3) - J^1(x^1_{min}, x^2, x^3)] \to 0 \tag{5.9}$$

where we have used the fact that $x^1_{max} \to \infty, x^1_{min} \to -\infty$. The only term that does not vanish is the first one, so we get

$$\int dx^1 dx^2 dx^3 [\frac{\partial J^0}{\partial x^0}] = \partial_0 \int dx^1 dx^2 dx^3 J^0 = \frac{d}{dt}[\int dx^1 dx^2 dx^3 J^0(t, x^1, x^2, x^3)] \tag{5.10}$$

Thus if we integrate J^0 over all *space*, at a fixed time t, then we get a number for each t, which we find does not change with t. We thus have a conserved quantity

$$Q \equiv [\int dx^1 dx^2 dx^3 J^0(t, x^1, x^2, x^3)] \tag{5.11}$$

Q is called the 'conserved charge'; it is a generalization of the charge found in electromagnetism.