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1 Introduction

The main purpose of this article is to offer a review on the basic ideas of quantum integrability as well as familiarize the reader, who has not necessarily a background on the subject, with the fundamental concepts.

Quantum integrability in 1+1 dimensions has been a very rich research subject, especially after the seminal works of the St. Petersburg group (see e.g. [1]–[5]) on the quantum inverse scattering method (QISM). We refer the interested reader to a number of lecture notes and review articles on algebraic Bethe ansatz, special topics on integrable models, or articles with emphasis on statistical and thermodynamic properties or applications to condensed matter physics (see e.g [6]–[11]). In these notes we are basically focusing on the algebraic content of quantum integrable systems giving particular emphasis on the quantum algebras and their connections to braid groups and Hecke algebras. We also review the quantum inverse scattering method and briefly discuss lattice integrable models with open boundary conditions.

The outline of the article is as follows: in the next section we introduce the basic notation on tensor products of matrices and vectors and we briefly review the \mathfrak{su}_2 algebra as well as its representations. We then introduce the Heisenberg model [12], describing first neighbors spin-spin interaction. In section 3 we present in more detail the XXX (isotropic) and XXZ (anisotropic Heisenberg) models. In particular, we give a first flavor on the corresponding spectra and eigenstates for small a number of sites. We also discuss the zero temperature phase diagram. The next section is basically devoted to the Yang-Baxter [13] equation and its solution, the so called R matrix. This is the fundamental equation within the QISM context. We introduce the equation and also provide systematic means for solving it via its structural similarity with the braid group. The braid group and certain quotients, such as the Hecke and Temperley-Lieb algebras [14, 15, 16],

are also discussed.

In section 5 we introduce the quantum Lax operator, and the fundamental algebraic equation governing the underlying quantum algebras (Yangians and q deformed Lie algebras) [17, 18]. We then construct tensorial representations of the underlying algebras, and eventually build the closed (periodic) transfer matrix of a spin chain-like system. We show the integrability of the system, and also extract the corresponding local Hamiltonian. In the next section we discuss in more detail the non-trivial co products arising in quantum algebras and we show how one can exploit them in order to investigate the symmetry of the associated R matrix. In section 7 we present representations of the $U_q(\mathfrak{sl}_2)$ algebra and discuss in detail the algebraic Bethe ansatz technique for diagonalizing the generalized XXZ spin chain. In the last section we discuss integrable lattice models with generic integrable boundary conditions [19]. The corresponding fundamental algebraic relation i.e. the reflection equation [20] is introduced and solutions (reflection matrices) are obtained with the help of the B -type braid group and its quotients [21, 22, 23, 24]. Tensorial representations of the reflection algebra are constructed and the open transfer matrix is introduced. Finally, the $U_q(\mathfrak{sl}_2)$ invariant open XXZ spin chain [25] is discussed and the corresponding quadratic Casimir is extracted from the open transfer matrix.

2 Preliminaries

2.1 Notation

Before we proceed with the presentation of the fundamental notions of quantum integrability it is necessary to introduce some basic notation.

Consider the tensor vector space $V \otimes V$ then define

$$\begin{aligned} A_1 &= A \otimes \mathbb{I} \\ B_2 &= \mathbb{I} \otimes B. \end{aligned} \tag{2.1}$$

We then attach subscripts on the various elements to define the respective vector space on which they act non-trivially. For example, suppose that $A, B \in \text{End}V$. In the described notation the tensor product between them can be written as

$$A \otimes B = A_1 B_2. \tag{2.2}$$

In general, consider the tensor sequence of N vector spaces $V \otimes V \otimes \dots \otimes V$ then define:

$$A_n = \mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \underbrace{A}_n \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}, \quad n \in \{1, 2, \dots, N\}. \quad (2.3)$$

We shall extensively use such notation subsequently when constructing one dimensional integrable quantum spin chains, which is one of the primary objectives of this review.

Some basic properties of the tensor product are listed below¹:

$$\begin{aligned} (A \otimes B) (C \otimes D) &= AC \otimes BD \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1} \\ (A \otimes B)^T &= A^T \otimes B^T. \end{aligned} \quad (2.4)$$

There is a simple rule that gives the tensor product of two matrices. Consider for simplicity the 2×2 matrices $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, the tensor product $A \otimes B$ is a 4×4 matrix defined as

$$\begin{aligned} A \otimes B &= \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \end{aligned} \quad (2.5)$$

In general for two $n \times n$ matrices A, B the corresponding tensor product is a $n^2 \times n^2$ matrix and the rule generalizes in a straightforward manner: $(A \otimes B)_{ij,kl} = a_{ij} b_{kl}$.

The tensor product of two vectors $a, b \in \mathbb{C}^2$ is derived as

$$a \otimes b = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{pmatrix}, \quad (2.6)$$

and in general for n column vectors $a, b \in \mathbb{C}^n$ we obtain an n^2 column vector: $(a \otimes b)_{i,j} = a_i b_j$.

¹Note that for the first of the properties listed we are focusing on non super symmetric algebras. In the super symmetric case it is modified in accordance to the fermionic and bosonic degrees of freedom.

2.2 The \mathfrak{su}_2 algebra: a brief review

The \mathfrak{su}_2 algebra is defined by the generators J^\pm , J^z and the exchange relations

$$\begin{aligned} [J^+, J^-] &= 2J^z \\ [J^z, J^\pm] &= \pm J^\pm. \end{aligned} \quad (2.7)$$

The spin $\frac{1}{2}$ representation of \mathfrak{su}_2 maps the three generators of the algebra to the three Pauli matrices. Indeed consider the spin $\frac{1}{2}$ representation $\pi : \mathfrak{su}_2 \hookrightarrow \text{End}(\mathbb{C}^2)$ such that:

$$\pi(J^z) = \frac{1}{2}\sigma^z, \quad \pi(J^\pm) = \sigma^\pm \quad (2.8)$$

and σ^\pm , σ^z are the familiar 2×2 Pauli matrices

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.9)$$

There is a homomorphism h defined as $h : \mathfrak{su}_2 \hookrightarrow \mathfrak{sl}_2$ such that

$$h(J^z) = J^z, \quad h(J^+) = J^+, \quad h(J^-) = -J^-, \quad (2.10)$$

so the two algebras \mathfrak{su}_2 , \mathfrak{sl}_2 will be equivalently used henceforth.

The typical spin s representation of \mathfrak{su}_2 is an $n = 2s + 1$ dimensional representation and may be expressed in terms of $n \times n$ matrices; infinite dimensional representations exist in terms of differential operators. Define first the $n \times n$ matrices e_{ij} such that

$$(e_{ij})_{kl} = \delta_{il} \delta_{jk}, \quad \text{and} \quad e_{ij} e_{kl} = \delta_{jk} e_{il}. \quad (2.11)$$

Consider the n dimensional matrix representation: $\rho : \mathfrak{su}_2 \hookrightarrow \text{End}(\mathbb{C}^n)$ such that:

$$\rho(J^z) = \sum_{k=1}^n \alpha_k e_{kk}, \quad \rho(J^+) = \sum_{k=1}^{n-1} C_k e_{kk+1}, \quad \rho(J^-) = \sum_{k=1}^{n-1} C_k e_{k+1k} \quad (2.12)$$

where we define

$$\alpha_k = \frac{n+1}{2} - k, \quad C_k = \sqrt{k(n-k)}. \quad (2.13)$$

The generic spin s representation of \mathfrak{sl}_2 in terms of differential operators may be expressed as:

$$J^z \hookrightarrow y \frac{d}{dy}, \quad J^+ \hookrightarrow y^{-1} \left(y \frac{d}{dy} + s \right), \quad J^- \hookrightarrow y \left(y \frac{d}{dy} - s \right). \quad (2.14)$$

In this case s may be any number –not necessarily an integer– and the space of functions is infinite dimensional. It is a straightforward exercise to show that (2.12), (2.14) satisfy the \mathfrak{sl}_2 exchange relations.

2.3 The Heisenberg model

We come now to the description of physical models associated to the \mathfrak{su}_2 algebra described above. The Heisenberg model was introduced as a natural physical description of magnetism in solid state physics [12]. Its one-dimensional version, the only one of interest here, has also the merit of having inaugurated the studies of quantum integrable systems and of the methods known as Bethe Ansatz [12, 2, 3].

The Heisenberg idea is to consider, on each lattice site, a quantum magnetic needle of spin $\frac{1}{2}$, fully free to rotate. Formally, this is represented by a two-dimensional local space of states \mathbb{C}^2 that can accommodate a spin up and a spin down components

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.15)$$

The full space of states is then consisting of sums of tensor products of such spins up and down on all sites of the lattice, here taken one dimensional and consisting of N sites:

$$W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2 \quad (2.16)$$

The local spin operators can be introduced by using Pauli matrices on each site

$$S_i^z = \frac{1}{2}\sigma_i^z, \quad S_i^{x,y} = \frac{1}{2}\sigma_i^{x,y} \quad (2.17)$$

where the Pauli matrices act on the i th site of the chain, according to 2.3. Recall that

$$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y), \quad \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y). \quad (2.18)$$

The Pauli matrices are given by (2.9), and clearly satisfy the \mathfrak{su}_2 commutation relations. Following (2.3), Pauli matrices acting on different sites commute with each other:

$$[\sigma_i^\xi, \sigma_j^{\xi'}] = 0, \quad i \neq j, \quad \xi \in \{x, y, z\}. \quad (2.19)$$

The magnetic needle is assumed sensitive to the nearest neighbor needles with the simplest possible coupling of magnetic dipoles

$$J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z, \quad (2.20)$$

then the corresponding Hamiltonian is obtained by summing on all the lattice sites

$$H = \sum_i (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z). \quad (2.21)$$

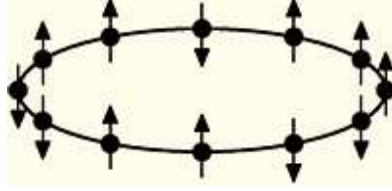


Figure 1: Spin chain with periodic boundary conditions.

Here the couplings J_ξ are taken constant through the lattice. This model is known as XYZ model.

Often, for computational reasons, it is convenient to impose some special conditions at the two edges of the lattice. These can also represent the interaction of the lattice with some environment. A very common choice, that will be adopted here, is to consider the first and last sites to be adjacent

$$N + 1 \equiv 1 \quad \Rightarrow \quad i + N \equiv i \quad (2.22)$$

this is known as periodic boundary conditions.

These conditions ‘close’ the chain (see Figure 1) and are a bit unphysical², but often extremely convenient; more physical are the open boundaries conditions, where no interaction between the first and the last sites is assumed (see last lecture).

The Hamiltonian (2.21) with constant coefficients can be exactly treated with functional methods. Indeed, this model is equivalent to the 8-vertex model solved by Baxter [13]. Here a different and simpler approach will be used –algebraic Bethe ansatz– that works for the so called XXX and XXZ models, obtained by identifying the coefficients $J_x = J_y = J_z$ and $J_x = J_y$ respectively. Note that the XYZ model can be also solved via the algebraic Bethe ansatz methodology, but after implementing certain modifications that involve specific local gauge transformations (see [3]). We shall not however discuss these subtle technical points here.

²These is no physical justification for the assumption that, in a piece of material, the atoms at opposite extremes should be considered as interacting. In spite of that, thermodynamic properties are independent of boundary conditions if interactions are short ranged therefore periodic boundary conditions can be used to obtain realistic results.

3 The XXX and XXZ quantum spin chains

3.1 The XXX model

Our main purpose now is to diagonalize local Hamiltonians describing interactions between first neighbors as described by (2.21). We shall first introduce the 2-site XXX or isotropic Heisenberg model. For the two-site problem, the basic observables are the spin operators at each site, $\vec{\sigma}_1 \equiv \vec{\sigma} \otimes \mathbb{I}$ and $\vec{\sigma}_2 \equiv \mathbb{I} \otimes \vec{\sigma}$. That is (see also [9]),

$$\begin{aligned}
 \sigma_1^x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & 1 & & \end{array} \right), \\
 \sigma_1^y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} & & -i & 0 \\ & & 0 & -i \\ \hline i & 0 & & \\ 0 & i & & \end{array} \right), \\
 \sigma_1^z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & \\ \hline & & -1 & 0 \\ & & 0 & -1 \end{array} \right), \tag{3.1}
 \end{aligned}$$

and

$$\begin{aligned}
 \sigma_2^x &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & 1 & 0 \end{array} \right), \\
 \sigma_2^y &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & -i & & \\ i & 0 & & \\ \hline & & 0 & -i \\ & & i & 0 \end{array} \right), \\
 \sigma_2^z &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & -1 & & \\ \hline & & 1 & 0 \\ & & 0 & -1 \end{array} \right). \tag{3.2}
 \end{aligned}$$

Recall also (2.19). These operators act on the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$, with elements $a \otimes b$ given explicitly in (2.6)

The 2-site Hamiltonian for the XXX model is then given by

$$H_{12} = -\frac{1}{2}(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z), \quad (3.3)$$

which describes the interaction of two spin 1/2 magnets.

It is easy to verify that the 2-site Hamiltonian may be alternatively written as:

$$H_{12} = -\mathcal{P} + c \mathbb{I}, \quad (3.4)$$

where

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z + \mathbb{I} \otimes \mathbb{I}), \quad (3.5)$$

is the *permutation operator*. The constant here is $c = \frac{1}{2}$.

Exercise 1: Show that \mathcal{P} satisfies the following properties:

$$\mathcal{P}(a \otimes b) = b \otimes a, \quad \forall a, b \in \mathbb{C}^2, \quad (3.6)$$

$$\mathcal{P}_{12} A_1 B_2 \mathcal{P}_{12} = A_2 B_1, \quad (3.7)$$

$$\mathcal{P}_{12}^2 = \mathbb{I}. \quad (3.8)$$

where A, B are 2×2 matrices.

One can then generalize the permutation operator \mathcal{P} acting on $\mathbb{C}^n \otimes \mathbb{C}^n$, and defined as

$$\mathcal{P} = \sum_{i,j=1}^n e_{ij} \otimes e_{ji}. \quad (3.9)$$

Exercise 2: Show explicitly that $\mathcal{P}^2 = \mathbb{I}$ holds for the generic case. Do the same for the properties of the permutation operator

$$\mathcal{P}_{12} A_1 \mathcal{P}_{12} = A_2 \Leftrightarrow \mathcal{P}_{12} A_1 = A_2 \mathcal{P}_{12}. \quad (3.10)$$

where A is any $n \times n$ matrix that can be expressed as: $A = \sum_{i,j=1}^n a_{ij} e_{ij}$.

Exercise 3: Show by inspection that the following states

$$\begin{aligned}
 |\psi_1\rangle &= |\uparrow\rangle \otimes |\uparrow\rangle, & |\psi_2\rangle &= |\downarrow\rangle \otimes |\downarrow\rangle \\
 |\psi_3\rangle &= |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \\
 |\psi_4\rangle &= |\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle,
 \end{aligned} \tag{3.11}$$

where $|\uparrow\rangle, |\downarrow\rangle$ are defined in (2.15), are eigenfunctions of the Hamiltonian. Find also the corresponding eigenvalues and discuss the possible degeneracies.

Exercise 4: Define the *co-product* of \mathfrak{sl}_2 as a mapping $\Delta : \mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$

$$\Delta(X) = X_1 + X_2 = X \otimes \mathbb{I} + \mathbb{I} \otimes X, \quad X \in \mathfrak{sl}_2. \tag{3.12}$$

Show that $\Delta(J^\pm), \Delta(J^z)$, satisfy the relations of the \mathfrak{sl}_2 algebra. In other words, show that the co-product is a tensor realization of the algebra.

Exercise 5: Show that the 2-site Hamiltonian enjoys the \mathfrak{su}_2 symmetry, that is:

$$[X_1 + X_2, H_{12}] = 0, \tag{3.13}$$

where $X \in \{\sigma^\pm, \sigma^z\}$.

The N-site Hamiltonian

The Hamiltonian can be generalized to the case of N ‘particles’, described by the N -site Hamiltonian:

$$H = \sum_{i=1}^N H_{i,i+1} = -\frac{1}{2} \sum_{i=1}^N (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z), \tag{3.14}$$

which clearly acts on $W = (\mathbb{C}^2)^{\otimes N}$, and $\dim W = 2^N$. The model is integrable and may be exactly solved [12]. In general the problem of diagonalizing the Hamiltonian is quite tedious, even with the use of numerical methods. However, one can find analytical solutions to this problem by using quite sophisticated and powerful techniques under the name of Bethe ansatz (see e.g. [1, 2, 6, 7, 8]), which will be one of the main subjects of the following sections. Here we shall focus on one of the many variations of the Bethe ansatz method that is the algebraic Bethe ansatz [1, 2].

The Hamiltonian (3.14) is manifestly translation invariant, made evident by shifting i into $i + 1$ in the sum. The translation operator corresponding to the translational symmetry is

$$\Pi = \mathcal{P}_{12} \mathcal{P}_{23} \dots \mathcal{P}_{L-1,L} = e^{-i\mathbb{P}} \quad (3.15)$$

where \mathcal{P} is the permutation operator; Π commutes with the Hamiltonian (see exercise in Lecture 5). The operator in the exponent is the momentum operator, and is defined up to 2π . The action of Π consists of shifting each site of the lattice by one step to the right.

Before concluding, we should note that there also exists a generalization of the co-product for the case of N particles,

$$\Delta^{(N)}(X) = \sum_{i=1}^N X_i = \sum_i \mathbb{I} \otimes \mathbb{I} \otimes \dots \otimes \underbrace{X}_i \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}. \quad (3.16)$$

Exercise 6: Show that the N -co-product commutes with the XXX Hamiltonian

$$[H, \Delta^{(N)}(X)] = 0, \quad X \in \{\sigma^\pm, \sigma^z\} \quad (3.17)$$

i.e. the XXX Hamiltonian is \mathfrak{su}_2 symmetric.

3.2 The XXZ model

We come now to the anisotropic model, that is the XXZ spin chain; we simply introduce an anisotropy along z by

$$H = -\frac{1}{2} \sum_{i=1}^N (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z). \quad (3.18)$$

This Hamiltonian acts on the Hilbert space defined in (2.16). The exact behavior of the system depends on the coupling constant Δ , that describes the anisotropy of the model.

We shall consider periodic boundary conditions on the chain as in (2.22); for local³ operators this is expressed by $\vec{\sigma}_{i+N} = \vec{\sigma}_i$ as is shown in figure (1). The XXZ Hamiltonian is also translation invariant and the translation operator is given by (3.15). The case $\Delta \neq 1$, or XXZ model was solved by Hans Bethe in 1931 [12], by using his famous

³A local operator acts on a single site only.

ansatz. The case $\Delta = 1$ corresponds to the properly named Heisenberg model or XXX model, discussed previously, which as already mentioned (see exercise in previous section) has the global symmetry $\mathfrak{su}(2)$. If $\Delta = 0$ the z component disappears and the model is called XY. If $\Delta \rightarrow \pm\infty$, one actually obtains the Ising model. One simple way to see this is to introduce a new Hamiltonian by dividing by the coupling; the first group of terms disappears

$$H_{\text{Ising}} = \lim_{\Delta \pm\infty} \frac{H}{\Delta} = -\frac{1}{2} \sum_{i=1}^N \sigma_i^z \sigma_{i+1}^z, \quad (3.19)$$

and the model is actually indistinguishable from the classical spins model

$$-\frac{1}{2} \sum_{i=1}^N \tau_i \tau_{i+1} \quad (3.20)$$

with $\tau_i = \pm 1$, known as the Ising model. For this reason, the XXZ model is indicated also as the Heisenberg-Ising model.

By the N co-product we can define the total spin operator S^ξ

$$\Delta(S^\xi) = S^\xi \sum_{i=1}^N S_i^\xi. \quad (3.21)$$

The third component is a symmetry of the model, no matter what the value of Δ is,

$$[H, S^z] = 0. \quad (3.22)$$

To verify this last statement, the relations of \mathfrak{su}_2 (satisfied by Pauli matrices) are essentially needed, plus the fact that different spaces in the tensor product sequence commute with each other.

If $\Delta = 1$, it is immediate to check, (see exercise in the previous section), that the x and y components also commute therefore

$$[H, S^\xi] = 0, \quad \xi \in \{x, y, z\} \quad (3.23)$$

and the Hamiltonian is fully $\mathfrak{su}(2)$ symmetric.

Because of the indicated symmetry, in the XXZ model the third component S_z is particularly important; it will also be used as an order parameter, because it specifies the different phases of the model itself.

The operator S_z is the sum of mutually commuting diagonal operators; each component contributes to the eigenvalues with $\pm\frac{1}{2}$ therefore the spectrum is

$$-\frac{N}{2}, -\frac{N}{2} + 1, -\frac{N}{2} + 2, \dots, \frac{N}{2} \quad (3.24)$$

There is a reflection symmetry

$$\sigma_i^z \rightarrow -\sigma_i^z \quad (3.25)$$

that makes the spectrum of H symmetric with respect to the values of S_z .

Example: two-site Hamiltonian.

Consider now the two-site Hamiltonian

$$H = -(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta \sigma^z \otimes \sigma^z) \quad (3.26)$$

The base vectors are ordered as $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, and for a two-sites lattice the Hamiltonian reads (the sum in (3.18) goes over $j=1$ and $j=2$)

$$H = \begin{pmatrix} -\Delta & 0 & 0 & 0 \\ 0 & \Delta & -2 & 0 \\ 0 & -2 & \Delta & 0 \\ 0 & 0 & 0 & -\Delta \end{pmatrix}. \quad (3.27)$$

This matrix is symmetric and has real entries therefore is diagonalizable and has real eigenvalues. It has the following set of eigenvectors and eigenstates:

$$\{|\uparrow\uparrow\rangle, -\Delta\}, \{|\downarrow\downarrow\rangle, -\Delta\}, \{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle, \Delta - 2\}, \{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle, \Delta + 2\} \quad (3.28)$$

We examine the ground state to understand the possible phases of the system.

When $\Delta > 1$, the ground state has energy $E = -\Delta$ and it can be one of the states $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ therefore the system is ferromagnetic and the total magnetization is $M = S^z = 2$. This extends immediately to an arbitrary number of sites.

If $\Delta < 1$, the ground state is $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$ with energy $E = \Delta - 2$ therefore the system has vanishing total magnetization $M = S^z = 0$. Notice that the Hamiltonian model is at zero temperature; the temperature has not been introduced neither any statistical ensemble. With that in mind, the ground state for $\Delta < 1$ is not a Gibbs mixture of

states, but is given as indicated: it has a global order (at least in the two-site case!) and is actually called anti-ferromagnetic state⁴.

The doubly degenerate level $E = -\Delta$ becomes triply degenerate if $\Delta = 1$ or $\Delta = -1$. The case $\Delta = 1$ is especially interesting, because the three coinciding eigenvalues $E = -\Delta = -1$ correspond also to the lowest energy. This marks a so called quantum phase transition, namely a transition not induced by thermal fluctuations but by the variation of a parameter in the Hamiltonian. It occurs at zero temperature only.

Some general comments are now in order. The XXX Hamiltonian is free of couplings apart from the overall sign. Given our sign choice, it is apparent that adjacent parallel spins lower the energy. This explains the custom to call ferromagnetic the Hamiltonian in (3.14) and anti-ferromagnetic the opposite one. The presence of Δ in the XXZ model spoils this distinction because the ferromagnetic/antiferromagnetic behavior depends on the coupling and, as standard in physics, the name is attached to the phase of the system. Here no temperature is introduced (namely we are at zero temperature) so the phase is dictated by the ground state: $\Delta > 1$, the ground state is ferromagnetic; $\Delta < 1$, the ground state is anti-ferromagnetic; $\Delta = 1$, critical case, the ground state is multi-degenerate.

In exercises 8, 9, 10 below the reader can construct slightly larger cases, and have a more accurate indication of the phases for the different values of Δ .

Exercise 7: Prove that the Hamiltonian (3.18) is real and symmetric therefore diagonalizable with real eigenvalues.

Exercise 8: Construct the three-site Hamiltonian and check that the ground-state for $\Delta > 1$ is ferromagnetic while for $\Delta < 1$ is anti-ferromagnetic with frustration⁵.

Exercise 9: Construct the four-site Hamiltonian and check that the groundstate for $\Delta > 1$ is ferromagnetic. Find the ground state for $\Delta < 1$, call it ϕ . This state is

⁴The regions $\Delta < -1$ and $-1 < \Delta < 1$ can be further discriminated if an external magnetic field is added to the Hamiltonian

⁵Frustration indicates the phenomenon where adjacent magnets tend to be antiparallel but geometry or topology forces a pair of them to be parallel. In the present case, on a periodic odd sites lattice, an alternating sequence of up and down spins is always frustrated. See later.

anti-ferromagnetic. Check that it is proportional to

$$\phi \propto |\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + \frac{-\Delta \pm \sqrt{\Delta^2 + 8}}{2} (|\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle) \quad (3.29)$$

Exercise 10: In the special $\Delta = 1$ case, the anti-ferromagnetic ground state for a four-site lattice can be obtained just using symmetry considerations and the repeated action of S^- on $|\uparrow\uparrow\uparrow\uparrow\rangle$. Find it and compare with the longer method of Exercise 9.

3.2.1 The Ising limits

The argument given before (3.19) is extremely useful to have an indication of the phases. For $\Delta \gg 1$, the Ising-like z component of the hamiltonian

$$\frac{H}{\Delta} = -\frac{1}{2} \sum_{i=1}^L \sigma_i^z \sigma_{i+1}^z + \mathcal{O}\left(\frac{1}{\Delta}\right) \quad (3.30)$$

contributes to the eigenvalues with 0 for each pair of parallel spins and with +1 for each pair of antiparallel spins; these values are obtained using the classical model (3.20). In other words, antiparallel spins increase the energy. This means that the ground state is one of the two saturated ferromagnets

$$|\uparrow\uparrow \dots \uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.31)$$

$$|\downarrow\downarrow \dots \downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.32)$$

If $\Delta \ll -1$, the situation is the opposite and adjacent antiparallel spins are favored so the ground state will be an anti-ferromagnet. Its actual form is not necessarily trivial, as shown in Exercise 9; moreover it depends on Δ and on the parity of the chain length N . First, we consider the Ising limit $\Delta \rightarrow -\infty$ with N even. In that case, the ground state is a succession of up and down spins

$$|\uparrow\downarrow\uparrow\downarrow \dots\rangle \quad (3.33)$$

$$|\downarrow\uparrow\downarrow\uparrow \dots\rangle.$$

These states are called Néel states. For N being even, both vectors are compatible with the periodic boundary conditions. If N is odd and if the site 1 has spin up, the site N has also spin up, however sites 1 and N are adjacent therefore two up spins meet. The perfect alternation of up and down cannot be realized. This phenomenon is called frustration (see also Exercise 8).

When Δ is finite, the situation is even more complicated: the vectors (3.33) are not eigenvectors of the Hamiltonian so they cannot be the ground state. The two-sites example is clear: in (3.28) the state $|\uparrow\downarrow\rangle$ is not eigenvector. It becomes eigenvector only at the limit, because the two eigenvalues $\Delta \pm 2$ become degenerate so one can take linear combinations of the corresponding eigenvectors.

4 Yang-Baxter equation and the braid group

4.1 Yang-Baxter equation

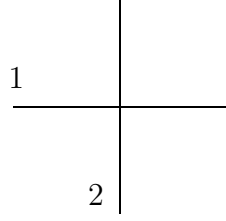
We wish now to introduce a more abstract and general formalism, that will later allow us to investigate the spectrum of the Hamiltonian (3.18) and its integrable properties. Within this formalism, we make use of a special matrix, usually indicated as R -matrix, and satisfies the Yang-Baxter equation [13]. This equation provides a set of very strong conditions on the model, implying its integrability. This section will be developed independently of the ideas introduced in the previous sections. The connection between the two formalisms will be done later.

Let us now introduce the fundamental relation in our context (quantum inverse scattering method), that is the Yang-Baxter equation (YBE) [4, 13]

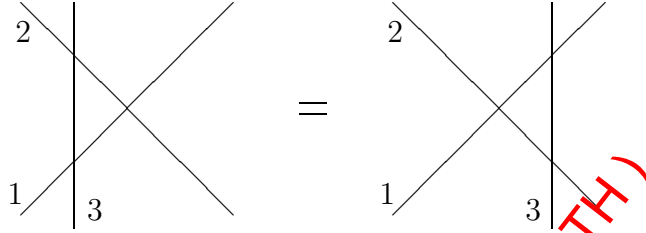
$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1 - \lambda_3) R_{23}(\lambda_2 - \lambda_3) = R_{23}(\lambda_2 - \lambda_3) R_{13}(\lambda_1 - \lambda_3) R_{12}(\lambda_1 - \lambda_2), \quad (4.1)$$

where $R(\lambda)$ is a matrix acting on $V \otimes V$. YBE acts on $V \otimes V \otimes V$, and according to the notation introduced earlier $R_{12} = R \otimes \mathbb{I}$, $R_{23} = \mathbb{I} \otimes R$ and so on. We set henceforth $\lambda_3 = 0$ for simplicity.

Graphically, one represents R_{12} as



The YBE is then simply represented as



From the physical viewpoint, as pointed out in [26] the Yang-Baxter equation describes the factorization of multi-particle scattering, a unique feature displayed by 2-d integrable systems (see figure above). On the other hand from a mathematical viewpoint the algebraic structures underlying the Yang-Baxter equation may be seen as deformations of the usual Lie algebras or their infinite dimensional extensions, the Kac-Moody algebras [27]. Such deformed algebraic structures are endowed with a non trivial co-product as we shall see, and are known as quantum groups or quantum algebras [18, 17].

Exercise 1: Show that [28]

$$R(\lambda) = \lambda \mathbb{I} + i\mathcal{P} \quad (4.2)$$

is a solution to the Yang-Baxter equation (recall \mathcal{P} is the permutation operator).

The YBE can be also written in an alternative form. First, define $\check{R} = \mathcal{P}R$ [18]; substituting in the YBE and exploiting the properties of \mathcal{P} , one finds that \check{R} satisfies

$$\check{R}_{12}(d) \check{R}_{23}(\lambda_1) \check{R}_{12}(\lambda_2) = \check{R}_{23}(\lambda_2) \check{R}_{12}(\lambda_1) \check{R}_{23}(d), \quad (4.3)$$

where $d \equiv \lambda_1 - \lambda_2$. By finding solutions of the YBE in the form above (4.3), then one automatically finds solutions to the original YBE. This will be the subject of the subsequent section.

Exercise 2: Let

$$R(\lambda) = \begin{pmatrix} \lambda + i & 0 & 0 & 0 \\ 0 & \lambda & i & 0 \\ 0 & i & \lambda & 0 \\ 0 & 0 & 0 & \lambda + i \end{pmatrix}. \quad (4.4)$$

Show that this R -matrix generates the XXX spin chain Hamiltonian

$$\left. \frac{d}{d\lambda} \check{R}_{12}(\lambda) \right|_{\lambda=0} = \mathcal{P} \propto H_{12}, \quad (4.5)$$

up to a constant shift in (3.3) or equivalently after taking $c = 0$ in (3.4).

The solution of the YBE is one of the primary objectives in this context, thus finding suitable methods to extract solutions in an elegant and economical way is a significant issue. We shall discuss below how one can identify solutions of the Yang-Baxter equations using a quite powerful technique involving the so called braid group (see e.g. [14, 15, 16, 18, 29]).

4.2 Braid groups

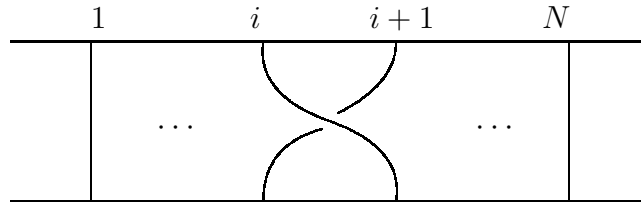
Definition 3.1. *The A-type Artin braid group is defined by generators g_i , $i = 1, 2, \dots, N-1$, and exchange relations:*

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, & i \in \{1, \dots, N-2\}, \\ [g_i, g_j] &= 0, & |i - j| > 1. \end{aligned} \quad (4.6)$$

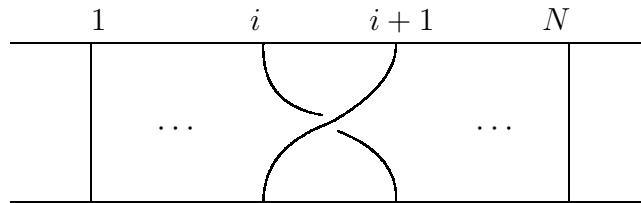
One easily observes the structural similarity between the ‘braid relation’ –the first of the relations above– and the modified YBE, which is satisfied by \check{R} . One can exploit this similarity, and search for candidate solutions of the YBE, within the representations of the braid group.

Graphical representation of the braid group: we can graphically depict the braid group, by defining suitable graphical representations for the generator g_i and its inverse g_i^{-1} .

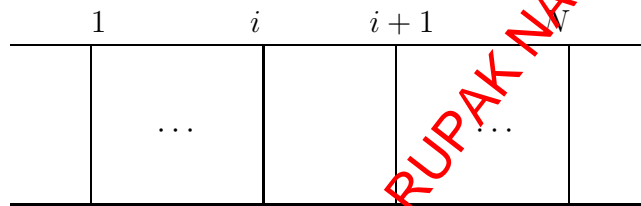
Depict g_i as



and g_i^{-1} as

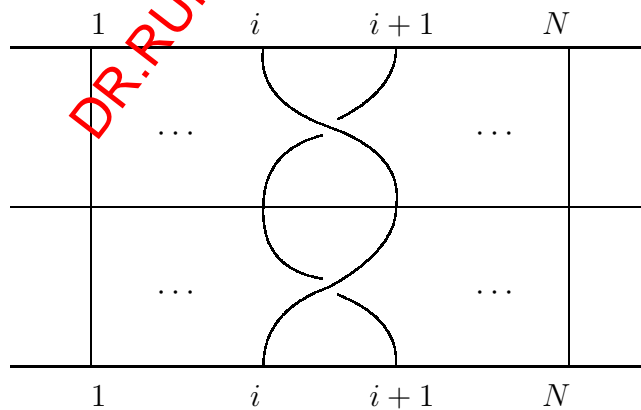


The group identity \mathbb{I} will be



and two diagrams that can be brought to coincide by “pulling the wires” will be considered as the same group element.

Using these graphical representations, one can prove the braid relations satisfied by the generators g_i . For example, it is easily seen that $g_i g_i^{-1} = \mathbb{I}$



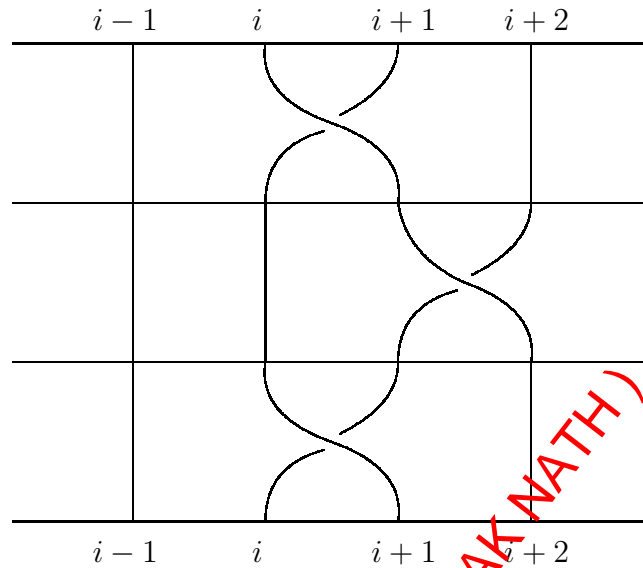
A quick look to the diagram of g_i^2 explains the origin of the name braid group.

We can also show the braid relation

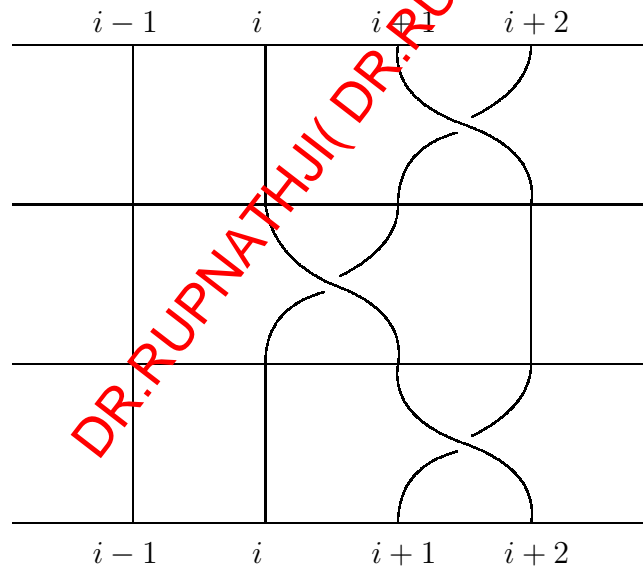
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \tag{4.7}$$

graphically.

Exploiting the graphical representation of g_i , the LHS becomes



while the RHS is



It is then straightforward to check the equality of the two sides by comparing the two graphical representations. We mentioned earlier in the text that representations of the braid group may provide solutions of the YBE. However, the braid group is too ‘big’ to be physical, hence we shall restrict ourselves to quotients of the braid group to search for solutions of the YBE [14, 15, 16].

Definition 3.2. The A -type Hecke algebra $H_N(q)$ is defined by the generators g_i , $i \in \{1, \dots, N-1\}$ and the braid relations presented above, plus an extra condition

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, & i \in \{1, \dots, N-2\}, \\ (g_i - q)(g_i + q^{-1}) &= 0, \\ [g_i, g_j] &= 0, & |i - j| > 1. \end{aligned} \quad (4.8)$$

It is clear that the Hecke algebra $H_N(q)$ is a quotient of the A -type braid group.

There is an alternative form of the Hecke algebra. Renaming the generators as $U_i = g_i - q$, we get

$$\begin{aligned} U_i U_{i+1} U_i - U_i &= U_{i+1} U_i U_{i+1} - U_{i+1} \\ U_i^2 &= -(q + q^{-1})U_i \\ [U_i, U_j] &= 0, & |i - j| > 0. \end{aligned} \quad (4.9)$$

You may check this as an exercise.

Definition 3.3. The Temperley-Lieb algebra $T_N(q)$ is a quotient of the Hecke algebra, and is defined by (4.9) and the additional requirement:

$$U_{i\pm 1} U_i U_{i\pm 1} = U_{i\pm 1}. \quad (4.10)$$

Exercise 3: The R -matrix for the XXZ spin chain is the following

$$R(\lambda) = \begin{pmatrix} \sinh(\lambda + i\mu) & 0 & 0 & 0 \\ 0 & \sinh \lambda & e^\lambda \sinh i\mu & 0 \\ 0 & e^{-\lambda} \sinh i\mu & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + i\mu) \end{pmatrix}. \quad (4.11)$$

Find \check{R} and show that it can be written in a form (up to an irrelevant overall factor)

$$\check{R} = e^\lambda g - e^{-\lambda} g^{-1}. \quad (4.12)$$

Show that if $U = g - q$ satisfies the Temperley-Lieb algebra, then \check{R} satisfies the Yang-Baxter equation.

Notice that the R matrix (4.11) is expressed in the so-called homogeneous gradation, there is also the principal gradation. The two are related via a simple gauge transformation as⁶

$$R_{12}^{(p)}(\lambda) = V_1(-\lambda) R_{12}^{(h)}(\lambda) V_1(\lambda), \quad V(\lambda) = \text{diag}(e^{\frac{\lambda}{2}}, e^{-\frac{\lambda}{2}}). \quad (4.14)$$

Baxterization: For any representation g_i of the A -type Hecke algebra we obtain a solution of the YBE, expressed as [18]:

$$\check{R}_{ii+1} = e^\lambda g_i - e^{-\lambda} g_i^{-1}. \quad (4.15)$$

Exercise 4: Suppose that the R -matrix has the form

$$\check{R}_{i,i+1}(\lambda) = a(\lambda)\mathbb{I} + b(\lambda)U_i, \quad (4.16)$$

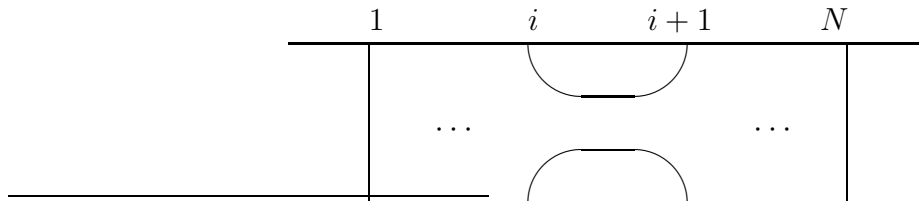
where $U_i \in H_N(q)$. Find $a(\lambda), b(\lambda)$ so that $\check{R}_{i,i+1}(\lambda)$ satisfies the Yang-Baxter equation.

Exercise 5: Show that

$$\begin{aligned} \left. \frac{d}{d\lambda} \check{R}_{12}^{XXZ}(\lambda) \right|_{\lambda=0} &\propto H_{12}^{XXZ} \\ &= \frac{1}{2}(\sigma_1^x \sigma_2^x + c_1 \sigma_1^y \sigma_2^y + \cosh(i\mu) \sigma_1^z \sigma_2^z + c_1(\sigma_1^z - \sigma_2^z) + c_2), \end{aligned} \quad (4.17)$$

that is, \check{R}^{XXZ} generates the XXZ spin chain Hamiltonian (up to an additive constant). Determine the constants c_1 and c_2 .

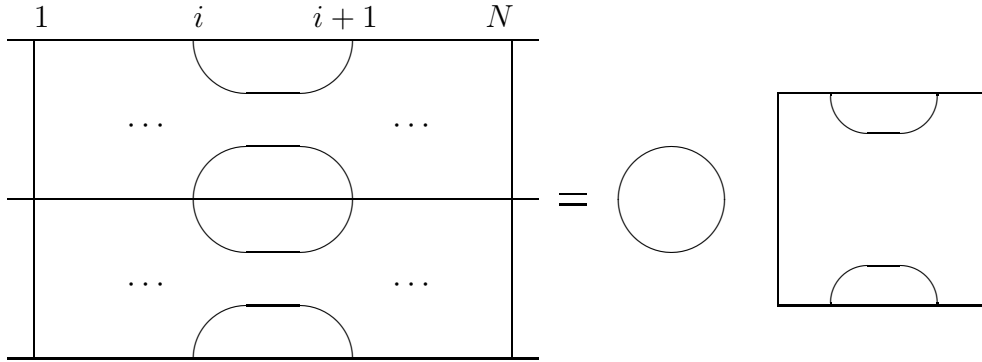
Graphical Representation of Temperley-Lieb algebra: just as in the case of the Hecke algebra, there is a nice graphical representation of the Temperley-Lieb algebra. The generator U_i is graphically depicted as:



⁶In general we have:

$$R_{12}^{(p)}(\lambda - \mu) = V_1(-\lambda) V_2(-\mu) R_{12}^{(h)}(\lambda) V_1(\lambda) V_2(\mu) \quad (4.13)$$

The relation $U_i^2 = -(q - q^{-1})U_i$ is represented as



The circle in the RHS of the graph above represents the constant $-(q + q^{-1})$.

Exercise 6: Prove the relation

$$U_i U_{i+1} U_i = U_i \quad (4.18)$$

by using the graphical representation of U_i .

Exercise 7: Consider the $n \times n$ matrix:

$$U = \sum_{i \neq j=1}^n (e_{ij} \otimes e_{ji} - q^{-\text{sgn}(i-j)} e_{ii} \otimes e_{jj}). \quad (4.19)$$

Show that it provides a representation of the Hecke algebra, $\pi : H_N(q) \hookrightarrow \text{End}((\mathbb{C}^n)^{\otimes N})$

$$\pi(U_i) = \mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \underbrace{U}_{i, i+1} \otimes \dots \otimes \mathbb{I}. \quad (4.20)$$

5 Quantum integrability

5.1 The quantum Lax operator

It is easy to verify that the XXX R -matrix may be expressed in terms of the spin $\frac{1}{2}$ representation of \mathfrak{sl}_2 . This observation gives us a motivation to introduce objects that are associated to higher representations of \mathfrak{sl}_2 . Of course such generalizations occur for any Lie algebra, but we use here the \mathfrak{sl}_2 algebra as a pedagogical example.

Take the R -matrix of the XXX spin chain [28],

$$R = \lambda \mathbb{I} + i\mathcal{P}. \quad (5.1)$$

The permutation operator may be also expressed as:

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sigma^z}{2} + \frac{1}{2} & \sigma^- \\ \sigma^+ & -\frac{\sigma^z}{2} + \frac{1}{2} \end{pmatrix}. \quad (5.2)$$

Inspired by the above form, we introduce a general matrix \mathbb{P} as

$$\mathbb{P} = \begin{pmatrix} J^z + \frac{1}{2} & J^- \\ J^+ & -J^z + \frac{1}{2} \end{pmatrix}, \quad (5.3)$$

where J^\pm, J^z are now abstract algebraic elements, which satisfy as will be clear below the \mathfrak{sl}_2 exchange relations (2.7). Now define the Lax operator L as

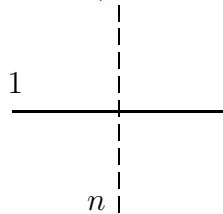
$$L = \lambda \mathbb{I} + i\mathbb{P}, \quad (5.4)$$

and assume L satisfies the following fundamental algebraic relation [2, 1, 5]

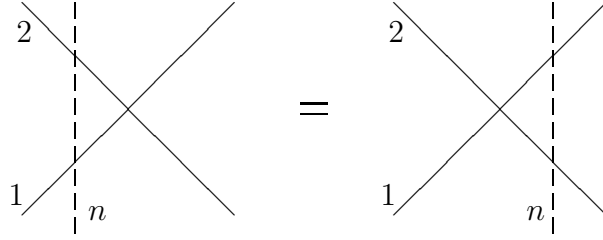
$$R_{12}(\lambda_1 - \lambda_2) L_{1n}(\lambda_1) L_{2n}(\lambda_2) = L_{2n}(\lambda_2) L_{1n}(\lambda_1) R_{12}(\lambda_1 - \lambda_2). \quad (5.5)$$

In (5.5) the indices 1, 2 traditionally denote the auxiliary space, and n denotes the quantum space on which the algebra generators act. In general $L \in \text{End}(V) \otimes \mathcal{A}$, where \mathcal{A} is the algebra defined by (5.5). Different choices of R matrix lead naturally to distinct algebras, as will be transparent later in the text.

Graphically, one represents the L operator as



The fundamental algebraic relation satisfied by L is then simply represented graphically as



For the particular choice of R matrix the YBE lives in: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{Y}$, with \mathcal{Y} being the Yangian of \mathfrak{sl}_2 defined by (5.5). The relation above holds for any R matrix associated to any Lie algebra, but we focus here in \mathfrak{sl}_2 for simplicity.

Substituting the explicit forms of R and L in (5.5), one finds that the following condition should hold

$$[\mathbb{P}_1, \mathbb{P}_2] = (\mathbb{P}_2 - \mathbb{P}_1)\mathcal{P}_{12} = \mathcal{P}_{12}(\mathbb{P}_1 - \mathbb{P}_2) \quad (5.6)$$

(here the quantum space for \mathbb{P} is omitted). It may be shown that the condition above leads to the \mathfrak{sl}_2 exchange relations.

Exercise 1: Consider \mathbb{P} defined in (5.3); based on relations (5.6) and notation (2.1) show that J^z , J^\pm satisfy the \mathfrak{sl}_2 exchange relations.

5.2 The q -deformed case: $U_q(\mathfrak{sl}_2)$

We come now to the q -deformed case, which corresponds to the XXZ spin chain and its generalizations. Take the R -matrix of the XXZ model (4.11), which can be also written as (homogeneous gradation)

$$R(\lambda) = e^\lambda R^+ - e^{-\lambda} R^-, \quad (5.7)$$

where R^+ , R^- are upper, lower triangular matrices:

$$R^+ = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad R^- = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -(q - q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \quad (5.8)$$

These matrices may be also written as

$$R^+ = \begin{pmatrix} q^{\frac{1}{2}(\sigma^z + 1)} & (q - q^{-1})\sigma^- \\ 0 & q^{\frac{1}{2}(-\sigma^z + 1)} \end{pmatrix}, \quad R^- = \begin{pmatrix} q^{-\frac{1}{2}(\sigma^z - 1)} & 0 \\ -(q - q^{-1})\sigma^+ & q^{\frac{1}{2}(\sigma^z - 1)} \end{pmatrix}. \quad (5.9)$$

Now suppose that the L -matrix can be also written in terms of upper/lower triangular matrices, giving rise as will be clear to upper/lower Borel subalgebras of $U_q(\mathfrak{sl}_2)$ [1, 5],

$$L = e^\lambda L^+ - e^{-\lambda} L^-, \quad (5.10)$$

where

$$L^+ = \begin{pmatrix} cA & B \\ 0 & cD \end{pmatrix}, \quad L^- = \begin{pmatrix} c^{-1}A^{-1} & 0 \\ C & c^{-1}D^{-1} \end{pmatrix}. \quad (5.11)$$

where c is an arbitrary constant. These matrices R, L obey (5.5). By taking various limits of it when $\lambda_i \rightarrow \pm\infty$, one arrives at the following set of equations:

- Limit $\lambda_1 - \lambda_2, \lambda_1, \lambda_2 \rightarrow \infty$

$$R_{12}^+ L_{1n}^+ L_{2n}^+ = L_{2n}^+ L_{1n}^+ R_{12}^+, \quad (5.12)$$

- Limit $\lambda_1, \lambda_2 \rightarrow -\infty$ and $\lambda_1 - \lambda_2 \rightarrow -\infty$

$$R_{12}^- L_{1n}^- L_{2n}^- = L_{2n}^- L_{1n}^- R_{12}^-, \quad (5.13)$$

- Limit $\lambda_1 \rightarrow \infty, \lambda_2 \rightarrow -\infty$

$$R_{12}^+ L_{1n}^+ L_{2n}^- = L_{2n}^- L_{1n}^+ R_{12}^+. \quad (5.14)$$

Note that the Lax operator can be also expressed in the principal gradation via:

$$L_{1n}^{(p)}(\lambda) = V_1(-\lambda) L_{1n}^{(h)}(\lambda) V_1(\lambda). \quad (5.15)$$

Exercise 2: Solve the above set of equations (5.12)-(5.13) and determine the various relations among A, B, C and D :

$$[B, C] = (q - q^{-1})(A^2 - D^2), \quad AD = DA = \mathbb{I}, \quad AB = qBA, \quad AC = q^{-1}CA. \quad (5.16)$$

By further imposing

$$A = q^{J^z} = D^{-1}, \quad B = (q - q^{-1})J^-, \quad C = (q - q^{-1})J^+, \quad (5.17)$$

these are in fact the relations of the deformed Lie group $U_q(SL_2)$. Show that the latter relations imply also:

$$\begin{aligned} [J^+, J^-] &= \frac{q^{2J^z} - q^{-2J^z}}{q - q^{-1}} \\ [J^z, J^\pm] &= \pm J^\pm. \end{aligned} \quad (5.18)$$

This is the so-called q -deformed \mathfrak{sl}_2 algebra denoted as $U_q(\mathfrak{sl}_2)$ [18]. Finally, taking the limit $q \rightarrow 1$ one recovers the familiar \mathfrak{sl}_2 relations. $U_q(\mathfrak{sl}_2)$ is a Hopf algebra and is equipped with a non-trivial co-product [18], $\Delta : U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ such that

$$\begin{aligned} \Delta(q^{\pm J^z}) &= q^{\pm J^z} \otimes q^{\pm J^z}, \\ \Delta(J^\pm) &= q^{-J^z} \otimes J^\pm + J^\pm \otimes q^{J^z}. \end{aligned} \quad (5.19)$$

More details on co-products of quantum algebras will be given in subsequent lecture.

Exercise 3: Show that the quantity

$$C = qq^{2J^z} + q^{-1}q^{-2J^z} + (q - q^{-1})^2 J^- J^+. \quad (5.20)$$

is the Casimir operator of $U_q(\mathfrak{sl}_2)$

Exercise 4: Show that if J^z, J^\pm satisfy the $U_q(\mathfrak{sl}_2)$ algebra then $\Delta(J^z), \Delta(J^\pm)$ also satisfy $U_q(\mathfrak{sl}_2)$.

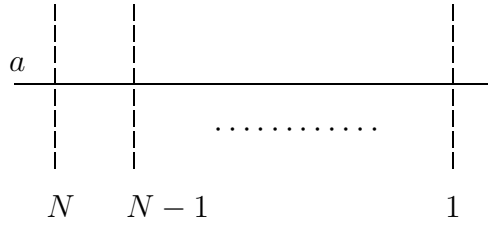
5.3 The transfer matrix: integrability

Our purpose now is to construct and solve 1-dimensional spin chain-like systems using the so-called quantum inverse scattering method (see e.g. [4, 5, 6, 7]). To achieve this we shall introduce tensor type representations of the fundamental algebraic relation (5.5).

We may introduce the so-called *monodromy* matrix, as

$$T_a(\lambda) = L_{aN}(\lambda) L_{aN-1}(\lambda) \cdots L_{a1}(\lambda). \quad (5.21)$$

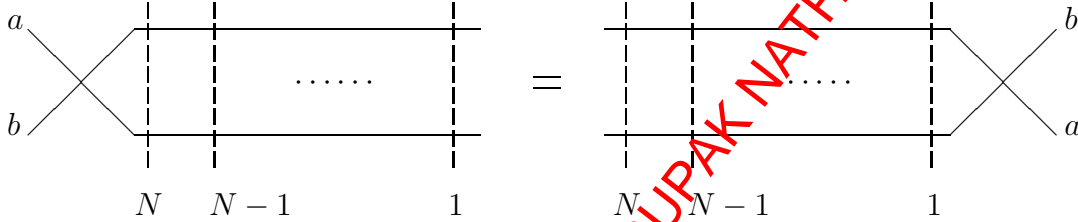
and apparently $T(\lambda) \in \text{End}(V) \otimes \mathcal{A}^{\otimes N}$. As customary we have suppressed the quantum spaces $1, \dots, N$ from the monodromy matrix. The monodromy matrix can be graphically represented as



and also satisfies the fundamental algebraic relation (FRT)

$$R_{12}(\lambda_1 - \lambda_2) T_1(\lambda_1) T_2(\lambda_2) = T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_1 - \lambda_2). \quad (5.22)$$

Graphically, the proof is immediate, if one takes into account the graphical representation of the $RLL = LLR$ relation, presented above.



On the other hand, FRT can be also proved algebraically, consider $N = 2$ for simplicity:

$$\begin{aligned} R_{ab} T_a T_b &= R_{ab} L_{a2} L_{a1} L_{b2} L_{b1} = R_{ab} L_{a2} L_{b2} L_{a1} L_{b1} \\ &= L_{b2} L_{a2} R_{ab} L_{a1} L_{b1} = L_{b2} L_{a2} L_{b1} L_{a1} R_{ab} \\ &= L_{b2} L_{b1} L_{a2} L_{a1} R_{ab} = T_b T_a R_{ab}. \end{aligned} \quad (5.23)$$

Tracing over the auxiliary space we get the *transfer matrix* $t(\lambda)$

$$t(\lambda) = \text{Tr}_a [T_a(\lambda)]. \quad (5.24)$$

and $t(\lambda) \in \mathcal{A}^{\otimes N}$. The transfer matrix constitutes a one-parameter family of commuting operators

$$[t(\lambda), t(\lambda')] = 0. \quad (5.25)$$

The proof goes as follows:

$$\begin{aligned} t(\lambda) t(\lambda') &= (\text{Tr}_a T_a(\lambda)) (\text{Tr}_b T_b(\lambda')) = \text{Tr}_{ab} [T_a(\lambda) T_b(\lambda')] \\ &= \text{Tr}_{ab} [R_{ab}(-\delta) R_{ab}(\delta) T_a(\lambda) T_b(\lambda')] \\ &= \text{Tr}_{ab} [R_{ba}(-\delta) T_b(\lambda') T_a(\lambda) R_{ab}(\delta)] \\ &= \text{Tr}_{ab} [T_b(\lambda') T_a(\lambda)] \end{aligned}$$

$$\begin{aligned}
&= (\mathrm{Tr}_b T_b(\lambda')) (\mathrm{Tr}_a T_a(\lambda)) \\
&= t(\lambda') t(\lambda).
\end{aligned} \tag{5.26}$$

This condition ensures that the system at hand is integrable.

The commutation relation (5.26) holds $\forall \lambda, \lambda' \in \mathbb{C}$, which implies that factors of formal series expansion commute with each other. Explicitly we have:

$$\sum_{n,m} \lambda^n \lambda'^m [t^{(n)}, t^{(m)}] = 0 \tag{5.27}$$

and this automatically yields:

$$[t^{(n)}, t^{(m)}] = 0 \quad \forall n, m. \tag{5.28}$$

It is clear that the elements $t^{(n)}$ are the so called charges in involution. Expansions based on other points are also possible. Below we shall derive the first two charges, i.e. the momentum and energy.

5.4 The momentum and the Hamiltonian

If we restrict our attention to the case where both auxiliary and quantum spaces are represented to the same vector space V , $L \leftrightarrow R$, then we obtain a local Hamiltonian as long as $R(0) \propto \mathcal{P}$. For instance in the $U_q(\mathfrak{sl}_2)$ case, when both auxiliary and quantum space correspond to the fundamental representation of the algebra, we deal with the familiar XXZ R -matrix (4.11) (we shall see a particular example below). Recall also, that since $R(0) \propto \mathcal{P}$ one may show that $t(0) \propto \Pi = \exp(-i\mathbb{P})$ where \mathbb{P} is the momentum of the system (3.15). To conclude, the momentum and Hamiltonian belong to the family of commuting operators obtained from the transfer matrix (see also [6]).

In particular we shall show

$$\left. \frac{d}{d\lambda} \ln(t(\lambda)) \right|_{\lambda=0} \propto H = \sum_{j=1}^N H_{jj+1}, \quad \text{where,} \quad H_{jj+1} \propto \left. \frac{d\check{R}_{jj+1}(\lambda)}{d\lambda} \right|_{\lambda=0} \tag{5.29}$$

imposing manifestly periodic boundary conditions: $H_{NN+1} = H_{N1}$.

Consider in general

$$R(0) \propto \mathcal{P} \quad (5.30)$$

Let us now compute:

$$\left. \frac{d \ln(t(\lambda))}{d\lambda} \right|_{\lambda=0} = t^{-1}(0)t'(0) \quad (5.31)$$

We now want to compute this expression in detail. First consider:

$$\begin{aligned} t(0) &= \text{Tr}_0 [R_{0N}(0) R_{0N-1}(0) \dots R_{01}(0)] \\ &\propto \text{Tr}_0 [\mathcal{P}_{0N} \mathcal{P}_{0N-1} \dots \mathcal{P}_{01}] \\ &= \text{Tr}_0 [\mathcal{P}_{N N-1} \mathcal{P}_{N N-2} \dots \mathcal{P}_{N1} \mathcal{P}_{0N}] \\ &= \mathcal{P}_{N N-1} \mathcal{P}_{N N-2} \dots \mathcal{P}_{N1} \text{Tr}_0 \mathcal{P}_{0N} \\ &= \mathcal{P}_{12} \mathcal{P}_{23} \dots \mathcal{P}_{N-1 N} = \Pi \end{aligned} \quad (5.32)$$

where we have used the fact that $\text{Tr}_0 \mathcal{P}_{0N} \propto \mathbb{I}$ for all known physical systems (for instance for all the solutions emanating from the Hecke algebras). Recall that $\Pi = e^{-i\mathbb{P}}$ is the translation operator and \mathbb{P} the momentum of the system. As a straightforward consequence we have:

$$\begin{aligned} t^{-1}(0) &\propto \mathcal{P}_{1N} \mathcal{P}_{1N-1} \dots \mathcal{P}_{13} \mathcal{P}_{12} \\ &= \mathcal{P}_{N-1 N} \mathcal{P}_{N-2 N-1} \dots \mathcal{P}_{23} \mathcal{P}_{12}. \end{aligned} \quad (5.33)$$

Now we have to compute the derivative of $t(\lambda)$, namely:

$$\begin{aligned} t'(0) &= \text{Tr}_0 \sum_i R_{0N}(0) \dots R'_{0i}(0) \dots R_{01}(0) = \\ &\propto \text{Tr}_0 \sum_i \mathcal{P}_{0N} \dots R'_{0i}(0) \dots \mathcal{P}_{01} \\ &= \text{Tr}_0 \sum_i \mathcal{P}_{N N-1} \dots \mathcal{P}_{N i+1} R'_{Ni}(0) \dots \mathcal{P}_{N1} \mathcal{P}_{0N} = \\ &= \sum_i \mathcal{P}_{N N-1} \dots \mathcal{P}_{N i+1} R'_{Ni}(0) \dots \mathcal{P}_{N1}. \end{aligned} \quad (5.34)$$

Collecting all previous results we can write down:

$$\begin{aligned} t^{-1}(0) t'(0) &\propto \sum_i \mathcal{P}_{N1} \mathcal{P}_{N2} \dots \mathcal{P}_{N N-2} \mathcal{P}_{N N-1} \mathcal{P}_{N N-1} \dots \mathcal{P}_{N i+1} \dots R'_{Ni} \dots \mathcal{P}_{N1} = \\ &= \dots \\ &= \sum_i \check{R}'_{i i+1}(0). \end{aligned} \quad (5.35)$$

We shall provide here as an example the Hamiltonian of the XXZ model. Let us recall the R -matrix of the XXZ chain (principal gradation):

$$R(\lambda) = \begin{pmatrix} \sinh(\lambda + i\mu) & 0 & 0 & 0 \\ 0 & \sinh \lambda & \sinh i\mu & 0 \\ 0 & \sinh i\mu & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + i\mu) \end{pmatrix} \quad (5.36)$$

Its derivative at the origin is

$$R'(0) = \begin{pmatrix} \cosh i\mu & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cosh i\mu \end{pmatrix} \quad (5.37)$$

and also

$$R(0) = \sinh i\mu \mathcal{P}, \quad R^{-1}(0) = (\sinh i\mu)^{-1} \mathcal{P} \quad (5.38)$$

therefore, recalling the expression of \mathcal{P} in (3.5), we get

$$\begin{aligned} R^{-1}(0) R'(0) &= \frac{1}{\sinh i\mu} \begin{pmatrix} \cosh i\mu & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cosh i\mu \end{pmatrix} \\ &\propto \frac{1}{2} (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \cosh i\mu (\mathbb{I} \otimes \mathbb{I} + \sigma^z \otimes \sigma^z)). \end{aligned} \quad (5.39)$$

Thus equation (5.31) becomes:

$$\begin{aligned} H &\propto \left. \frac{d \ln(t(\lambda))}{d\lambda} \right|_{\lambda=0} \\ &\propto -\frac{1}{2} \sum_i [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh i\mu (\sigma_i^z \sigma_{i+1}^z + \mathbb{I}_i \mathbb{I}_{i+1})]. \end{aligned} \quad (5.40)$$

This last expression corresponds to our XXZ Hamiltonian (3.18) (up to a constant), by fixing $\Delta = \cosh i\mu$. Such an observation is extremely important because thanks to (5.28) we can construct a series of conserved quantities, simply by computing the following derivatives:

$$\left. \frac{d^{(n)} \ln(t(\lambda))}{d^{(n)} \lambda} \right|_{\lambda=0} \propto H^{(n)}. \quad (5.41)$$

In the next section we will study the form and the properties of the ground state of the XXZ model, by introducing the Bethe Ansatz technique.

Exercise 5: Show that $[\Pi, H] = 0$.

6 Review on quantum algebras

6.1 Quantum algebras and non-trivial co-products

It will be instructive for what follows to examine two basic classes of deformed algebras arising in the context of integrable systems that is the so called Yangian and the q -deformed algebras. It is worth mentioning that the deformed algebras underlying any integrable system play an essential role in the context of algebraic Bethe ansatz for finding the associated spectra, as will be transparent in the following. Also, linear intertwining relations between the R matrix and co-products of the algebra elements can be used for the derivation of R matrices –we shall briefly discuss this issue later in the text. We shall focus here on these algebraic structures, and exhibit how their non-trivial co-products emerge naturally in the context of quantum integrability.

6.1.1 The Yangian

We shall first consider the Yangian (for a review on Yangians see e.g. [17, 30, 31]). The \mathfrak{gl}_n Yangian \mathcal{Y} , is a non abelian algebra with generators $Q_{ab}^{(p)}$ and defining relations given below

$$\begin{aligned}
 \left[Q_{ab}^{(0)}, Q_{cd}^{(0)} \right] &= i\delta_{cb}Q_{ad}^{(0)} - i\delta_{ad}Q_{cb}^{(0)} \\
 \left[Q_{ab}^{(0)}, Q_{cd}^{(1)} \right] &= i\delta_{cb}Q_{ad}^{(1)} - i\delta_{ad}Q_{cb}^{(1)} \\
 \left[Q_{ab}^{(1)}, Q_{cd}^{(1)} \right] &= i\delta_{cb}Q_{ad}^{(2)} - i\delta_{ad}Q_{cb}^{(2)} + \frac{ih^2}{4}Q_{ad}^{(0)}\left(\sum_e Q_{ce}^{(0)}Q_{eb}^{(0)}\right) - \frac{ih^2}{4}\left(\sum_e Q_{ae}^{(0)}Q_{ed}^{(0)}\right)Q_{cb}^{(0)} \\
 a, b \in \{1, 2, \dots, n\}
 \end{aligned} \tag{6.1}$$

and also relations

$$\begin{aligned}
 &\left[Q_{ab}^{(0)}, \left[Q_{cd}^{(1)}, Q_{ef}^{(1)} \right] \right] - \left[Q_{ab}^{(1)}, \left[Q_{cd}^{(0)}, Q_{ef}^{(1)} \right] \right] \\
 &= \frac{h^2}{4} \sum_{p,q} \left(\left[Q_{ab}^{(0)}, \left[Q_{cp}^{(0)}Q_{pd}^{(0)}, Q_{eq}^{(0)}Q_{qf}^{(0)} \right] \right] - \left[Q_{ap}^{(0)}Q_{pb}^{(0)}, \left[Q_{cd}^{(0)}, Q_{eq}^{(0)}Q_{qf}^{(0)} \right] \right]. \tag{6.2}
 \end{aligned}$$

The Yangian is endowed with a co-product $\Delta : \mathcal{Y} \hookrightarrow \mathcal{Y} \otimes \mathcal{Y}$ such that

$$\begin{aligned}\Delta(Q_{ab}^{(0)}) &= Q_{ab}^{(0)} \otimes \mathbb{I} + \mathbb{I} \otimes Q_{ab}^{(0)} \\ \Delta(Q_{ab}^{(1)}) &= Q_{ab}^{(1)} \otimes \mathbb{I} + \mathbb{I} \otimes Q_{ab}^{(1)} + \frac{\hbar}{2} \sum_{d=1}^n (Q_{ad}^{(0)} \otimes Q_{db}^{(0)} - Q_{db}^{(0)} \otimes Q_{ad}^{(0)}),\end{aligned}\quad (6.3)$$

Define also the opposite co-product $\Delta' : \mathcal{Y} \hookrightarrow \mathcal{Y}^{\otimes 2}$:

$$\Delta' = \sigma \circ \Delta \quad (6.4)$$

where σ is the ‘shift operator’, $\sigma : a \otimes b \hookrightarrow b \otimes a$. We may also define the l co-products $\Delta^{(l)}, \Delta'^{(l)} : \mathcal{Y} \hookrightarrow \mathcal{Y}^{\otimes l}$ as

$$\Delta^{(l)} = (\text{id} \otimes \Delta^{(l-1)})\Delta, \quad \Delta'^{(l)} = (\text{id} \otimes \Delta'^{(l-1)})\Delta'. \quad (6.5)$$

and obtain the l co-product.

The asymptotic behavior of the monodromy matrix T as $\lambda \rightarrow \infty$ provides tensor product realizations of \mathcal{Y} . Let us briefly review how this process works. Recall that the operators L and T are treated as $n \times n$ matrices with entries being elements of $\mathcal{Y}, \mathcal{Y}^{\otimes N}$ respectively. L in particular is given by (5.4) with $\mathbb{P}_{ab} \in \mathfrak{gl}_n$. The monodromy matrix T as $\lambda \rightarrow \infty$ may be written as (for simplicity we suppress the ‘auxiliary’ space index 0 from T in the following)

$$T(\lambda \rightarrow \infty) \propto \mathbb{I} + \sum_{m=0}^{\infty} \lambda^{-m-1} t^{(m)}. \quad (6.6)$$

Exchange relations among the charges $t_{ab}^{(m)}$ (the entries of $t^{(m)}$) may be derived by virtue of the fundamental algebraic relation (5.22), as $\lambda_i \rightarrow \infty$. To extract the Yangian generators we study the asymptotic expansion (6.6) keeping higher orders in the $\frac{1}{\lambda}$ expansion. Recalling the form of L we conclude that

$$T(\lambda \rightarrow \infty) \propto \mathbb{I} + \frac{i}{\lambda} \sum_{i=1}^N \mathbb{P}_{0i} - \frac{1}{\lambda^2} \sum_{i>j=1}^N \mathbb{P}_{0i} \mathbb{P}_{0j} + \mathcal{O}\left(\frac{1}{\lambda^3}\right). \quad (6.7)$$

Now consider the quantities below written as combinations of $t^{(p)}$, $p \in \{0, 1\}$,

$$\mathbb{Q}^{(0)} = t^{(0)}, \quad \mathbb{Q}^{(1)} = t^{(1)} - \frac{1}{2} \mathbb{Q}^{(0)} \mathbb{Q}^{(0)} \quad (6.8)$$

where the form of $t^{(p)}$ is defined by (6.6), (6.7). Then $\mathbb{Q}^{(p)}$ may be expressed as (set here $h = -1$ in (6.3))

$$\mathbb{Q}^{(0)} = i \sum_{i=1}^N \mathbb{P}_{0i}, \quad \mathbb{Q}^{(1)} = \frac{1}{2} \sum_{i=1}^N \mathbb{P}_{0i}^2 + \frac{1}{2} \sum_{j>i=1}^N (\mathbb{P}_{0i} \mathbb{P}_{0j} - \mathbb{P}_{0j} \mathbb{P}_{0i}). \quad (6.9)$$

Note that for simplicity both quantum and auxiliary indices in $\mathbb{Q}^{(p)}$ (6.9) are omitted. The entries of the matrices $\mathbb{Q}^{(p)}$ are the non-local charges $\mathbb{Q}_{ab}^{(p)} \in \mathcal{Y}^{\otimes N}$ being co-product realizations of the Yangian, i.e.

$$\mathbb{Q}_{ab}^{(p)} = \Delta^{(N)}(\mathbb{Q}_{ab}^{(p)}) \quad p \in \{0, 1\}, \quad a, b \in \{1, \dots, n\}. \quad (6.10)$$

6.1.2 The $U_q(\mathfrak{sl}_2)$ algebra

We saw in the preceding lecture that by taking appropriate limits of $\lambda \rightarrow \pm\infty$, one derives the $U_q(\mathfrak{sl}_2)$ relations. The monodromy matrix can be also used to derive the co-product for the q deformed case. We shall also derive here linear intertwining relations among the L , R matrices and the co-products of the associated deformed algebra.

First, take the limit $\lambda \rightarrow \infty$, $T(\lambda)$ becomes

$$\begin{aligned} T^+ &= L_{0N}^+ L_{0N-1}^+ \cdots L_{01}^+ \\ &= \begin{pmatrix} A_N & B_N \\ 0 & D_N \end{pmatrix} \begin{pmatrix} A_{N-1} & B_{N-1} \\ 0 & D_{N-1} \end{pmatrix} \cdots \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix}. \end{aligned} \quad (6.11)$$

Take $N = 2$ for simplicity, then

$$T^+ = \begin{pmatrix} A_2 A_1 & A_2 B_1 + B_2 D_1 \\ 0 & D_1 D_2 \end{pmatrix}. \quad (6.12)$$

By identifying

$$A = cq^{J^z}, \quad B = (q - q^{-1})J^-, \quad D = cq^{-J^z}, \quad (6.13)$$

one reads then from the elements of the monodromy matrix the co-products (5.19) (see also e.g. [5, 18]). If the co-product is known, one can also construct the N co-product, by iteration via (6.5).

By using these notations and ideas, we can derive important relations between L , R and the co-products of the algebra elements. Start from the fundamental algebraic relation (5.5), and take $\lambda_1 \rightarrow \infty$, $\lambda_2 \rightarrow \lambda$, to get

$$R_{12}^+ L_{1n}^+ L_{2n}(\lambda) = L_{2n}(\lambda) L_{1n}^+ R_{12}^+, \quad (6.14)$$

where

$$R_{12}^+ = \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & \delta_2 \end{pmatrix}, \quad L_{1n}^+ = \begin{pmatrix} A_n & B_n \\ 0 & D_n \end{pmatrix}. \quad (6.15)$$

Now α, β, δ are in the fundamental representation of $U_q(\mathfrak{sl}_2)$, while A, B, D are abstract elements of the algebra, that is

$$\begin{aligned} \alpha &= cq^{\frac{\sigma^z}{2}} & A &= cq^{J^z} \\ \delta &= cq^{-\frac{\sigma^z}{2}} & D &= cq^{-J^z} \\ \beta &= (q - q^{-1})\sigma^- & B &= (q - q^{-1})J^- \end{aligned} \quad (6.16)$$

Now define the representation $\pi : U_q(\mathfrak{sl}_2) \hookrightarrow \text{End}(\mathbb{C}^2)$ such as

$$\pi(J^z) = \frac{\sigma^z}{2}, \quad \pi(J^\pm) = \sigma^\pm. \quad (6.17)$$

and also

$$\pi(A) = \alpha, \quad \pi(B) = \beta, \quad \pi(D) = \delta. \quad (6.18)$$

Substituting in (6.14), we get

$$\begin{pmatrix} \alpha_2 A_n & \alpha_2 B_n + \beta_2 D_n \\ 0 & \delta_1 D_n \end{pmatrix} L_{2n}(\lambda) = L_{2n}(\lambda) \begin{pmatrix} \alpha_2 A_n & A_n \beta_2 + \delta_2 B_n \\ 0 & \delta_2 D_n \end{pmatrix}. \quad (6.19)$$

The latter suggests the L 'commutes' with each one of the entries of the right and left matrices. More precisely, the 'commutation' with the first entry of the matrices reads as

$$\alpha_2 A_n L_{2n}(\lambda) = L_{2n}(\lambda) \alpha_2 A_n, \quad (6.20)$$

or,

$$(q^{\frac{\sigma^z}{2}} \otimes q^{J^z}) L_{2n}(\lambda) = L_{2n}(\lambda) (q^{\frac{\sigma^z}{2}} \otimes q^{J^z}). \quad (6.21)$$

Since

$$\Delta(q^{J^z}) = q^{J^z} \otimes q^{J^z} \Rightarrow (\pi \otimes \text{id})\Delta(q^{J^z}) = q^{\frac{\sigma^z}{2}} \otimes q^{J^z}. \quad (6.22)$$

therefore (6.21) may be also expressed as:

$$(\pi \otimes \text{id})\Delta(q^{J^z}) L(\lambda) = L(\lambda) (\pi \otimes \text{id})\Delta(q^{J^z}). \quad (6.23)$$

Reading the second entry, we have

$$(\alpha_2 B_n + \beta_2 D_n) L_{2n}(\lambda) = L_{2n}(\lambda) (A_n \beta_2 + \delta_2 B_n). \quad (6.24)$$

Substituting the explicit forms of α , β , A , D ,

$$[\sigma^- \otimes q^{-J^z} + q^{\frac{\sigma^z}{2}} \otimes S^-] L_{2n}(\lambda) = L_{2n}(\lambda) [\sigma^- \otimes q^{J^z} + q^{-\frac{\sigma^z}{2}} \otimes S^-]. \quad (6.25)$$

It is easy to observe that the term inside the parenthesis on the RHS is just $(\pi \otimes \text{id})\Delta(J^-)$, while the LHS is $(\pi \otimes \text{id})\Delta'(J^-)$. Equation (6.14) then becomes

$$(\pi \otimes \text{id})\Delta'(J^-) L(\lambda) = L(\lambda) (\pi \otimes \text{id})\Delta(J^-). \quad (6.26)$$

From the asymptotics as $\lambda \rightarrow -\infty$ we obtain a relation similar to (6.26) for J^+ , so we conclude:

$$\begin{aligned} (\pi \otimes \text{id})\Delta'(X) L(\lambda) &= L(\lambda) (\pi \otimes \text{id})\Delta(X) \\ (\pi \otimes \pi)\Delta'(X) R(\lambda) &= R(\lambda) (\pi \otimes \pi)\Delta(X) \end{aligned} \quad X \in U_q(\mathfrak{sl}_2). \quad (6.27)$$

The second of the equations above suggests that the R -matrix satisfies linear intertwining relations with the co-products of the underlying quantum algebra. In fact, such types of linear exchange relations (6.27) may be used in order to extract R -matrices associated to particular quantum algebras (Yangian or affine q deformed algebras) see e.g. [18, 32]. Note that extra linear exchange relations involving the affine part of the associated quantum affine algebra are needed in order to fully identify the relevant R -matrix. Similar relations are obtained for the Yangian from the asymptotics of the FRT equation, but are left for the interested reader as exercise. Although this is a systematic and elegant means to solve the YBE, we shall not further pursue this issue in this article.

At this point, recall that $\check{R} = \mathcal{P}R$, and consequently

$$(\pi \otimes \pi)\Delta(X) \check{R}(\lambda) = \check{R}(\lambda) (\pi \otimes \pi)\Delta(X) \quad X \in U_q(\mathfrak{sl}_2). \quad (6.28)$$

This equation is very important since it shows that \check{R} commutes with the generators of $U_q(\mathfrak{sl}_2)$, in the co-product realization; \check{R} is called for obvious reasons the quantum invariant R matrix [18].

Exercise 1: Consider the \check{R} -matrix of the XXZ spin chain, and check explicitly that the above commutation relations (6.27) hold $\forall X \in U_q(\mathfrak{sl}_2)$.

7 Algebraic Bethe ansatz

7.1 $U_q(\mathfrak{sl}_2)$ representations

Let us briefly review here various representation of the $U_q(\mathfrak{sl}_2)$ algebra. First, recall the L -matrix (principal gradation)

$$L(\lambda) = \begin{pmatrix} e^\lambda A - e^{-\lambda} D & (q - q^{-1})B \\ (q - q^{-1})C & e^\lambda D - e^{-\lambda} A \end{pmatrix} \quad (7.1)$$

The spin s representation is $n = 2s + 1$ dimensional defined in terms of $n \times n$ matrices as:

$$\begin{aligned} A &= \sum_{k=1}^n q^{a_k} e_{kk} = D^{-1} && (\sim q^{S_z}) \\ B &= \sum_{k=1}^{n-1} \tilde{C}_k e_{k+1k} && (\sim S^-) \\ C &= \sum_{k=1}^{n-1} \tilde{C}_k e_{kk+1} && (\sim S^+), \end{aligned} \quad (7.2)$$

where

$$\begin{aligned} a_k &= q^{\frac{n+1}{2} - k}, & \tilde{C}_k &= \sqrt{[k]_q [n-k]_q}, \\ & [k]_q &= \frac{q^k - q^{-k}}{q - q^{-1}}. \end{aligned} \quad (7.3)$$

Exercise 1: Prove that A, B, C, D satisfy $U_q(SL_2)$.

The Heisenberg-Weyl group: let us first introduce the Heisenberg-Weyl group, defined by elements \mathbb{X}, \mathbb{Y} that satisfy

$$\mathbb{X} \mathbb{Y} = q \mathbb{Y} \mathbb{X}. \quad (7.4)$$

All the entries of the L matrix may be then expressed in terms of the Heisenberg-Weyl elements as follows:

$$A = D^{-1} = \mathbb{X}, \quad B = \frac{1}{q - q^{-1}} (q^{-s} \mathbb{X}^{-1} - q^s \mathbb{X}) \mathbb{Y}^{-1}, \quad C = \frac{1}{q - q^{-1}} (q^{-s} \mathbb{X} - q^s \mathbb{X}^{-1}) \mathbb{Y}. \quad (7.5)$$

Let us also focus for a moment on the special case where q is root of unity, i.e. $q^p = 1$, $q = e^{i\mu}$, $\mu = \frac{2k\pi}{p}$, where k, p integers. In this case the algebra admits a p dimensional representation, known as the cyclic representation [33]. More specifically, one more restriction is applied so one may obtain a representation with no highest (lowest) weight

$$\mathbb{X}^p = \mathbb{Y}^p = 1$$

then the generators \mathbb{X} , \mathbb{Y} may be expressed as p dimensional matrices

$$\mathbb{X} = \sum_{k=1}^p q^{-k} e_{kk}, \quad \mathbb{Y} = \sum_{k=1}^{p-1} e_{k, k+1} + e_{p1}. \quad (7.6)$$

Sine-Gordon and Liouville models: in what follows we shall briefly review how the lattice sine-Gordon [34] and Liouville models [35] are obtained in a natural way from the XXZ L matrix. Also the q harmonic oscillator realization will be obtained from the generalized XXZ form. The generators \mathbb{X} and \mathbb{Y} may be associated with an infinite dimensional representation in terms of some lattice ‘fields’. Consider $\mathbb{Z} = \mathbb{X}\mathbb{Y}$, then \mathbb{Z} also satisfies

$$\mathbb{X}\mathbb{Z} = q\mathbb{Z}\mathbb{X}. \quad (7.7)$$

Parametrizing

$$\mathbb{X}_n = e^{-i\Phi_n}, \quad \mathbb{Z}_n = e^{i\Pi_n}, \quad (7.8)$$

where apparently Φ_n, Π_n are canonical

$$[\Phi_n, \Pi_m] = i\mu\delta_{nm}, \quad q^{s-1/2} = -im. \quad (7.9)$$

The parameter s of the representation is associated to the mass scale of the system. Also by multiplying by $im\sigma^x$ (we are allowed to multiply with σ^x because this leaves the XXZ R matrix invariant) one obtains the lattice sine-Gordon L matrix [34]

$$L_{an}^{S.G} = \begin{pmatrix} h_+(\phi_n)e^{i\Pi_n} & -2im \sinh(\lambda + i\Phi_n) \\ -2im \sinh(\lambda - i\Phi_n) & h_-(\Phi_n)e^{-i\Pi_n} \end{pmatrix}, \quad (7.10)$$

where

$$h_{\pm}(\Phi_n) = 1 + m^2 e^{\pm 2i\Phi_n + i\mu}. \quad (7.11)$$

Consider also the following limiting process [35]

$$i\Phi_n \rightarrow i\Phi_n + c, \quad \mu\lambda \rightarrow \mu\lambda + c, \quad m \rightarrow 0, \quad e^{-c} \rightarrow \infty, \quad m^2 e^{-2c} \rightarrow \alpha^2 \quad (7.12)$$

one obtains the lattice Liouville L matrix

$$L_{an}^{Lv}(\lambda) = \begin{pmatrix} e^{i\Pi_n} & \alpha e^{-\mu\lambda - i\Phi_n} \\ 2\alpha \sinh(\mu\lambda - i\Phi_n) & h(\Phi_n) e^{-i\Pi_n} \end{pmatrix} \quad (7.13)$$

$$h(\Phi_n) = 1 + \alpha^2 e^{-2i\Phi_n + i\mu}.$$

The interesting observation is that the entailed L operator (7.13) has a non trivial spectral (λ) dependence a fact that allows the application of Bethe ansatz techniques for the derivation of the spectrum (see also [35]).

The classical limit of the aforementioned L matrices gives the corresponding classical Lax operators satisfying the zero curvature condition, and giving rise to the classical equations of motion of the relevant models. Let us briefly review the connection between the quantum (lattice) versions and the classical sine-Gordon and Liouville models. Consider the following classical limit [34, 35], the spacing $\alpha \rightarrow 0$, set $\mu \rightarrow h\mu$ such that $\frac{1}{h}[\cdot] \rightarrow \{\cdot\}$, and

$$\Phi_n \rightarrow \frac{\beta}{2}\phi(x) - \frac{\pi}{2}, \quad \Pi_n \rightarrow \alpha \frac{\beta}{4}\pi(x), \quad \tilde{m} \rightarrow \alpha \tilde{m}, \quad \mu\lambda \rightarrow u + \frac{i\pi}{2} \quad (7.14)$$

\tilde{m} is the continuum mass and β corresponds to the coupling constant of the sine Gordon model, and for the Liouville model we set $\frac{\beta}{2} = 1$, following the normalization of [35]. Bearing in mind the expressions above we obtain as $\alpha \rightarrow 0$

$$L(u) = 1 - \alpha U(u) + \mathcal{O}(\alpha^2) \quad (7.15)$$

then the quantities $U(u)$ written below provides the Lax operator for the classical continuum counterparts of the lattice sine-Gordon and Liouville models. More precisely for the sine Gordon model:

$$U(u) = \frac{1}{2} \begin{pmatrix} -i\frac{\beta}{2}\pi(x) & -\tilde{m} \sinh(u + i\frac{\beta}{2}\phi(x)) \\ \tilde{m} \sinh(u - i\frac{\beta}{2}\phi(x)) & i\frac{\beta}{2}\pi(x) \end{pmatrix}, \quad (7.16)$$

whereas for the Liouville model the Lax operator reads

$$U(u) = \frac{1}{2} \begin{pmatrix} -i\pi(x) & -2e^{-u - i\phi(x)} \\ 4 \sinh(u - i\phi(x)) & i\pi(x) \end{pmatrix}. \quad (7.17)$$

The Lax operators satisfy the classical analogue of the fundamental relation [36]. More precisely, U satisfies classical linear exchange relations described in [36]. We shall not

further discuss this topic here given that is beyond the intended scope of the present article.

Similar limiting process to (7.12) leads to the q -harmonic oscillator L matrix starting from (7.1) (see also [37]). In fact, by simply multiplying the Liouville L matrix with an anti-diagonal matrix we obtain the following

$$L_{an}(\lambda) = \begin{pmatrix} e^{\mu\lambda}V_n - e^{-\mu\lambda}V_n^{-1} & a_n^+ \\ a_n & -e^{-\mu\lambda}V_n \end{pmatrix} \quad (7.18)$$

where the operators a_n , a_n^+ , V_n are expressed in terms of \mathbb{X}_n , \mathbb{Y}_n as

$$V_n = \mathbb{X}_n, \quad a_n^+ = (\mathbb{X}_n^{-1} - q\mathbb{X}_n)\mathbb{Y}_n^{-1}, \quad a_n = \mathbb{Y}_n \mathbb{X}_n \quad (7.19)$$

and they satisfy the q harmonic oscillator algebra i.e.

$$a_n^+ a_n = 1 - qV_n^2, \quad a_n a_n^+ = 1 - q^{-1}V_n^2. \quad (7.20)$$

7.2 Algebraic Bethe ansatz

Having introduced all the necessary algebraic setting we are now in a position to describe the algebraic Bethe ansatz method. This can be basically applied for representations of Lie and deformed Lie algebras with highest (lowest) weight. For representations with no highest (lowest) weight the method can be applied with certain modifications, which however will not be discussed here (see e.g.[3]). We shall extract below the spectrum and Bethe ansatz equations for the whole hierarchy of the spin s representations of $U_q(\mathfrak{sl}_2)$.

The main objective within QISM is the diagonalization of the transfer matrix. This will be achieved by means of the algebraic Bethe ansatz method [2, 6, 7]. We shall essentially exploit the exchange relations emanating from the fundamental algebraic relation (5.22) in order to determine the spectrum of the transfer matrix as well as the corresponding eigenstates. Recall that the transfer matrix is given by

$$t(\lambda) = \text{Tr}_0[L_{0N}(\lambda) \cdots L_{01}(\lambda)] = \text{Tr}_0 T_0(\lambda), \quad (7.21)$$

where

$$L_{0n}(\lambda) = \begin{pmatrix} \sinh(\lambda + \frac{i\mu}{2} + i\mu J^z) & \sinh i\mu J^- \\ \sinh i\mu J^+ & \sinh(\lambda + \frac{i\mu}{2} - i\mu J^z) \end{pmatrix}. \quad (7.22)$$

The first step is to determine a reference state also called the “pseudo-vacuum” in the anti-ferromagnetic case. Let $|\omega\rangle$ be the state annihilated by J^+ and $|\Omega\rangle$ be the tensor product of N such states:

$$\begin{aligned} J_n^+ |\omega\rangle_n &= 0 \\ |\Omega\rangle &= |\omega\rangle \otimes \cdots \otimes |\omega\rangle. \end{aligned} \quad (7.23)$$

Applying then $T(\lambda)$ to $|\Omega\rangle$ we get rid of the C 's, since they annihilate the state

$$\begin{aligned} T(\lambda)|\Omega\rangle &= \begin{pmatrix} A_N & B_N \\ C_N & D_N \end{pmatrix} \cdots \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} |\omega\rangle \otimes \cdots \otimes |\omega\rangle \\ &= \begin{pmatrix} A_N & B_N \\ 0 & D_N \end{pmatrix} \cdots \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} |\omega\rangle \otimes \cdots \otimes |\omega\rangle \\ &= \begin{pmatrix} A_N \cdots A_1 & \mathcal{B} \\ 0 & D_N \cdots D_1 \end{pmatrix} |\Omega\rangle \\ &= \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{D} \end{pmatrix} |\Omega\rangle. \end{aligned} \quad (7.24)$$

The exact form of \mathcal{B} is not required, since we are going to trace over the monodromy matrix, so only $\mathcal{A} + \mathcal{D}$ is needed. However, the action of A_n (or D_n in the same spirit) on $|\omega\rangle$ is known, and is just

$$\sinh\left(\lambda + \frac{i\mu}{2} \pm i\mu s\right) |\omega\rangle_n = \sinh\left(\lambda + \frac{i\mu}{2} \pm i\mu s\right) |\omega\rangle_n. \quad (7.25)$$

Thus we conclude that the action of \mathcal{A} and \mathcal{D} on $|\Omega\rangle$ is

$$\begin{aligned} \mathcal{A}|\Omega\rangle &= \sinh\left(\lambda + \frac{i\mu}{2} + i\mu s\right)^N |\Omega\rangle \\ \mathcal{D}|\Omega\rangle &= \sinh\left(\lambda + \frac{i\mu}{2} - i\mu s\right)^N |\Omega\rangle. \end{aligned} \quad (7.26)$$

The action of the transfer matrix on our pseudo-vacuum is then known

$$\begin{aligned} t(\lambda)|\Omega\rangle &= (\mathcal{A} + \mathcal{D})|\Omega\rangle \\ &= \left(\sinh\left(\lambda + \frac{i\mu}{2} + i\mu s\right)^N + \sinh\left(\lambda + \frac{i\mu}{2} - i\mu s\right)^N \right) |\Omega\rangle. \end{aligned} \quad (7.27)$$

The next step is to make the following ansatz for a general Bethe state $|\Psi\rangle$:

$$|\Psi\rangle = \mathcal{B}(\lambda_1) \cdots \mathcal{B}(\lambda_M) |\Omega\rangle. \quad (7.28)$$

We would like to find the action of $\mathcal{A} + \mathcal{D}$ on $|\Psi\rangle$. Since we already know how \mathcal{A} and \mathcal{D} act on $|\Omega\rangle$, we only need to determine the exchange relations between \mathcal{A} , \mathcal{D} and \mathcal{B} .

As discussed the monodromy matrix satisfies (5.22), with the R -matrix being the XXZ matrix:

$$R_{12}(\lambda) = \begin{pmatrix} \sinh(\lambda + i\mu) & 0 & 0 & 0 \\ 0 & \sinh \lambda & \sinh i\mu & 0 \\ 0 & \sinh i\mu & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + i\mu) \end{pmatrix}. \quad (7.29)$$

After some algebra, the fundamental algebraic relation gives the commutation relations between \mathcal{A} , \mathcal{D} and \mathcal{B} . For example, one derives ($\delta = \lambda_1 - \lambda_2$):

$$\begin{aligned} [\mathcal{A}(\lambda_1), \mathcal{A}(\lambda_2)] &= [\mathcal{B}(\lambda_1), \mathcal{B}(\lambda_2)] = [\mathcal{C}(\lambda_1), \mathcal{C}(\lambda_2)] \\ a(\delta)\mathcal{B}(\lambda_1)\mathcal{A}(\lambda_2) &= b(\delta)\mathcal{A}(\lambda_2)\mathcal{B}(\lambda_1) + c\mathcal{B}(\lambda_2)\mathcal{A}(\lambda_1) \\ a(\delta)\mathcal{B}(\lambda_2)\mathcal{D}(\lambda_1) &= b(\delta)\mathcal{D}(\lambda_1)\mathcal{B}(\lambda_2) + c\mathcal{B}(\lambda_1)\mathcal{D}(\lambda_2). \end{aligned} \quad (7.30)$$

The last terms on the RHS of the last two equations are “unwanted”. Acting with $(\mathcal{A}(\lambda) + \mathcal{D}(\lambda))$ on $|\Psi\rangle$ we have

$$\begin{aligned} (\mathcal{A}(\lambda) + \mathcal{D}(\lambda))|\Psi\rangle &= (\mathcal{A}(\lambda) + \mathcal{D}(\lambda))\mathcal{B}(\lambda_1) \cdots \mathcal{B}(\lambda_M)|\Omega\rangle \\ &= \left[\frac{a(-\delta_1)}{b(-\delta_1)} \cdots \frac{a(-\delta_M)}{b(-\delta_M)} \mathcal{B}(\lambda_1) \cdots \mathcal{B}(\lambda_M)\mathcal{A}(\lambda) + (\cdots) \right. \\ &\quad \left. + \frac{a(\delta_1)}{b(\delta_1)} \cdots \frac{a(\delta_M)}{b(\delta_M)} \mathcal{B}(\lambda_1) \cdots \mathcal{B}(\lambda_M)\mathcal{D}(\lambda) + (\cdots) \right] |\Omega\rangle, \end{aligned} \quad (7.31)$$

where $\delta_i = \lambda - \lambda_i$, and the (\cdots) stand for the “unwanted” terms. One sees that if these terms vanish, then $|\Psi\rangle$ is an eigenstate of the transfer matrix, with known eigenvalues.

Since we know how \mathcal{A} and \mathcal{D} act on $|\Omega\rangle$, we may write then the above equation as

$$\begin{aligned} t(\lambda)|\Psi\rangle &= \left[\prod_{i=1}^M \frac{\sinh(\lambda - \lambda_i - i\mu)}{\sinh(\lambda - \lambda_i)} \sinh(\lambda + i\mu s + \frac{i\mu}{2})^N \right. \\ &\quad \left. + \prod_{i=1}^M \frac{\sinh(\lambda - \lambda_i + i\mu)}{\sinh(\lambda - \lambda_i)} \sinh(\lambda - i\mu s + \frac{i\mu}{2})^N \right. \\ &\quad \left. + (\cdots \cdots) \right] |\Psi\rangle. \end{aligned} \quad (7.32)$$

It is therefore relevant to examine the conditions for the unwanted terms to vanish. In fact, this is merely true for some values of λ , which are denoted by λ_i and are called

“Bethe roots”, and satisfy the following set of equations

$$\left(\frac{\sinh(\lambda_i + i\mu s)}{\sinh(\lambda_i - i\mu s)} \right)^N = \prod_{i \neq j}^M \frac{\sinh(\lambda_i - \lambda_j + i\mu)}{\sinh(\lambda_i - \lambda_j - i\mu)}, \quad (7.33)$$

which are called *Bethe Ansatz Equations* (BAE). As long as λ_i 's satisfy the BAE the unwanted terms vanish. Moreover, these equations guarantee the analyticity of the eigenvalues, and provide all the physical information regarding the considered system.

Exercise 2: Work out the details of the procedure for the XXZ model and convince yourself that the unwanted terms really do vanish.

Having obtained the eigenvalues of the transfer matrix, which we now denote as $\Lambda(\lambda)$, one can obtain the energy and momentum eigenvalues of the system. The energy is proportional to (see also [6, 7]) (we focus now on the spin 1/2 representation)

$$E \propto \left. \frac{d}{d\lambda} \ln \Lambda(\lambda) \right|_{\lambda=0}, \quad (7.34)$$

and for the XXZ spin-1/2 model is found to be

$$E = -\frac{1}{2\pi} \sum_{j=1}^M \frac{\mu \sinh i\mu}{\sinh(\lambda_j + \frac{i\mu}{2}) \sinh(\lambda_j - \frac{i\mu}{2})}, \quad (7.35)$$

while the momentum is proportional to $\ln \Lambda(0)$ and for the XXZ spin-1/2 model is found to be

$$P = -\sum_{j=1}^M \ln \frac{\sinh(\lambda_j + \frac{i\mu}{2})}{\sinh(\lambda_j - \frac{i\mu}{2})}. \quad (7.36)$$

Finally, the Bethe states are \mathfrak{sl}_2 highest weight states and are also eigenstates of J^z with eigenvalue

$$J^z = \frac{N}{2} - M. \quad (7.37)$$

8 Reflection equation and open boundaries

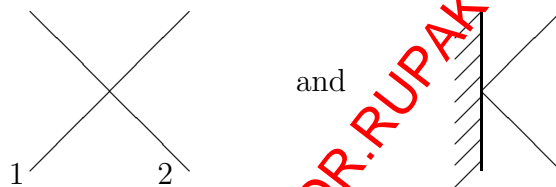
8.1 The reflection equation

So far we have just considered systems with periodic boundary conditions. To incorporate generic boundary conditions that still preserve integrability we have to deal with another

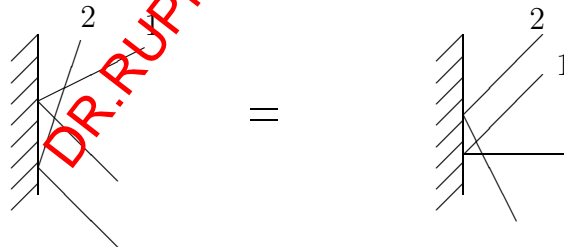
quadratic algebra called the reflection algebra. In this context open spin chain like systems may be also considered, that is spin chains with non trivial boundaries attached at their edges (see e.g. [20, 19, 38]). The starting point is to introduce a description of scattering of particles on a boundary, compatible with the bulk consistency relations discussed in Lecture 3. The boundary scattering is described by the so called reflection matrix, which satisfies the basic equation called *reflection equation* (RE), or *boundary Yang-Baxter* [20, 19] equation

$$R_{12}(\lambda_1 - \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)K_2(\lambda_2) = K_2(\lambda_2)R_{12}(\lambda_1 + \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 - \lambda_2), \quad (8.1)$$

where the R -matrix obeys the Yang-Baxter equation introduced in earlier lecture. The K -matrix contains all the information about the reflection of the particle on the boundary. Graphically, the R and K matrices are represented as



respectively. Using this graphical representation, the reflection equation can be represented as



In a number of situations the “reflection” matrix K will indeed encapsulate boundary conditions on the spin chain derived from the corresponding transfer matrices. The situation may be subtler in other instances, i.e. dynamical reflection algebras, but we shall not further comment on these cases.

We can now find solutions of the reflection equation of the form

$$K(\lambda) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (8.2)$$

where $\alpha, \beta, \gamma, \delta$ are c-numbers. This way, one may obtain several c-numbers solutions of the reflection equation. For instance the generic non-diagonal solution for the XXZ (sine-Gordon model) found in [39, 40] in the homogeneous gradation

$$K(\lambda) = \begin{pmatrix} \sinh(-\lambda + i\xi)e^\lambda & \kappa \sinh(2\lambda) \\ \kappa \sinh(2\lambda) & \sinh(\lambda + i\xi)e^{-\lambda} \end{pmatrix}. \quad (8.3)$$

The K matrix in the principal gradation may be obtained via a gauge transformation

$$K^{(p)}(\lambda) = V(-\lambda) K^{(h)}(\lambda) V(-\lambda) \quad (8.4)$$

However, one would also like to find solutions where the elements of \mathbb{K} are now operators and not c-numbers. These type of operatorial solutions are of the generic form [19]

$$\mathbb{K}(\lambda) = L(\lambda) K^-(\lambda) L^{-1}(-\lambda), \quad (8.5)$$

where $K^-(\lambda)$ is a c-number solution and $L(\lambda)$ satisfies (5.5) We also define the corresponding modified monodromy matrix [19]

$$\mathbb{T}(\lambda) = T(\lambda) K^-(\lambda) T^{-1}(-\lambda), \quad (8.6)$$

where $T(\lambda)$ is the already known monodromy matrix

$$T(\lambda) = L_{0N}(\lambda) \cdots L_{01}(\lambda) \quad (8.7)$$

Moreover, we introduce K^+ as

$$K^+(\lambda) = MK^t(-\lambda - i\rho), \quad (8.8)$$

where ρ is the crossing parameter and for the $\mathfrak{gl}_n, U_q(\mathfrak{gl}_n)$ cases is $\rho = \frac{n}{2}$, t stands for transposition, and K is a solution of the reflection equation. Also M is in general a diagonal matrix such that

$$\left[R_{12}(\lambda), M_1 M_2 \right] = 0 \quad (8.9)$$

and for the XXZ model in particular

$$\begin{aligned} M &= \mathbb{I} && \text{principal gradation,} \\ M &= \text{diag}(q, q^{-1}) && \text{homogeneous gradation.} \end{aligned} \quad (8.10)$$

This way, we are able to define the open transfer matrix as

$$t(\lambda) = \text{Tr}_0[K^+(\lambda) \mathbb{T}(\lambda)] = \text{Tr}_0[K^+(\lambda) T(\lambda) K^-(\lambda) T^{-1}(-\lambda)]. \quad (8.11)$$

One can show then that the integrability condition, that is

$$[t(\lambda), t(\lambda')] = 0. \quad (8.12)$$

8.2 Solutions of the RE and B -type braid group

Just as in the case of the closed spin chains, we can systematically search for solutions of the reflection equation by exploiting the similarity of the latter with the B -type braid group (see e.g [21, 22, 23, 24] and references within).

Definition 8.1. *The B -type braid group consists of $N - 1$ generators g_i , which satisfy the already known braid relations*

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \quad i \in \{1, \dots, N - 2\}, \\ [g_i, g_j] &= 0, \quad |i - j| > 1, \end{aligned} \quad (8.13)$$

plus an additional generator g_0 which satisfies

$$\begin{aligned} g_0 g_1 g_0 g_1 &= g_1 g_0 g_1 g_0, \\ [g_i, g_0] &= 0, \quad i > 1. \end{aligned} \quad (8.14)$$

The similarity of the latter relation with the ‘modified’ reflection equation

$$\check{R}_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) \check{R}_{12}(\lambda_1 + \lambda_2) K_1(\lambda_2) = K_1(\lambda_2) \check{R}_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) \check{R}_{12}(\lambda_1 - \lambda_2) \quad (8.15)$$

suggests that finding representations of the B -type braid group is equivalent to finding solutions of the RE. However the B -type braid group is too big to be physical, therefore we shall restrict our attention to certain quotients, which will be introduced below.

Definition 8.2. *The affine Hecke algebra is defined by generators g_i , g_0 that satisfy the B -type conditions plus*

$$(g_i - q)(g_i + q^{-1}) = 0, \quad (\text{Hecke condition}). \quad (8.16)$$

Definition 8.3. *The Cyclotomic algebra is a quotient of the affine Hecke algebra obtained by imposing the extra constraint*

$$\prod_{i=1}^n (g_0 - \gamma_i) = 0, \quad (8.17)$$

where γ_i are free parameters.

Definition 8.4. *The B-type Hecke algebra $B_N(q, Q)$ is a quotient of the affine Hecke algebra satisfying the extra condition:*

$$(g_0 - Q_1)(g_0 - Q_2) = 0, \quad (8.18)$$

where usually Q_2 is taken to be $-Q_1^{-1}$.

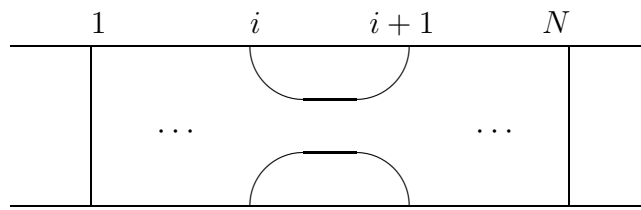
For the next definition it is convenient to introduce the following alternative generators:

$$\begin{aligned} U_0 &= g_0 - Q, \\ U_i &= g_i - q, \end{aligned} \quad (8.19)$$

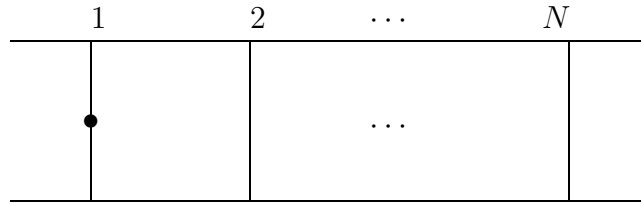
Definition 8.5. *The boundary Temperley-Lieb (blob) algebra $\mathcal{B}_N(q, Q)$ with generators U_i, U_0 is a quotient of the B-type Hecke algebra with exchange relations*

$$\begin{aligned} U_{i\pm 1} U_i U_{i\pm 1} &= U_{i\pm 1} \\ U_i^2 &= -(q + q^{-1})U_i \\ U_1 U_0 U_1 &= \kappa U_1, \quad \kappa = qQ^{-1} + q^{-1}Q \\ U_0^2 &= -(Q + Q^{-1})U_0. \end{aligned} \quad (8.20)$$

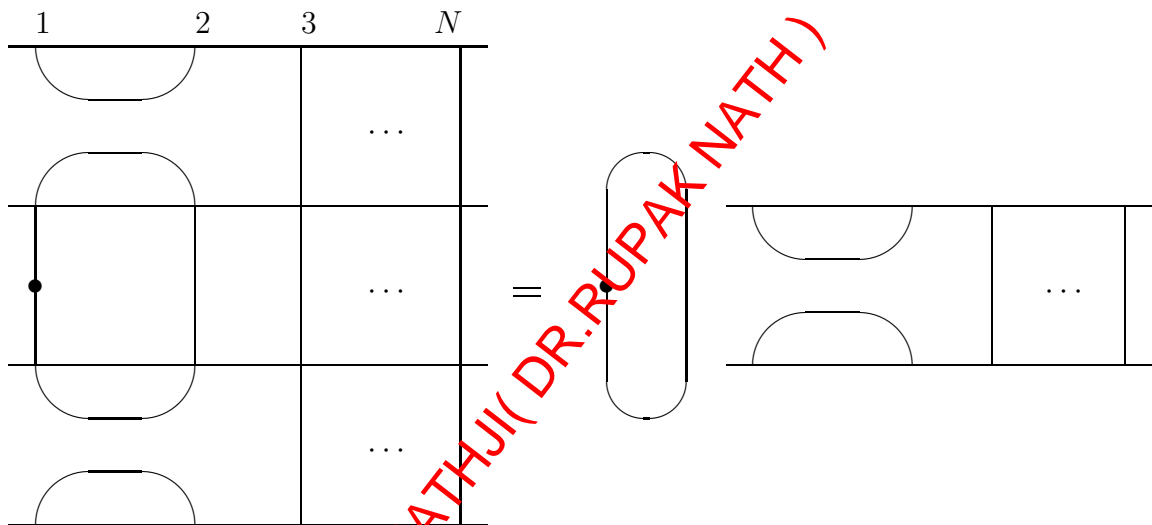
Graphical Representation of the boundary Temperley-Lieb algebra: recall the generator U_i is defined as



Depict also U_0 as



The relation $U_1 U_0 U_1 = \kappa U_1$ is represented then as



The closed line with the dot in the RHS of the graph above stands for the constant κ .

Exercise 1: Consider the matrices

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} -Q^{-1} & c \\ c^{-1} & -Q \end{pmatrix}. \quad (8.21)$$

Show that they provide a representation of the boundary Temperley-Lieb algebra, $\pi : \mathcal{B}_N(q, Q) \hookrightarrow \text{End}((\mathbb{C}^2)^{\otimes N})$

$$\begin{aligned} \pi(U_i) &= \mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \underbrace{U}_{i, i+1} \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I} \\ \pi(U_0) &= \underbrace{e}_{1^{st}} \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}. \end{aligned} \quad (8.22)$$

Exercise 2: Take the definition of the transfer matrix for the open spin chains with boundaries. Suppose also that

$$\begin{aligned}\hat{R}_{ii+1} &= a(\lambda)U_i + b(\lambda) \\ K^- &= x(\lambda)\mathbb{I} + y(\lambda)U_0 \\ K^+ &= M \quad \text{and} \quad \text{tr}_0[M_0\check{R}_{N0}(0)] \propto \mathbb{I}\end{aligned}\tag{8.23}$$

where

$$\begin{aligned}y(\lambda) &= 2 \sinh i\mu \sinh 2\lambda \\ x(\lambda) &= (Q + Q^{-1}) \cosh(2\lambda + i\mu) - \cosh(2i\mu\gamma) - \kappa \cosh(2\lambda),\end{aligned}\tag{8.24}$$

with $Q = ie^{i\mu m}$, γ free parameters. Show that (see also [24, 41])

$$H = \left. \frac{d}{d\lambda} t(\lambda) \right|_{\lambda=0} \propto \sum_{i=1}^N U_i + c_1 U_0 + c_2.\tag{8.25}$$

8.3 The $U_q(\mathfrak{sl}_2)$ invariant spin chain

We shall now deal with the $U_q(\mathfrak{sl}_2)$ invariant open spin chain [25] (see also relevant discussion on quantum symmetries in general, and associated properties in [42]). We shall exhibit the symmetry of the spin chain following a logic quite different from the one presented in the original work of [25]. Consider the simplest case of boundary conditions, that is $K^- \propto \mathbb{I}$ and $K^+ = M$ (8.10).

We shall focus here on the homogeneous gradation. Recall that in this case the L matrix may be decomposed in upper/lower triangular matrices giving rise to upper lower Borel sub-algebras in a natural way. The tensor representation of the reflection equations is

$$\mathbb{T}(\lambda) = T(\lambda) \hat{T}(\lambda).\tag{8.26}$$

Recall the linear algebraic relations (6.27). It is then clear that

$$(\pi \otimes \text{id}^{\otimes N}) \Delta'^{(N+1)}(X) T(\lambda) = T(\lambda) (\pi \otimes \text{id}^{\otimes N}) \Delta^{(N+1)}(X), \quad X \in U_q(\mathfrak{sl}_2).\tag{8.27}$$

Also it is easy to verify, recalling the form of \hat{T} , that

$$(\pi \otimes \text{id}^{\otimes N}) \Delta^{(N+1)}(X) \hat{T}(\lambda) = \hat{T}(\lambda) (\pi \otimes \text{id}^{\otimes N}) \Delta'^{(N+1)}(X).\tag{8.28}$$

Combining the relations above and bearing in mind the form of \mathbb{T} for the particular choice of boundaries ($K \propto \mathbb{I}$) we conclude:

$$(\pi \otimes \text{id}^{\otimes N})\Delta^{(N+1)}(X) \mathbb{T}(\lambda) = \mathbb{T}(\lambda) (\pi \otimes \text{id}^{\otimes N})\Delta^{(N+1)}(X), \quad X \in U_q(\mathfrak{sl}_2). \quad (8.29)$$

Let us first consider $X = J^z$ then the relation above becomes

$$\left(\frac{\sigma^z}{2} \otimes \mathbb{I} + \mathbb{I} \otimes \Delta^{(N)}(J^z)\right) \mathbb{T}(\lambda) = \mathbb{T}(\lambda) \left(\frac{\sigma^z}{2} \otimes \mathbb{I} + \mathbb{I} \otimes \Delta^{(N)}(J^z)\right). \quad (8.30)$$

Multiply both sides with $M \otimes \mathbb{I}$ and take the trace over the auxiliary space then obtain

$$\begin{aligned} [\Delta^{(N)}(J^z), \text{tr}_0\{M_0\mathbb{T}_0(\lambda)\}] &= \frac{1}{2}(\text{tr}_0\{\sigma_0^z\mathbb{T}_0(\lambda)\} - \text{tr}_0\{\mathbb{T}_0(\lambda)\sigma_0^z\}) = 0 \\ \Rightarrow [\Delta^{(N)}(J^z), t(\lambda)] &= 0. \end{aligned} \quad (8.31)$$

Equation (8.30) can be equivalently expressed as

$$q^{\pm \frac{\sigma^z}{2}} \otimes \Delta^{(N)}(q^{\pm J^z}) \mathbb{T}(\lambda) = \mathbb{T}(\lambda) q^{\pm \frac{\sigma^z}{2}} \otimes \Delta^{(N)}(q^{\pm J^z}), \quad (8.32)$$

and this form is more convenient in what follows.

Consider now $X = J^\pm$ then (8.29) becomes

$$\left(q^{\frac{\sigma^z}{2}} \otimes \Delta^{(N)}(J^\pm) + \sigma^\pm \otimes \Delta(q^{-J^z})\right) \mathbb{T}(\lambda) = \mathbb{T}(\lambda) \left(q^{\frac{\sigma^z}{2}} \otimes \Delta^{(N)}(J^\pm) + \sigma^\pm \otimes \Delta(q^{-J^z})\right) \quad (8.33)$$

Multiply both sides with $Mq^{-\frac{\sigma^z}{2}} \otimes \mathbb{I}$ then the latter expression becomes

$$\begin{aligned} \left[\Delta^{(N)}(J^\pm), M\mathbb{T}(\lambda)\right] &= \\ M\mathbb{T}(\lambda)q^{-\frac{\sigma^z}{2}}\sigma^\pm \otimes \Delta^{(N)}(q^{-J^z}) - Mq^{-\frac{\sigma^z}{2}}\sigma^\pm \otimes \Delta(q^{-J^z})\mathbb{T}(\lambda). \end{aligned} \quad (8.34)$$

After taking the trace over the auxiliary trace, bearing in mind (8.32), and appropriately moving the elements within the trace we conclude

$$\left[\Delta^{(N)}(J^\pm), t(\lambda)\right] = 0, \quad (8.35)$$

and with this we conclude our proof on the $U_q(\mathfrak{sl}_2)$ symmetry of the transfer matrix. The proof can be easily generalized for any higher rank quantum algebra.

In general, special choice of K -matrix may suitably break down the symmetry, and depending on the structure of the reflection matrix just part of the algebra or particular combinations of the algebra elements may commute with the transfer matrix (see e.g.

[43, 44, 45, 41]). Although this is a particularly interesting issue we shall not further discuss it here.

To obtain the local Hamiltonian we focus on the case where both auxiliary and quantum spaces are represented by the fundamental representation of the considered Lie algebra, that is $L(\lambda) \leftrightarrow R(\lambda)$. The Hamiltonian in this case is obtained from the derivative of the transfer matrix, and has the universal form in terms of the Temperley-Lieb generators (see also Exercise 2), recall (4.16)

$$H \propto \sum_{i=1}^{N-1} U_i, \quad (8.36)$$

U_i is any TL algebra representation. Here we are focusing on $U_q(\mathfrak{sl}_2)$ (XXZ model), but the later expression of the local Hamiltonian (8.36) is universal, i.e. independent of the choice of the representation of the Temperley-Lieb algebra. For the XXZ chain ($U_q(\mathfrak{sl}_2)$) in particular the Hamiltonian may be expressed in terms of Pauli matrices as:

$$H \propto \frac{1}{2} \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(i\mu) \sigma_i^z \sigma_{i+1}^z \right) - \frac{\sinh(i\mu)}{2} (\sigma_N^z - \sigma_1^z) - (N-1) \frac{\cosh(i\mu)}{2} \quad (8.37)$$

and it is manifestly $U_q(\mathfrak{sl}_2)$ invariant [46].

We shall finally identify in a simple manner the quadratic Casimir operators of $U_q(\mathfrak{sl}_2)$. Focus for simplicity on the case where $N = 1$, then the asymptotic behavior of \mathbb{T}

$$\mathbb{T}^\pm = L^\pm \hat{L}^\pm \quad (8.38)$$

More precisely (recall $\hat{L}(\lambda) = L^{-1}(-\lambda)$)

$$\mathbb{T}^+ = \begin{pmatrix} q^{\frac{1}{2}+c} q^{J^z} & (q - q^{-1}) J^- \\ 0 & q^{\frac{1}{2}+c} q^{-J^z} \end{pmatrix} \begin{pmatrix} q^{\frac{1}{2}-c} q^{J^z} & 0 \\ (q - q^{-1}) J^+ & q^{\frac{1}{2}-c} q^{-J^z} \end{pmatrix} \quad (8.39)$$

and

$$\mathbb{T}^- = \begin{pmatrix} q^{-\frac{1}{2}-c} q^{-J^z} & 0 \\ (q^{-1} - q) J^+ & q^{-\frac{1}{2}-c} q^{J^z} \end{pmatrix} \begin{pmatrix} q^{-\frac{1}{2}+c} q^{-J^z} & (q^{-1} - q) J^- \\ 0 & q^{-\frac{1}{2}+c} q^{J^z} \end{pmatrix} \quad (8.40)$$

then the asymptotics of the transfer matrix provides the quadratic Casimir operators of $U_q(\mathfrak{sl}_2)$

$$t^\pm = \text{tr}\{M\mathbb{T}^\pm\} \Rightarrow$$

$$\begin{aligned}
t^+ &\propto qq^{2J^z} + q^{-1}q^{2J^z} + (q - q^{-1})^2 J^- J^+ \\
t^- &\propto q^{-1}q^{2J^z} + qq^{-2J^z} + (q - q^{-1})^2 J^+ J^-.
\end{aligned} \tag{8.41}$$

It is clear that for generic N all algebra elements $X \rightarrow \Delta^{(N)}(X)$, $X \in U_q(\mathfrak{sl}_2)$. It is thus clear that the study of the asymptotic behavior of a spin chain may provide in a very simple and elegant manner all the Casimir operators associated to any Lie or deformed Lie algebra (see also [37, 47] for more details).

Exercise 3: Consider the XXZ representation of the Temperley-Lieb algebra, $\rho : T_N(q) \hookrightarrow \text{End}((\mathbb{C}^2)^{\otimes N})$ such that

$$\rho(U_i) = \mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \underbrace{U}_{i, i+1} \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}, \tag{8.42}$$

where the matrix U is defined in (8.21); consider also the spin $\frac{1}{2}$ representation of $U_q(\mathfrak{sl}_2)$. Show that:

$$[\rho(U_i), \Delta^{(N)}(x)] = 0, \quad x \in \{\sigma^\pm, q^{\frac{\sigma^z}{2}}\}, \tag{8.43}$$

i.e. the $U_q(\mathfrak{sl}_2)$ algebra is central to the Temperley-Lieb algebra.

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