

Recently, the theory of symmetric spaces has come to play an increased role in the physics of integrable systems and in quantum transport problems. In addition, it provides a classification of random matrix theories. In this paper we give a self-contained introduction to symmetric spaces and their main characteristics. We take an algebraic approach; therefore it is not necessary to know almost anything about differential geometry to be able to follow the outline.

DR.RUPNATHJIK(DR.RUPAK NATH)

1 Introduction

Recently, the study of symmetric spaces has gained renewed interest in physics. This is mainly due to two developments. The first of these connects random matrix theories to symmetric spaces, and provides a new classification of the former. The second development started in the eighties with the work of Olshanetsky and Perelomov [1], who demonstrated the deep connection between some quantum integrable systems and root systems of Lie algebras.

The connection that has emerged between random matrix theories and symmetric spaces was mentioned already by Dyson [2]. The integration manifolds defining the symmetry classes of the random matrix ensembles are usually symmetric spaces with positive or negative curvature. In contrast, the integration manifold could also be the *algebra* of matrices spanning a symmetric space of zero curvature. These issues will be discussed below and in more detail in a forthcoming paper [3].

Although Dyson was the first to recognize that the integration manifolds in random matrix theory actually are symmetric spaces, the subsequent emergence of new random matrix symmetry classes and their classification in terms of Cartan's symmetric spaces is relatively recent [4, 5, 6, 7, 8]. Until recently, random matrix ensembles used for physical applications were known to correspond to ten of the eleven classes of symmetric spaces in Cartan's classification. In a recent paper, Ivanov [8] found realizations of the remaining class, the algebra $\mathbf{SO}(2\mathbf{n} + 1)$ as well as $\mathbf{SO}(4\mathbf{n} + 2)/\mathbf{U}(2\mathbf{n} + 1)$ in disordered vortices in p -wave superconductors (this author splits the Cartan class DIII into DIII-even and DIII-odd; in that case there are twelve Cartan classes and these algebras correspond to B and DIII-odd). Thereby each Cartan class is realized in some physical system.

In the early eighties, Olshanetsky and Perelomov showed that these same symmetric spaces, through their root systems, are related to integrable Calogero–Sutherland models. In [1] the authors demonstrated that it is the symmetry of the underlying root systems that make these models integrable for certain values of the coupling constants in the Calogero–Sutherland potential. These special values are determined by the multiplicities of the various types of roots (long, ordinary, and short) in the corresponding root system. This deep and beautiful connection between Lie algebras and integrability of many-particle systems is described in [1]. In addition, the same authors showed that the dynamics of these quantum integrable systems is related to free diffusion on symmetric spaces.

The above scenario was applied to the physics of disordered wires in [9], where it was demonstrated that the symmetry class of the random matrix ensemble used in modeling the transfer matrix of a disordered wire determines a particular Calogero–Sutherland model, through the above mentioned common connection to a particular symmetric space. The

Calogero–Sutherland Hamiltonian so defined maps onto the radial part of the Laplace–Beltrami operator on the underlying symmetric space [1]. This is the connection to the dynamics of the quantum system mentioned in the preceding paragraph. The Laplace–Beltrami operator defines free diffusion on the symmetric space. The equation of free diffusion becomes the Dorokhov–Mello–Pereyra–Kumar (DMPK) equation, a differential equation describing the evolution of the transmission eigenvalues of the disordered wire with increasing length of the wire. As a consequence, it was demonstrated in [9, 10] that known properties of the underlying symmetric space and of the integrable Calogero–Sutherland model can be exploited in solving the DMPK equation for a disordered wire.

In a forthcoming publication [3] we will discuss the present status of the various applications of symmetric spaces to transport problems and in the field of integrable systems, indicating some new directions of research. In addition, we will discuss the applications of random matrix theory and the classification of the ensembles of random matrix theory implied by the Cartan classification of symmetric spaces.

The theory of symmetric spaces has a long history in mathematics. Here we will give a brief, self-contained introduction to symmetric spaces, listing in the process some references that are more easily accessible to physicists than the standard reference, the book by Helgason [11]. The reader is referred to this book for a rigorous treatment. We will concentrate on the issues that will be of relevance in our forthcoming paper [3]. For physicists with little background in differential geometry we recommend the book by Gilmore [12] (especially Chapter 9) for an introduction of exceptional clarity. Our treatment will be somewhat rigorous; however, we skip proofs that can be found in the mathematical literature and concentrate on simple examples that illustrate the concepts presented.

In section 2, after reviewing the basics about Lie groups, we will present some of the most important properties of root systems. In section 3 we define symmetric spaces and discuss their main characteristics: defining involutive automorphisms, spherical decomposition of the group elements, and the metric on the Lie algebra. We also discuss the algebraic structure of the coset space.

In section 4 we show how to obtain all the real forms of a complex semisimple Lie algebra. The same techniques will then be used to classify the real forms of the symmetric spaces in section 5. In this section we also define the curvature of a symmetric space, and discuss triplets of symmetric spaces with positive, zero and negative curvature, all corresponding to the same symmetric subgroup. We will see why curved symmetric spaces arise from semisimple groups, whereas the flat spaces are associated to non-semisimple groups. In addition, in section 5 we will define restricted root systems. The restricted root systems are associated to symmetric spaces, just like ordinary root systems are associated to groups. As we will discuss in detail in [3], they are key objects when considering the integrability of Calogero–Sutherland models.

In the last section of the paper we will discuss Casimir and Laplace operators on symmetric spaces and mention some known properties of the eigenfunctions of the latter, so called zonal spherical functions. These functions play a prominent role in many physical applications. In every section we work out several simple examples that illustrate the material presented. This paper contains the basis for understanding the developments to be discussed in more detail in [3].

2 Lie groups and root spaces

In this introductory section we define the basic concepts relating to Lie groups. We will build on the material presented here when we discuss symmetric spaces in the next main section of the paper. The reader with a solid background in group theory may want to skip most or all of this section.

2.1 Lie groups and manifolds

A manifold can be thought of as the generalization of a surface, but we do not in general consider it as embedded in a higher-dimensional euclidean space. A short introduction to differentiable manifolds can be found in ref. [13], and a more elaborate one in refs. [14] and [15] (Ch. III). The points of an N -dimensional manifold can be labelled by real coordinates (x^1, \dots, x^N) . Suppose that we take an open set U_α of this manifold, and we introduce local real coordinates on it. Let ψ_α be the function that attaches N real coordinates to each point in the open set U_α . Suppose now that the manifold is covered by overlapping open sets, with local coordinates attached to each of them. If for each pair of open sets U_α, U_β , the function $\psi_\alpha \circ \psi_\beta^{-1}$ is differentiable in the overlap region $U_\alpha \cap U_\beta$, it means that we can go smoothly from one coordinate system to another in this region. Then the manifold is differentiable.

Consider a group G acting on a space V . We can think of G as being represented by matrices, and of V as a space of vectors on which these matrices act. A group element $g \in G$ transforms the vector $v \in V$ into $gv = v'$.

If G is a Lie group, it is also a differentiable manifold. The fact that a Lie group is a differentiable manifold means that for two group elements $g, g' \in G$, the product $(g, g') \in G \times G \rightarrow gg' \in G$ and the inverse $g \rightarrow g^{-1}$ are smooth (C^∞) mappings, that is, these mappings have continuous derivatives of all orders.

Example: The space \mathbf{R}^n is a smooth manifold and at the same time an abelian group. The “product” of elements is addition $(x, x') \rightarrow x + x'$ and the inverse of x is $-x$. These operations are smooth.

Example: The set $GL(n, R)$ of nonsingular real $n \times n$ matrices M , $\det M \neq 0$, with matrix multiplication $(M, N) \rightarrow MN$ and multiplicative matrix inverse $M \rightarrow M^{-1}$ is a non-abelian group manifold. Any such matrix can be represented as $M = e^{\sum_i t^i X_i}$ where X_i are generators of the $\mathbf{GL}(n, \mathbf{R})$ algebra and t^i are real parameters.

2.2 The tangent space

In each point of a differentiable manifold, we can define the tangent space. If a curve through a point P in the manifold is parametrized by $t \in \mathbf{R}$

$$x^a(t) = x^a(0) + \lambda^a t \quad a = 1, \dots, N \quad (1)$$

where $P = (x^1(0), \dots, x^N(0))$, then $\lambda = (\lambda^1, \dots, \lambda^N) = (\dot{x}^1(0), \dots, \dot{x}^N(0))$ is a tangent vector at P . Here $\dot{x}^a(0) = \left. \frac{d}{dt} x^a(t) \right|_{t=0}$. The space spanned by all tangent vectors at P is the tangent space. In particular, the tangent vectors to the coordinate curves (the curves obtained by keeping all the coordinates fixed except one) through P are called the natural basis for the tangent space.

Example: In euclidean 3-space the natural basis is $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$. On a patch of the unit 2-sphere parametrized by polar coordinates it is $\{\hat{e}_\theta, \hat{e}_\phi\}$.

For a Lie group, the tangent space at the origin is spanned by the generators, that play the role of (contravariant) vector fields (also called derivations), expressed in local coordinates on the group manifold as $X = X^a(x) \partial_a$ (for an introduction to differential geometry see ref. [16], Ch. 5, or [15]). Here the partial derivatives $\partial_a = \frac{\partial}{\partial x^a}$ form a basis for the vector field. That the generators span the tangent space at the origin can easily be seen from the exponential map. Suppose X is a generator of a Lie group. The exponential map then maps X onto e^{tX} , where t is a parameter. This mapping is a one-parameter subgroup, and it defines a curve $x(t)$ in the group manifold. The tangent vector of this curve at the origin is then

$$\left. \frac{d}{dt} e^{tX} \right|_{t=0} = X \quad (2)$$

All the generators together span the tangent space at the origin (the identity element).

2.3 Coset spaces

The isotropy subgroup G_{v_0} of a group G at the point $v_0 \in V$ is the subset of group elements that leave v_0 fixed. The set of points that can be reached by applying elements $g \in G$ to v_0 is the orbit of G at v_0 , denoted Gv_0 . If $Gv_0 = V$ for one point v_0 , then this is true for every $v \in V$. We then say that G acts transitively on V .

In general, a symmetric space can be represented as a coset space. Suppose H is a subgroup of a Lie group G . The coset space G/H is the set of subsets of G of the form gH , for $g \in G$. G acts on this coset space: $g_1(gH)$ is the coset $(g_1g)H$.

If G acts transitively on V , then $V = Gv$ for any $v \in V$. Since the isotropy subgroup G_{v_0} leaves a fixed point v_0 invariant, $gG_{v_0}v_0 = gv_0$, $v \in V$, we see that the action of the group G on V defines a bijective action of elements of G/G_{v_0} on V . Therefore the space V on which G acts transitively, can be identified with G/G_{v_0} , since there is one-to-one correspondence between the elements of V and G/G_{v_0} . There is a natural mapping from the group element g onto the point gv_0 on the manifold.

Example: The $SO(2)$ subgroup of $SO(3)$ is the isotropy subgroup at the north pole of a unit 2-sphere imbedded in 3-dimensional space, since it keeps the north pole fixed. On the other hand, the north pole is mapped onto any point on the surface of the sphere by elements of the coset $SO(3)/SO(2)$. This can be seen from the explicit form of the coset representatives. As we will see in eq. (54) in subsection 3.5, the general form of the elements of the coset is

$$M = \exp \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{I_2 - XX^T} & X \\ -X^T & \sqrt{1 - X^T X} \end{pmatrix} \quad (3)$$

where C is the matrix

$$C = \begin{pmatrix} t^2 \\ t^1 \end{pmatrix} \quad (4)$$

and t^1, t^2 are real coordinates. I_2 in eq. (3) is the 2×2 unit matrix. For the coset space $SO(3)/SO(2)$, M is equal to

$$M = \exp\left(\sum_{i=1}^2 t^i L_i\right), \quad L_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (5)$$

The third $SO(3)$ generator

$$L_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6)$$

spans the algebra of the stability subgroup $SO(2)$, that keeps the north pole fixed:

$$\exp(t^3 L_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (7)$$

The generators L_i ($i = 1, 2, 3$) satisfy the $SO(3)$ commutation relations $[L_i, L_j] = \frac{1}{2}\epsilon_{ijk}L_k$. Note that since the L_i and the t^i are real, $C^\dagger = C^T$.

In (3), M is a general representative of the coset $SO(3)/SO(2)$. By expanding the exponential we see that the explicit form of M is

$$M = \begin{pmatrix} 1 + (t^2)^2 \frac{(\cos\sqrt{(t^1)^2+(t^2)^2}-1)}{(t^1)^2+(t^2)^2} & t^1 t^2 \frac{(\cos\sqrt{(t^1)^2+(t^2)^2}-1)}{(t^1)^2+(t^2)^2} & t^2 \frac{\sin\sqrt{(t^1)^2+(t^2)^2}}{\sqrt{(t^1)^2+(t^2)^2}} \\ t^1 t^2 \frac{(\cos\sqrt{(t^1)^2+(t^2)^2}-1)}{(t^1)^2+(t^2)^2} & 1 + (t^1)^2 \frac{(\cos\sqrt{(t^1)^2+(t^2)^2}-1)}{(t^1)^2+(t^2)^2} & t^1 \frac{\sin\sqrt{(t^1)^2+(t^2)^2}}{\sqrt{(t^1)^2+(t^2)^2}} \\ -t^2 \frac{\sin\sqrt{(t^1)^2+(t^2)^2}}{\sqrt{(t^1)^2+(t^2)^2}} & -t^1 \frac{\sin\sqrt{(t^1)^2+(t^2)^2}}{\sqrt{(t^1)^2+(t^2)^2}} & \cos\sqrt{(t^1)^2+(t^2)^2} \end{pmatrix} \quad (8)$$

Thus the matrix $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is given in terms of the components of C by (cf. eq. (55)):

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^2 \frac{\sin\sqrt{(t^1)^2+(t^2)^2}}{\sqrt{(t^1)^2+(t^2)^2}} \\ t^1 \frac{\sin\sqrt{(t^1)^2+(t^2)^2}}{\sqrt{(t^1)^2+(t^2)^2}} \end{pmatrix} \quad (9)$$

Defining now $z = \cos\sqrt{(t^1)^2 + (t^2)^2}$, we see that the variables x, y, z satisfy the equation of the 2-sphere:

$$x^2 + y^2 + z^2 = 1 \quad (10)$$

When the coset space representative M acts on the north pole it is easily seen that the orbit is all of the 2-sphere:

$$M \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & x \\ \cdot & \cdot & y \\ \cdot & \cdot & z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (11)$$

This shows that there is one-to-one correspondence between the elements of the coset and the 2-sphere. The coset $SO(3)/SO(2)$ can therefore be identified with a unit 2-sphere imbedded in 3-dimensional space.

2.4 The Lie algebra and the adjoint representation

A Lie algebra \mathbf{G} is a vector space over a field F . Multiplication in the Lie algebra is given by the bracket $[X, Y]$. It has the following properties:

- [1] If $X, Y \in \mathbf{G}$, then $[X, Y] \in \mathbf{G}$,
- [2] $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z]$ for $\alpha, \beta \in F$,
- [3] $[X, Y] = -[Y, X]$,
- [4] $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (the Jacobi identity).

The algebra \mathbf{G} generates a group through the exponential mapping. A general group element is

$$M = \exp \left(\sum_i t^i X_i \right); \quad t^i \in F, X_i \in \mathbf{G} \quad (12)$$

We define a mapping $\text{ad}X$ from the Lie algebra to itself by $\text{ad}X : Y \rightarrow [X, Y]$. The mapping $X \rightarrow \text{ad}X$ is a representation of the Lie algebra called the adjoint representation. It is easy to check that it is an automorphism: it follows from the Jacobi identity that $[\text{ad}X_i, \text{ad}X_j] = \text{ad}[X_i, X_j]$. Suppose we choose a basis $\{X_i\}$ for \mathbf{G} . Then

$$\text{ad}X_i(X_j) = [X_i, X_j] = C_{ij}^k X_k \quad (13)$$

where we sum over k . The C_{ij}^k are called structure constants. Under a change of basis, they transform as mixed tensor components. They define the matrix $(M_i)_{jk} = C_{ik}^j$ associated with the adjoint representation of X_i . One can show that there exists a basis for any complex semisimple algebra in which the structure constants are real. This means the adjoint representation is real. Note that the dimension of the adjoint representation is equal to the dimension of the group.

Example: Let's construct the adjoint representation of $SU(2)$. The generators in the defining representation are

$$J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_{\pm} = \frac{1}{2} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) \quad (14)$$

and the commutation relations are

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_3 \quad (15)$$

The structure constants are therefore $C_{3+}^+ = -C_{+3}^+ = -C_{3-}^- = C_{-3}^- = 1$, $C_{+-}^3 = -C_{-+}^3 = 2$ and the adjoint representation is given by $(M_3)_{++} = 1$, $(M_3)_{--} = -1$, $(M_+)_{+3} = -1$, $(M_+)_{3-} = 2$, $(M_-)_{-3} = 1$, $(M_-)_{3+} = -2$, and all other matrix elements equal to 0:

$$M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_+ = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_- = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (16)$$

These representation matrices are real, have the same dimension as the group, and satisfy the $SU(2)$ commutation relations $[M_3, M_{\pm}] = \pm M_{\pm}$, $[M_+, M_-] = 2M_3$.

2.5 Semisimple algebras and root spaces

In this paragraph we will briefly recall the basic facts about root spaces and the classification of complex simple Lie algebras, to set the stage for our discussion of real forms of Lie algebras and finally symmetric spaces.

An *ideal*, or *invariant subalgebra* \mathbf{I} is a subalgebra such that $[\mathbf{G}, \mathbf{I}] \subset \mathbf{I}$. An abelian ideal also satisfies $[\mathbf{I}, \mathbf{I}] = 0$. A *simple* Lie algebra has no proper ideal. A *semisimple* Lie algebra is the direct sum of simple algebras, and has no proper abelian ideal (by proper we mean different from $\{0\}$).

A Lie algebra is a linear vector space over a field F , with an antisymmetric product defined by the Lie bracket (cf. subsection 2.4). If F is the field of real, complex or quaternion numbers, the Lie algebra is called a real, complex or quaternion algebra. A complexification of a real Lie algebra is obtained by taking linear combinations of its elements with complex coefficients. A real Lie algebra \mathbf{H} is a real form of the complex algebra \mathbf{G} if \mathbf{G} is the complexification of \mathbf{H} .

In any simple algebra there are two kinds of generators: there is a maximal abelian subalgebra, called the *Cartan subalgebra* $\mathbf{H}_0 = \{H_1, \dots, H_r\}$, $[H_i, H_j] = 0$ for any two elements of the Cartan subalgebra. There are also raising and lowering operators denoted E_α . α is an r -dimensional vector $\alpha = (\alpha_1, \dots, \alpha_r)$ and r is the *rank* of the algebra.¹ The latter are eigenoperators of the H_i in the adjoint representation belonging to eigenvalue α_i : $[H_i, E_\alpha] = \alpha_i E_\alpha$. For each eigenvalue, or *root* α , there is another eigenvalue $-\alpha$ and a corresponding eigenoperator $E_{-\alpha}$ under the action of H_i .

Suppose we represent each element of the Lie algebra by an $n \times n$ matrix. Then $[H_i, H_j] = 0$ means the matrices H_i can all be diagonalized simultaneously. Their eigenvalues μ_i are given by $H_i|\mu\rangle = \mu_i|\mu\rangle$, where the eigenvectors are labelled by the *weight vectors* $\mu = (\mu_1, \dots, \mu_r)$ [17].

A weight whose first non-zero component is positive is called a positive weight. Also, a weight μ is greater than another weight μ' if $\mu - \mu'$ is positive. Thus we can define the highest weight as the one which is greater than all the others. The highest weight is unique in any representation.

The roots $\alpha_i \equiv \alpha(H_i)$ of the algebra \mathbf{G} are the weights of the adjoint representation. Recall that in the adjoint representation, the states on which the generators act are defined by the generators themselves, and the action is defined by

$$X_a|X_b\rangle \equiv \text{ad}X_a(X_b) \equiv [X_a, X_b] \tag{17}$$

The roots are functionals on the Cartan subalgebra satisfying

¹The rank of an algebra is defined through the secular equation (see subsection 6.1). For a non-semisimple algebra, the maximal number of mutually commuting generators can be greater than the rank of the algebra.

$$\text{ad}H_i(E_\alpha) = [H_i, E_\alpha] = \alpha(H_i)E_\alpha \quad (18)$$

where H_i is in the Cartan subalgebra. The eigenvectors E_α are called the root vectors. These are exactly the raising and lowering operators $E_{\pm\alpha}$ for the weight vectors μ . There are canonical commutation relations defining the system of roots belonging to each simple rank r -algebra. These are summarized below: ²

$$[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha_i H_i \quad (19)$$

One can prove the fundamental relation [16, 17]

$$\frac{2\alpha \cdot \mu}{\alpha^2} = -(p - q) \quad (20)$$

where α is a root, μ is a weight, and p, q are positive integers such that $E_\alpha|\mu + p\alpha\rangle = 0$, $E_{-\alpha}|\mu - q\alpha\rangle = 0$ ³. This relation gives rise to the strict properties of root lattices, and permits the complete classification of all the complex (semi)simple algebras.

Eq. (20) is true for any representation, but has particularly strong implications for the adjoint representation. In this case μ is a root. As a consequence of eq. (20), the possible angle between two root vectors of a simple Lie algebra is limited to a few values: these turn out to be multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (see e.g. [17], Ch. VI). The root lattice is invariant under reflections in the hyperplanes orthogonal to the roots (the Weyl group). As we will shortly see, this is true not only for the root lattice, but for the weight lattice of any representation.

²For the reader who wants to understand more about the origin of the structure of Lie algebras, we recommend Chapter 7 of Gilmore [12].

³Here the scalar product \cdot can be defined in terms of the metric on the Lie algebra. For the adjoint representation, μ is a root β and

$$\frac{2\alpha \cdot \beta}{\alpha^2} = \frac{2K(H_\alpha, H_\beta)}{K(H_\alpha, H_\alpha)} \equiv \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} \quad (21)$$

where K denotes the Killing form (see paragraph 3.4). There is always a unique element H_α in the algebra such that $K(H, H_\alpha) = \alpha(H)$ for each $H \in \mathbf{H}_0$ (see for example [16], Ch. 10). In general for a linear form μ on the Lie algebra,

$$\frac{2\alpha \cdot \mu}{\alpha^2} = \frac{2\mu(H_\alpha)}{\alpha(H_\alpha)} \quad (22)$$

Then μ is a highest weight for some representation if and only if this expression is an integer for each positive root α .

Note that the roots α are real-valued linear functionals on the Cartan subalgebra. Therefore they are in the space dual to \mathbf{H}_0 . A subset of the positive roots span the root lattice. These are called simple roots. Obviously, since the roots are in the space dual to \mathbf{H}_0 , the number of simple roots is equal to the rank of the algebra.

The same relation (20) determines the highest weights of all irreducible representations. Setting $p = 0$, choosing a positive integer q , and letting α run through the simple roots, $\alpha = \alpha^i$ ($i = 1, \dots, r$), we find the highest weights μ^i of all the irreducible representations corresponding to the given value of q [17]. For example, for $q = 1$ we get the highest weights of the r fundamental representations of the group, each corresponding to a simple root α^i . For higher values of q we get the highest weights of higher-dimensional representations of the same group.

The set of all possible simple root systems are classified by means of Dynkin diagrams, each of which correspond to an equivalence class of isomorphic Lie algebras. The classical Lie algebras $\mathbf{SU}(\mathbf{n} + \mathbf{1}, \mathbf{C})$, $\mathbf{SO}(2\mathbf{n} + \mathbf{1}, \mathbf{C})$, $\mathbf{Sp}(2\mathbf{n}, \mathbf{C})$ and $\mathbf{SO}(2\mathbf{n}, \mathbf{C})$ correspond to root systems A_n , B_n , C_n , and D_n , respectively. In addition there are five exceptional algebras corresponding to root systems E_6 , E_7 , E_8 , F_4 and G_2 . Each of these complex algebras in general has several real forms associated with it (see section 4). These real forms correspond to the same Dynkin diagram and root system as the complex algebra. Since we will not make reference to Dynkin diagrams in the following, we will not discuss them here. The interested reader can find sufficient material for example in the book by Georgi [17].

The (semi)simple complex algebra \mathbf{G} decomposes into a direct sum of root spaces [16]:

$$\mathbf{G} = \mathbf{H}_0 \oplus \sum_{\alpha} \mathbf{G}_{\alpha} \quad (23)$$

where $\mathbf{G}_{\alpha} = \{E_{\alpha}\}$, $\mathbf{G}_{-\alpha} = \{E_{-\alpha}\}$. This will be evident in the example given below.

Example: The root system A_{n-1} corresponds to the complex Lie algebra $\mathbf{SL}(\mathbf{n}, \mathbf{C})$ and all its real forms. In a later section we will see how to construct all the real forms associated with a given complex Lie algebra. Let's see here explicitly how to construct the root lattice of $\mathbf{SU}(\mathbf{3}, \mathbf{C})$, which is one of the real forms of $\mathbf{SL}(\mathbf{3}, \mathbf{C})$.

The generators are determined by the commutation relations. In physics it is common to write the commutation relations in the form

$$[T_i, T_j] = if_{ijk}T_k \quad (24)$$

(an alternative form is to define the generators as $X_i = iT_i$ and write the commutation relations as $[X_i, X_j] = -f_{ijk}X_k$) where f_{ijk} are structure constants for the algebra $\mathbf{SU(3, C)}$.

Using the notation $g = e^{it^a T_a}$ for the group elements (with t^a real and a sum over a implied), the generators T_a in the fundamental representation of this group are hermitean⁴:

$$\begin{aligned}
T_1 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T_2 &= \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T_3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T_4 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & T_5 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & T_6 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & (25) \\
T_7 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & T_8 &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned}$$

In high energy physics the matrices $2T_a$ are known as Gell–Mann matrices. The generators are normalized in such a way that $\text{tr}(T_a T_b) = \frac{1}{2} \delta_{ab}$. Note that T_1, T_2, T_3 form an $\mathbf{SU(2, C)}$ subalgebra. We take the Cartan subalgebra to be $\mathbf{H_0} = \{T_3, T_8\}$. The rank of this group is $r = 2$.

Let's first find the weight vectors of the fundamental representation. To this end we look for the eigenvalues μ_i of the operators in the abelian subalgebra $\mathbf{H_0}$:

$$T_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T_8 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (26)$$

therefore the eigenvector $(100)^T$ corresponds to the state $|\mu\rangle$ where

⁴Note that we have written an explicit factor of i in front of the generators in the expression for the group elements. This is often done for compact groups; since the Killing form (subsection 3.4) has to be negative definite, the coordinates of the algebra spanned by the generators must be purely imaginary. Here we use this notation because it is conventional. If we absorb the factor of i into the generators, we get *antihermitean* matrices $X_a = iT_a$; we will do this in the example in subsection 3.1 to comply with eq. (35). Of course, the matrices in the *algebra* are always antihermitean.

$$\mu \equiv (\mu_1, \mu_2) = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right) \quad (27)$$

is distinguished by its eigenvalues under the operators H_i of the Cartan subalgebra. In the same way we find that $(0\ 1\ 0)^T$ and $(0\ 0\ 1)^T$ correspond to the states labelled by weight vectors

$$\mu' = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \quad \mu'' = \left(0, -\frac{1}{\sqrt{3}} \right) \quad (28)$$

respectively. μ , μ' , and μ'' are the weights of the fundamental representation $\rho = D$ and they form an equilateral triangle in the plane. The highest weight of the representation D is $\mu = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right)$.

There is also another fundamental representation \bar{D} of the algebra $\mathbf{SU}(3, \mathbf{C})$, since it generates a group of rank 2. Indeed, from eq. (24), for $p = 0$, $q = 1$, there is one highest weight μ^i , and one fundamental representation for each simple root α^i . The highest weight $\bar{\mu}$ of the representation \bar{D} is

$$\bar{\mu} = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}} \right) \quad (29)$$

The highest weights of the representations corresponding to any positive integer q can be obtained as soon as we know the simple roots. Then, by operating with lowering operators on this weight, we obtain other weights, on which we can further operate with lowering operators until we have obtained all the weights in the representation. For an example of this procedure see [17], Ch. IX.

Let's see now how to obtain the roots of $\mathbf{SU}(3, \mathbf{C})$. Each root vector E_α corresponds to either a raising or a lowering operator: E_α is the eigenvector belonging to the root $\alpha_i \equiv \alpha(H_i)$ under the adjoint representation of H_i , like in eq. (32). Each raising or lowering operator is a linear combination of generators T_i that takes one state of the fundamental representation to another state of the same representation: $E_{\pm\alpha}|\mu\rangle = N_{\pm\alpha,\mu}|\mu \pm \alpha\rangle$. Therefore the root vectors α will be differences of weight vectors in the fundamental representation. We find the raising and lowering operators $E_{\pm\alpha}$ to be

$$\begin{aligned}
E_{\pm(1,0)} &= \frac{1}{\sqrt{2}}(T_1 \pm iT_2) \\
E_{\pm(\frac{1}{2}, \frac{\sqrt{3}}{2})} &= \frac{1}{\sqrt{2}}(T_4 \pm iT_5) \\
E_{\pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})} &= \frac{1}{\sqrt{2}}(T_6 \pm iT_7)
\end{aligned} \tag{30}$$

These form the subspaces \mathbf{G}_α in eq. (23). In the fundamental representation, we find using the Gell–Mann matrices that these are matrices with only one non–zero element. For example, the raising operator E_α that corresponds to the root $\alpha = (1, 0)$ is

$$E_{+(1,0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{31}$$

This operator takes us from the state $|\mu'\rangle = |-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle$ to the state $|\mu\rangle = |\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle$. The components of the root vectors of $\mathbf{SU}(3, \mathbf{C})$ are the eigenvalues α_i of these under the adjoint representation of the Cartan subalgebra. That is,

$$H_i|E_\alpha\rangle \equiv \text{ad}_{H_i}(E_\alpha) \equiv [H_i, E_\alpha] = \alpha_i|E_\alpha\rangle \tag{32}$$

This way we easily find the roots; we can either explicitly use the structure constants of $SU(3)$ in $[T_a, T_b] = if_{abc}T_c = -if_{ab}^cT_c$ (note the explicit factor of i due to our conventions regarding the generators), or we can use an explicit representation for H_i, E_α like in eqs. (25), (30), (31), to calculate the commutators:

$$\begin{aligned}
\text{ad}_{H_1}(E_{\pm(1,0)}) &= [H_1, E_{\pm(1,0)}] = [T_3, \frac{1}{\sqrt{2}}(T_1 \pm iT_2)] = \frac{1}{\sqrt{2}}(iT_2 \pm T_1) = \pm E_{\pm(1,0)} \equiv \alpha_1^\pm E_{\pm(1,0)} \\
\text{ad}_{H_2}(E_{\pm(1,0)}) &= [H_2, E_{\pm(1,0)}] = [T_8, \frac{1}{\sqrt{2}}(T_1 \pm iT_2)] = 0 \equiv \alpha_2^\pm E_{\pm(1,0)}
\end{aligned} \tag{33}$$

The root vector corresponding to the raising operator $E_{+(1,0)}$ is thus $\alpha = (\alpha_1^+, \alpha_2^+) = (1, 0)$ and the root vector corresponding to the lowering operator $E_{-(1,0)}$ is $-\alpha = (\alpha_1^-, \alpha_2^-) = (-1, 0)$. These root vectors are indeed the differences between the weight vectors $\mu = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and $\mu' = (-\frac{1}{2}, \frac{1}{2\sqrt{3}})$ of the fundamental representation.

In the same way we find the other root vectors $(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, $(\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, and $(0, 0)$ (with multiplicity 2), by operating with H_1 and H_2 on the remaining $E_{\pm\alpha}$'s and on the H_i 's. The last root with multiplicity 2 has as its components the eigenvalues under H_1 , H_2 of the states $|H_1\rangle$ and $|H_2\rangle$: $H_i|H_j\rangle = [H_i, H_j] = 0$; $i, j \in \{1, 2\}$. The root vectors form a regular hexagon in the plane. The positive roots are $(1, 0)$, $\alpha^1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\alpha^2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The latter two are simple roots. $(1, 0)$ is not simple because it is the sum of the other positive roots. There are two simple roots, since the rank of $SU(3)$ is 2 and the root lattice is two-dimensional.

The root lattice of $SU(3)$ is invariant under reflections in the hyperplanes orthogonal to the root vectors. This is true of any weight or root lattice; the symmetry group of reflections in hyperplanes orthogonal to the roots is called the *Weyl group*. It is obtained from eq. (20): since for any root α and any weight μ , $2(\alpha \cdot \mu)/\alpha^2$ is the integer $q - p$,

$$\mu' = \mu - \frac{2(\alpha \cdot \mu)}{\alpha^2} \alpha \quad (34)$$

is also a weight. Eq. (34) is exactly the above mentioned reflection, as can easily be seen.

We have just shown by an example how to obtain a root system of type A_n . In general for any simple algebra the commutation relations determine the Cartan subalgebra and raising and lowering operators, that in turn determine a unique root system, and correspond to a given Dynkin diagram. In this way we can classify all the simple algebras according to the type of root system it possesses. The root systems for the four infinite series of classical non-exceptional Lie groups can be characterized as follows [17] (denote the r -dimensional space spanned by the roots by \mathcal{V} and let $\{e_1, \dots, e_n\}$ be a canonical basis in \mathbf{R}^n):

A_{n-1} : Let \mathcal{V} be the hyperplane in \mathbf{R}^n that passes through the points $(1, 0, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, \dots , $(0, 0, \dots, 0, 1)$ (the endpoints of the e_i , $i = 1, \dots, n$). Then the root lattice contains the vectors $\{e_i - e_j, i \neq j\}$.

B_n : Let \mathcal{V} be \mathbf{R}^n ; then the roots are $\{\pm e_i, \pm e_i \pm e_j, i \neq j\}$.

C_n : Let \mathcal{V} be \mathbf{R}^n ; then the roots are $\{\pm 2e_i, \pm e_i \pm e_j, i \neq j\}$.

D_n : Let \mathcal{V} be \mathbf{R}^n ; then the roots are $\{\pm e_i \pm e_j, i \neq j\}$.

The root lattice BC_n , that we will discuss in conjunction with restricted root systems, is the union of B_n and C_n . It is characterized as follows:

BC_n : Let \mathcal{V} be \mathbf{R}^n ; then the roots are $\{\pm e_i, \pm 2e_i, \pm e_i \pm e_j, i \neq j\}$.

Because this system contains both e_i and $2e_i$, it is called non-reduced (normally the only root collinear with α is $-\alpha$). However, it is irreducible in the usual sense, which means it is not the direct sum of two disjoint root systems B_n and C_n . This can be seen from the root multiplicities (cf. Table 1).

The semisimple algebras are direct sums of simple ones. That means the simple constituent algebras commute with each other, and the root systems are direct sums of the corresponding simple root systems. Therefore, knowing the properties of the simple Lie algebras, we also know the semisimple ones.

3 Symmetric spaces

In the previous section, we have reminded ourselves of some elementary facts concerning root spaces and the classification of the complex semisimple algebras. In this section we will define and discuss symmetric spaces.

A symmetric space is associated to an involutive automorphism of a given Lie algebra. As we will see, several different involutive automorphisms can act on the same algebra. Therefore we normally have several different symmetric spaces deriving from the same Lie algebra. The involutive automorphism defines a symmetric subalgebra and a remaining complementary subspace of the algebra. Under general conditions, the complementary subspace is mapped onto a symmetric space through the exponential map. In the following subsections we make these statements more precise. We discuss how the elements of the Lie group can act as transformations on the elements of the symmetric space. This naturally leads to the definition of two coordinate systems on symmetric spaces: the spherical and the horospheric coordinate systems. The radial coordinates associated to each element on a symmetric space through their spherical or horospheric decomposition will be of relevance when we discuss the radial parts of differential operators on symmetric spaces in section 6. In the same section we explain why these operators are important in applications to physical problems, and in [3] we will discuss some of their uses.

At the end of this section we define the metric tensor on a Lie algebra in terms of the Killing form. The latter is defined as a symmetric bilinear trace form on the adjoint representation, and is therefore expressible in terms of the structure constants. We will give several examples of Killing forms later, as we discuss the various real forms of a Lie algebra. The metric tensor will serve to define the curvature tensor on a symmetric space (subsection 5.1). It is also needed in computing the Jacobian of the transformation to radial coordinates. This Jacobian is relevant in calculating the radial part of the Laplace-Beltrami operator (see paragraph 6.2). Finally we discuss the general algebraic form of

coset representatives in subsection 3.5.

3.1 Involutive automorphisms

An automorphism of a Lie algebra \mathbf{G} is a mapping from \mathbf{G} onto itself such that it preserves the algebraic operations on the Lie algebra. For example, if σ is an automorphism, it preserves multiplication: $[\sigma(X), \sigma(Y)] = \sigma([X, Y])$, for $X, Y \in \mathbf{G}$.

Suppose that the linear automorphism $\sigma : \mathbf{G} \rightarrow \mathbf{G}$ is such that $\sigma^2 = 1$, but σ is not the identity. That means that σ has eigenvalues ± 1 , and it splits the algebra \mathbf{G} into orthogonal eigensubspaces corresponding to these eigenvalues. Such a mapping is called an *involutive automorphism*.

Suppose now that \mathbf{G} is a compact simple Lie algebra, σ is an involutive automorphism of \mathbf{G} , and $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ where

$$\sigma(X) = X \text{ for } X \in \mathbf{K}, \quad \sigma(X) = -X \text{ for } X \in \mathbf{P} \quad (35)$$

From the properties of automorphisms mentioned above, it is easy to see that \mathbf{K} is a subalgebra, but \mathbf{P} is not. In fact, the commutation relations

$$[\mathbf{K}, \mathbf{K}] \subset \mathbf{K}, \quad [\mathbf{K}, \mathbf{P}] \subset \mathbf{P}, \quad [\mathbf{P}, \mathbf{P}] \subset \mathbf{K} \quad (36)$$

hold. A subalgebra \mathbf{K} satisfying (36) is called a *symmetric subalgebra*. If we now multiply the elements in \mathbf{P} by i (the “Weyl unitary trick”), we construct a new noncompact algebra $\mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P}$. This is called a *Cartan decomposition*, and \mathbf{K} is a maximal compact subalgebra of \mathbf{G}^* . The coset spaces G/K and G^*/K are *symmetric spaces*.

Example: Suppose $G = SU(n, C)$, the group of unitary complex matrices with determinant $+1$. The algebra of this group then consists of complex antihermitean⁵ matrices of zero trace (this follows by differentiating the identities $UU^\dagger = 1$ and $\det U = 1$ with respect to t where $U(t)$ is a curve passing through the identity at $t = 0$); a group element is written as $g = e^{t^a X_a}$ with t^a real. Therefore any matrix X in the Lie algebra of this group can be written $X = A + iB$, where A is real, skew-symmetric, and traceless and B is real, symmetric and traceless. This means the algebra can be decomposed as $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$, where \mathbf{K} is the compact connected subalgebra $\mathbf{SO}(n, \mathbf{R})$ consisting of real, skew-symmetric and

⁵See the footnote in subsection 2.5.

traceless matrices, and \mathbf{P} is the subspace of matrices of the form iB , where B is real, symmetric, and traceless. \mathbf{P} is not a subalgebra.

Referring to the example for $\mathbf{SU}(3, \mathbf{C})$ in subsection 2.5 we see, setting $X_a = iT_a$, that the $\{X_a\}$ split into two sets under the involutive automorphism σ defined by complex conjugation $\sigma = K$. This splits the compact algebra \mathbf{G} into $\mathbf{K} \oplus \mathbf{P}$, since \mathbf{P} consists of imaginary matrices:

$$\begin{aligned} \mathbf{K} &= \{X_2, X_5, X_7\} = \left\{ \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\} \\ \mathbf{P} &= \{X_1, X_3, X_4, X_6, X_8\} \\ &= \left\{ \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \frac{i}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right\} \end{aligned} \quad (37)$$

\mathbf{K} spans the real subalgebra $\mathbf{SO}(3, \mathbf{R})$. Setting $X_2 \equiv L_3$, $X_5 \equiv L_2$, $X_7 \equiv L_1$, the commutation relations for the subalgebra are $[L_i, L_j] = \frac{1}{2}\epsilon_{ijk}L_k$. The Cartan subalgebra $i\mathbf{H}_0 = \{X_3, X_8\}$ is here entirely in the subspace \mathbf{P} .

Going back to the general case of $\mathbf{G} = \mathbf{SU}(n, \mathbf{C})$, we obtain from \mathbf{G} by the Weyl unitary trick the non-compact algebra $\mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P}$. $i\mathbf{P}$ is now the subspace of real, symmetric, and traceless matrices B . The Lie algebra $\mathbf{G}^* = \mathbf{SL}(n, \mathbf{R})$ is then the set of $n \times n$ real matrices of zero trace, and generates the linear group of transformations represented by real $n \times n$ matrices of unit determinant.

The involutive automorphism that split the algebra \mathbf{G} above was defined to be complex conjugation $\sigma = K$. The involutive automorphism that splits \mathbf{G}^* is defined by $\tilde{\sigma}(g) = (g^T)^{-1}$ for $g \in G^*$, as we will now see. On the level of the algebra, $\tilde{\sigma}(g) = (g^T)^{-1}$ means $\tilde{\sigma}(X) = -X^T$. Suppose now $g = e^{tX} \in G^*$ with X real and traceless and t a real parameter. If now X is an element of the subalgebra \mathbf{K} , we then have $\tilde{\sigma}(X) = +X$, i.e. $-X^T = X$ and X is skew-symmetric. If instead $X \in i\mathbf{P}$, we have $\tilde{\sigma}(X) = -X^T = -X$, i.e. X is symmetric. The decomposition $\mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P}$ is the usual decomposition of a $\mathbf{SL}(n, \mathbf{R})$ matrix in symmetric and skew-symmetric parts.

$G/K = SU(n, C)/SO(n, R)$ is a symmetric space of compact type, and the related symmetric space of non-compact type is $G^*/K = SL(n, R)/SO(n, R)$.

3.2 The action of the group on the symmetric space

Let G be a semisimple Lie group and K a compact symmetric subgroup. As we saw in the preceding paragraph, the coset spaces G/K and G^*/K represent symmetric spaces. Just as we have defined a Cartan subalgebra and the rank of a Lie algebra, we can define, in an exactly analogous way, a Cartan subalgebra and the rank of a symmetric space. A Cartan subalgebra of a symmetric space is a maximal abelian subalgebra of the subspace \mathbf{P} (see paragraph 5.2), and the rank of a symmetric space is the number of generators in this subalgebra.

If G is connected and $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ where \mathbf{K} is a compact symmetric subalgebra, then each group element can be decomposed as $g = kp$ (right coset decomposition) or $g = pk$ (left coset decomposition), with $k \in K = e^{\mathbf{K}}$, $p \in P = e^{\mathbf{P}}$. P is not a subgroup, unless it is abelian and coincides with its Cartan subalgebra. However, if the involutive automorphism that splits the algebra is denoted σ , one can show ([18], Ch. 6) that $gp\sigma(g^{-1}) \in P$. This defines G as a transformation group on P . Since $\sigma(k^{-1}) = k^{-1}$ for $k \in K$, this means

$$p' = kpk^{-1} \in P \quad (38)$$

if $k \in K$, $p \in P$. Now suppose there are no other elements in G that satisfy $\sigma(g) = g$ than those in K . This will happen if the set of elements satisfying $\sigma(g) = g$ is connected. Then P is isomorphic to G/K . Also, G acts transitively on P in the manner defined above (cf. subsection 2.3). The tangent space of G/K at the origin (identity element) is spanned by the subspace \mathbf{P} of the algebra.

3.3 Radial coordinates

In this paragraph we define two coordinate systems frequently used on symmetric spaces. Let $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ be a Cartan decomposition of a semisimple algebra and let $\mathbf{H}_0 \subset \mathbf{P}$ be a maximal abelian subalgebra in the subspace \mathbf{P} . Define M to be the subgroup of elements in K such that

$$M = \{k \in K : kHk^{-1} = H, H \in \mathbf{H}_0\} \quad (39)$$

This set is called the centralizer of \mathbf{H}_0 in K . Under conjugation by $k \in K$, each element H of the Cartan subalgebra is preserved. Further, denote

$$M' = \{k \in K : kHk^{-1} = H', H, H' \in \mathbf{H}_0\} \quad (40)$$

This is a larger subgroup than M that preserves the Cartan subalgebra as a whole, but not necessarily each element separately, and is called the normalizer of \mathbf{H}_0 in K . If K is a compact symmetric subgroup of G , one can show ([18], Ch. 6) that every element p of $P \simeq G/K$ is conjugated with some element $h = e^H$ for some $H \in \mathbf{H}_0$ by means of the adjoint representation⁶ of the stationary subgroup K :

$$p = khk^{-1} = kh\sigma(k^{-1}) \quad (42)$$

where $k \in K/M$ and H is defined up to the elements in the factor group M'/M . This factor group coincides with the Weyl group that was defined in eq. (34). It transforms an element of the algebra \mathbf{H}_0 into another element of the same algebra. In fact, this means that every element $g \in G$ can be decomposed as $g = pk = k'hk'^{-1}k = k'hk''$. This is very much like the Euler angle decomposition of $SO(p)$.

Thus, if x_0 is the fixed point of the subgroup K , an arbitrary point $x \in P$ can be written

$$x = khk^{-1}x_0 = khx_0 \quad (43)$$

The coordinates $(k(x), h(x))$ are called spherical coordinates. $k(x)$ is the angular coordinate and $h(x)$ is the *spherical radial coordinate* of the point x . Eq. (42) defines the so called spherical decomposition of the elements in the coset space. Of course, a similar reasoning is true for the space $P^* \simeq G^*/K$.

This means every matrix p in the coset space G/K can be *diagonalized* by a similarity transformation by the subgroup K , and the radial coordinates are exactly the set of eigenvalues of the matrix p . These “eigenvalues” are not necessarily real numbers. This is easily seen in the example in eq. (37). It can also be seen in the adjoint representation. Suppose the algebra $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ is compact. From eq. (13), in the adjoint representation $H_i \in \mathbf{H}_0$ has the form

⁶Note that

$$e^K H e^{-K} = e^{\text{ad}K} H \equiv \sum_{n=0}^{\infty} \frac{(\text{ad}K)^n}{n!} H \quad (41)$$

by K , and we can choose the Cartan subalgebra in any way we please. However, the number of elements that we can diagonalize simultaneously will always be equal to the rank of the symmetric space.

There is also another coordinate system valid only for spaces of the type $P^* \sim G^*/K$. This coordinate system is called *horospheric* and is based on the so called *Iwasawa decomposition* of the algebra:

$$\mathbf{G} = \mathbf{N}^+ \oplus \mathbf{H}_0 \oplus \mathbf{K} \quad (46)$$

Here \mathbf{K} , \mathbf{H}_0 , \mathbf{N}^+ are three subalgebras of \mathbf{G} . \mathbf{K} is a maximal compact subalgebra, \mathbf{H}_0 is a Cartan subalgebra, and

$$\mathbf{N}^+ = \sum_{\alpha \in R^+} \mathbf{G}_\alpha \quad (47)$$

is an algebra of raising operators corresponding to the positive roots $\alpha(H) > 0$ with respect to \mathbf{H}_0 . As a consequence, the group elements can be decomposed $g = nhk$, in an obvious notation. This means that if x_0 is the fixed point of K , any point $x \in G^*/K$ can be written

$$x = nhkx_0 = nhx_0 \quad (48)$$

The coordinates $(n(x), h(x))$ are called horospheric coordinates and the element $h = h(x)$ is called the *horospheric projection* of the point x or the *horospheric radial coordinate*.

3.4 The metric on a Lie algebra

A metric tensor can be defined on a Lie algebra [11, 12, 16, 18]. If $\{X_i\}$ form a basis for the Lie algebra \mathbf{G} , it is defined by

$$g_{ij} = K(X_i, X_j) \equiv \text{tr}(\text{ad}X_i \text{ad}X_j) = C_{is}^r C_{jr}^s \quad (49)$$

The symmetric bilinear form $K(X_i, X_j)$ is called the *Killing form*. It is intrinsically associated with the Lie algebra, and since the Lie bracket is invariant under automorphisms of the algebra, so is the Killing form.

Example: The generators $X_7 \equiv L_1$, $X_5 \equiv L_2$, $X_2 \equiv L_3$ of $SO(3)$ given in eq. (37) obey the commutation relations $[L_i, L_j] = C_{ij}^k L_k = \frac{1}{2}\epsilon_{ijk} L_k$. From eq. (49), the metric for this algebra is $g_{ij} = -\frac{1}{2}\delta_{ij}$. The generators and the structure constants can be normalized so that the metric takes the canonical form $g_{ij} = -\delta_{ij}$.

Just like we defined the Killing form $K(X_i, X_j)$ for the algebra \mathbf{G} in eq. (49) using the adjoint representation, we can define a similar trace form K_ρ and a metric tensor g_ρ for any representation ρ by

$$g_{\rho,ij} = K_\rho(X_i, X_j) = \text{tr}(\rho(X_i)\rho(X_j)) \quad (50)$$

where $\rho(X)$ is the matrix representative of the Lie algebra element X . If ρ is an automorphism of \mathbf{G} , $K_\rho(X_i, X_j) = K(X_i, X_j)$.

Suppose the Lie algebra is semisimple (this is true for all the classical Lie algebras except the Lie algebras $GL(n, C)$, $U(n, C)$). According to Cartan's criterion, *the Killing form is non-degenerate for a semisimple algebra*. This means that $\det g_{ij} \neq 0$, so that the inverse of g_{ij} , denoted by g^{ij} , exists. Since it is also real and symmetric, it can be reduced to canonical form $g_{ij} = \text{diag}(-1, \dots, -1, 1, \dots, 1)$ with p -1 's and $(n - p)$ $+1$'s, where n is the dimension of the algebra.

p is an invariant of the quadratic form. In fact, for any real form of a complex algebra, the trace of the metric, called the *character* of the particular real form (see below and in [12]) distinguishes the real forms from each other (though it can be degenerate for the classical Lie algebras [12]). The character ranges from $-n$, where n is the dimension of the algebra, to $+r$, where r is its rank. All the real forms of the algebra have a character that lies in between these values. In subsection 4.1 we will see several explicit examples of Killing forms.

A famous theorem by Weyl states that *a simple Lie group G is compact, if and only if the Killing form on \mathbf{G} is negative definite*. Otherwise it is non-compact. This is actually quite intuitive and natural (see [12], Ch. 9, paragraph I.2). On a compact algebra, the metric can be chosen to be minus the Killing form, if it is required to be positive-definite.

The metric on the Lie algebra can be extended to the whole coset space $P \simeq G/K$, $P^* \simeq G^*/K$ as follows. At the origin of G/K and G^*/K , the identity element I , the metric is identified with the metric in the algebra, restricted to the respective tangent spaces \mathbf{P} , $i\mathbf{P}$. Since the group acts transitively on the coset space (cf. paragraph 2.3), and the orbit of the origin is the entire space, we can use a group transformation to map the metric at the origin to any point M in the space. The metric tensor at M will depend on

the coset representative M . It is given by

$$g_{rs}(M) = g_{ij}(I) \frac{\partial x^i(I)}{\partial x^r(M)} \frac{\partial x^j(I)}{\partial x^s(M)} \quad (51)$$

where $g_{ij}(I)$ is the metric at the origin (identity element) of the coset space. (51) follows from the invariance of the line element $ds^2 = g_{ij}dx^i dx^j$ under translations. If $\{X_i\}$ is a basis in the tangent space, and $dM = \exp(dx^i X_i)$ is a coset representative infinitesimally close to the identity, we need to know how dx^i transforms under translations by the coset representative M . We will not discuss that here, but some generalities can be found for example in Ch. 9, paragraph V.4. of ref. [12]. In general, it is not an easy problem unless the coset has rank 1.

Example: The line element ds^2 on the radius-1 2-sphere $SO(3)/SO(2)$ in polar coordinates is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. The metric at the point (θ, ϕ) is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix} \quad (52)$$

where the rows and columns are labelled in the order θ, ϕ .

The distance between points on the symmetric space is defined as follows. The length of a vector $X = \sum_i t^i X_i$ in the tangent space \mathbf{P} (this object is well-defined because \mathbf{P} is endowed with a definite metric) is identified with the length of the geodesic connecting the identity element in the coset space with the element $M = \exp(X)$ [12].

3.5 The algebraic structure of symmetric spaces

Except for the two algebras $\mathbf{SL}(\mathbf{n}, \mathbf{R})$ and $\mathbf{SU}^*(2\mathbf{n})$ (and their dual spaces related by the Weyl trick), for which the subspace representatives of \mathbf{K} , \mathbf{P} and $i\mathbf{P}$ consist of square, irreducible matrices (for $\mathbf{SL}(\mathbf{n}, \mathbf{R})$, we saw this in the example in subsection 3.1 and for $\mathbf{SU}(\mathbf{n}, \mathbf{C})$ explicitly in eq. (37)), the matrix representatives of the subalgebra \mathbf{K} and of the subspaces \mathbf{P} and $i\mathbf{P}$ in the fundamental representation consist of block-diagonal matrices $X \in \mathbf{K}$, $Y \in \mathbf{P}$, $Y' \in i\mathbf{P}$ of the form [12]

$$X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & C \\ -C^\dagger & 0 \end{pmatrix}, \quad Y' = \begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^\dagger & 0 \end{pmatrix}, \quad (53)$$

in the Cartan decomposition. Here $A^\dagger = -A$, $B^\dagger = -B$ and $\tilde{C} = iC$. In fact, for *any* finite-dimensional representation, the matrix representatives of \mathbf{K} and \mathbf{P} are antihermitean (thus they become antisymmetric if the representation of \mathbf{P} is real) and as a consequence, those of $i\mathbf{P}$ are hermitean (symmetric in case the representation of $i\mathbf{P}$ is real) [12]. This is true irrespective of whether the matrix representatives are block-diagonal or square.

The exponential maps of the subspaces \mathbf{P} and $i\mathbf{P}$ are isomorphic to coset spaces G/K and G^*/K , respectively (see for example [11, 18]). The exponential map of the algebra maps the subspaces \mathbf{P} and $i\mathbf{P}$ into unitary and hermitean matrices, respectively. In the fundamental representation, these spaces are mapped onto [12]

$$\begin{aligned}\exp(\mathbf{P}) &= \exp\begin{pmatrix} 0 & C \\ -C^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{I - XX^\dagger} & X \\ -X^\dagger & \sqrt{I - XX^\dagger} \end{pmatrix} \\ \exp(i\mathbf{P}) &= \exp\begin{pmatrix} 0 & \tilde{C} \\ \tilde{C}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{I + \tilde{X}\tilde{X}^\dagger} & \tilde{X} \\ \tilde{X}^\dagger & \sqrt{I + \tilde{X}\tilde{X}^\dagger} \end{pmatrix}\end{aligned}\quad (54)$$

where X is a spherical and \tilde{X} a hyperbolic function of the submatrix C :

$$X = C \frac{\sin\sqrt{C^\dagger C}}{\sqrt{C^\dagger C}}, \quad \tilde{X} = \tilde{C} \frac{\sinh\sqrt{\tilde{C}^\dagger \tilde{C}}}{\sqrt{\tilde{C}^\dagger \tilde{C}}}\quad (55)$$

This shows explicitly that the range of parameters parametrizing the two cosets is bounded for the compact coset and unbounded for the non-compact coset, respectively. We already saw an explicit example of these formulas in subsection 2.3.

4 Real forms of semisimple algebras

In this section we will introduce the tools needed to find all the real forms of any (semi)simple algebra. The same tools will then be used in the next section to find the real forms of a symmetric space. When thinking of a real form, it is convenient to visualize it in terms of its metric. As we saw in paragraph 3.4 the trace of the metric is called the character of the real form and it distinguishes the real forms from each other. In the following subsection we discuss various real forms of an algebra and we see how to go from one form to another. In each case, we compute the metric and the character explicitly. We also give the simplest

$$J_3 = \frac{1}{2\sqrt{2}}\tau_3, \quad J_{\pm} = \frac{1}{4}(\tau_1 \pm i\tau_2) \quad (61)$$

where in the defining representation of $\mathbf{SU}(2, \mathbf{C})$

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (62)$$

The normalization is such that

$$[J_3, J_{\pm}] = \pm \frac{1}{\sqrt{2}}J_{\pm}, \quad [J_+, J_-] = \frac{1}{\sqrt{2}}J_3 \quad (63)$$

In equation (16) we constructed the adjoint representation of this algebra, albeit with a different normalization. Using the present normalization to set the entries of the metric equal to 1, we see that the non-zero structure constants are $C_{3+}^+ = -C_{+3}^+ = -C_{3-}^- = C_{-3}^- = C_{+-}^3 = -C_{-+}^3 = \frac{1}{\sqrt{2}}$. The entries of the metric are given by eq. (49), $g_{ij} = K(J_i, J_j) = C_{is}^r C_{jr}^s$ with summation over repeated indices, so we see that the metric of the normal real form $\mathbf{SU}(2, \mathbf{R})$ in this basis is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (64)$$

where the rows and columns are labelled by 3, +, - respectively. This corresponds to eq. (57).

To pass now to a diagonal metric, we just have to set

$$\begin{aligned} \Sigma_3 &= J_3 \\ \Sigma_1 &= \frac{J_+ + J_-}{\sqrt{2}} = \frac{1}{2\sqrt{2}}\tau_1 \\ \Sigma_2 &= \frac{J_+ - J_-}{\sqrt{2}} = \frac{i}{2\sqrt{2}}\tau_2 \end{aligned} \quad (65)$$

like in eq. (58). The commutation relations then become

$$[\Sigma_1, \Sigma_2] = -\frac{1}{\sqrt{2}}\Sigma_3, \quad [\Sigma_2, \Sigma_3] = -\frac{1}{\sqrt{2}}\Sigma_1, \quad [\Sigma_3, \Sigma_1] = \frac{1}{\sqrt{2}}\Sigma_2 \quad (66)$$

These commutation relations characterize the algebra $\mathbf{SO}(\mathbf{2}, \mathbf{1}; \mathbf{R})$. From here we find the structure constants $C_{12}^3 = -C_{21}^3 = C_{23}^1 = -C_{32}^1 = -C_{31}^2 = C_{13}^2 = -\frac{1}{\sqrt{2}}$ and the diagonal metric of the normal real form with rows and columns labelled 3, 1, 2 (in order to comply with the notation in eq. (59)) is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (67)$$

which is to be compared with eq. (59). According to eq. (58), the Cartan decomposition of \mathbf{G}^* is $\mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P}$ where $\mathbf{K} = \{\Sigma_2\}$ and $i\mathbf{P} = \{\Sigma_3, \Sigma_1\}$. The Cartan subalgebra of $i\mathbf{P}$ consists of Σ_3 .

Finally, we arrive at the compact real form by multiplying Σ_3 and Σ_1 with i . Setting $i\Sigma_1 = \tilde{\Sigma}_1$, $\Sigma_2 = \tilde{\Sigma}_2$, $i\Sigma_3 = \tilde{\Sigma}_3$ the commutation relations become those of the special orthogonal group:

$$[\tilde{\Sigma}_1, \tilde{\Sigma}_2] = -\frac{1}{\sqrt{2}}\tilde{\Sigma}_3, \quad [\tilde{\Sigma}_2, \tilde{\Sigma}_3] = -\frac{1}{\sqrt{2}}\tilde{\Sigma}_1, \quad [\tilde{\Sigma}_3, \tilde{\Sigma}_1] = -\frac{1}{\sqrt{2}}\tilde{\Sigma}_2 \quad (68)$$

The last commutation relation in eq. (66) has changed sign whereas the others are unchanged. C_{31}^2 , C_{13}^2 , and consequently g_{33} and g_{11} change sign and we get the metric for $\mathbf{SO}(\mathbf{3}, \mathbf{R})$:

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (69)$$

This is the compact real form. The subspaces of the compact algebra $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ are $\mathbf{K} = \{\tilde{\Sigma}_2\}$ and $\mathbf{P} = \{\tilde{\Sigma}_3, \tilde{\Sigma}_1\}$. Weyl's theorem states that a simple Lie group G is compact, if and only if the Killing form on \mathbf{G} is negative definite; otherwise it is non-compact. In the present example, we see this explicitly.

4.2 The classification machinery

To classify all the real forms of any complex Lie algebra, with characters lying between the character of the normal real form and the compact real form (the intermediate real forms obviously have an indefinite metric), it suffices to enumerate all the involutive automorphisms of its compact real form. A detailed and almost complete account of these procedures for the non-exceptional groups can be found in [12], Chapter 9, paragraph 3. To summarize, if \mathbf{G} is the compact real form of a complex semisimple Lie algebra $\mathbf{G}^{\mathbf{C}}$, \mathbf{G}^* runs through all its associated non-compact real forms \mathbf{G}^* , \mathbf{G}'^* , ... with corresponding maximal compact subgroups \mathbf{K} , \mathbf{K}' , ... and complementary subspaces $i\mathbf{P}$, $i\mathbf{P}'$, ... as σ runs through all the involutive automorphisms of \mathbf{G} .

One such automorphism is complex conjugation $\sigma_1 = K$, which is used to split the compact real algebra into subspaces \mathbf{K} and \mathbf{P} in eq. (60). (To avoid confusion: the generators can be complex even though the field of real numbers is used to multiply the generators in a real form of an algebra. If the generators are also real, we speak of a real representation. However, whether we consider the field to be \mathbf{R} and the generators to be complex, or the opposite, also depends on our definition of basis.) The involutive automorphisms σ satisfy $\sigma\mathbf{G}\sigma^{-1} = \mathbf{G}$, $\sigma^2 = 1$, which implies that σ either commutes or anticommutes with the elements of the compact algebra \mathbf{G} : if $\sigma X\sigma^{-1} = X'$, then $\sigma X'\sigma^{-1} = X$, and we get $X' = \pm X$ for $X, X' \in \mathbf{G}$ (see the example below). One can show [19] (Ch. VII), that it suffices to consider the following three possibilities for σ : $\sigma_1 = K$, $\sigma_2 = I_{p,q}$ and $\sigma_3 = J_{p,p}$ where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad J_{p,p} = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix} \quad (70)$$

and I_p denotes the $p \times p$ unit matrix. By operating with one (or two successive ones) of these automorphisms on the elements of \mathbf{G} , we can construct the subspaces \mathbf{K} and \mathbf{P} , and \mathbf{K} and $i\mathbf{P}$ of the corresponding non-compact real form \mathbf{G}^* . A complex algebra and all its real forms (the compact and the various non-compact ones) correspond to the same root lattice and Dynkin diagram.

Example: The normal real form of the complex algebra $\mathbf{G}^{\mathbf{C}} = \mathbf{SL}(n, \mathbf{C})$ is the non-compact algebra $\mathbf{G}^* = \mathbf{SL}(n, \mathbf{R})$. As we saw in subsection 3.1, this algebra can be decomposed as $\mathbf{K} \oplus i\mathbf{P}$ where \mathbf{K} is the algebra consisting of real, skew-symmetric and traceless $n \times n$ matrices and $i\mathbf{P}$ is the algebra consisting of real, symmetric and traceless $n \times n$ matrices. Under the Weyl unitary trick we constructed, in a previous example, this algebra from the compact real form of $\mathbf{G}^{\mathbf{C}}$, $\mathbf{SU}(n, \mathbf{C}) = \mathbf{G} = \mathbf{K} \oplus \mathbf{P}$.

Starting with the compact real form \mathbf{G} , we can construct all the various non-compact real forms \mathbf{G}^* , \mathbf{G}'^* , ... from it, by applying the involutive automorphisms $\sigma_1, \sigma_2, \sigma_3$ to the elements of \mathbf{G} . All the real forms related to the root system A_{n-1} are obtained by applying the three involutions to $\mathbf{G} = \mathbf{SU}(\mathbf{n}, \mathbf{C})$:

σ_1) The involutive automorphism $\sigma_1 = K$ (complex conjugation) splits $\mathbf{G} = \mathbf{SU}(\mathbf{n}, \mathbf{C})$ into $\mathbf{K} \oplus \mathbf{P}$ (we recall this from the example in paragraph 3.1). The non-compact real form obtained this way, by the Weyl unitary trick, is exactly the normal real form $\mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P} = \mathbf{SL}(\mathbf{n}, \mathbf{R})$.

σ_2) A general matrix in the Lie algebra $\mathbf{SU}(\mathbf{n}, \mathbf{C})$ can be written in the form

$$X = \begin{pmatrix} A & B \\ -B^\dagger & C \end{pmatrix} \quad (71)$$

where A, C are complex $p \times p$ and $q \times q$ matrices satisfying $A^\dagger = -A, C^\dagger = -C, \text{tr}A + \text{tr}C = 0$ (since the determinant of the group elements must be +1), and B is an arbitrary complex $p \times q$ matrix ($p + q = n$). In eq. (71), the matrices A, B and C are all linear combinations of submatrices in *both* subspaces $\mathbf{K} = \{\frac{1}{\sqrt{2}}(E_{\alpha_i} - E_{-\alpha_i})\}$ and $\mathbf{P} = \{iH_j, \frac{i}{\sqrt{2}}(E_{\alpha_i} + E_{-\alpha_i})\}$. The action of the involution $\sigma_2 = I_{p,q}$ on X is

$$I_{p,q} X I_{p,q}^{-1} = \begin{pmatrix} A & -B \\ B^\dagger & C \end{pmatrix} \quad (72)$$

Therefore, we see that the subspaces \mathbf{K}' and \mathbf{P}' are given by the matrices

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \in \mathbf{K}', \quad \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} \in \mathbf{P}' \quad (73)$$

Indeed, we see that $I_{p,q}$ transforms the Lie algebra elements in \mathbf{K}' into themselves, and those in \mathbf{P}' into minus themselves. The transformation by $I_{p,q}$ mixes the subspaces \mathbf{K} and \mathbf{P} , and splits the algebra in a different way into $\mathbf{K}' \oplus \mathbf{P}'$. The matrices

$$\begin{pmatrix} A & iB \\ -iB^\dagger & C \end{pmatrix} \in \mathbf{K}' \oplus i\mathbf{P}' \quad (74)$$

define the non-compact real form \mathbf{G}'^* . This algebra is called $\mathbf{SU}(\mathbf{p}, \mathbf{q}; \mathbf{C})$ and its maximal

compact subalgebra \mathbf{K}' is $\mathbf{SU}(\mathbf{p}) \otimes \mathbf{SU}(\mathbf{q}) \otimes \mathbf{U}(1)$.

σ_3) By the involutive automorphism $\sigma_3\sigma_1 = J_{p,p}K$ one constructs in a similar way (for details see [12]) a third non-compact real form (for even $n = 2p$) $\mathbf{G}''^* = \mathbf{K}'' \oplus i\mathbf{P}''$ associated to the algebra $\mathbf{G} = \mathbf{SU}(2\mathbf{p}, \mathbf{C})$. \mathbf{G}''^* is the algebra $\mathbf{SU}^*(2\mathbf{p})$ and its maximal compact subalgebra is $\mathbf{USp}(2\mathbf{p})$.⁷

This procedure, summarized in the formula below, exhausts all the real forms of the simple algebras.

$$\mathbf{G}^{\mathbf{C}} \rightarrow \mathbf{G} = \mathbf{K} \oplus \mathbf{P} \begin{array}{l} \nearrow^{\sigma_1} \\ \xrightarrow{\sigma_2} \\ \searrow^{\sigma_3} \end{array} \begin{array}{l} \mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P} \\ \mathbf{G}'^* = \mathbf{K}' \oplus i\mathbf{P}' \\ \mathbf{G}''^* = \mathbf{K}'' \oplus i\mathbf{P}'' \end{array} \quad (77)$$

Example: Note that it may not always be possible to apply all the above involutions $\sigma_1, \sigma_2, \sigma_3$ to the algebra. For example, complex-conjugation σ_1 does not do anything to $\mathbf{SO}(2\mathbf{n} + 1, \mathbf{R})$, because it is represented by real matrices, neither is σ_3 a symmetry of this algebra, since the adjoint representation is odd-dimensional and σ_3 has to act on a $2p \times 2p$ matrix. The only possibility remains $\sigma_2 = I_{p,q}$. For a second, even more concrete example, let's look at the algebra $\mathbf{SO}(3, \mathbf{R})$, belonging to the root lattice B_1 . This algebra is spanned by the generators L_1, L_2, L_3 given in subsection 2.3. A general element of the algebra is

$$X = \mathbf{t} \cdot \mathbf{L} = \frac{1}{2} \begin{pmatrix} t^3 & t^2 \\ -t^2 & t^1 \\ -t^2 & -t^1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} & t^3 \\ -t^3 & \end{pmatrix} \oplus \frac{1}{2} \begin{pmatrix} & t^2 \\ -t^2 & -t^1 \\ & t^1 \end{pmatrix} \quad (78)$$

⁷ The algebra $\mathbf{SU}^*(2\mathbf{p})$ is represented by complex $2p \times 2p$ matrices of the form

$$X = \begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \quad (75)$$

where $\text{tr}A + \text{tr}A^* = 0$. $\mathbf{USp}(2\mathbf{p})$ denotes the complex $2p \times 2p$ matrix algebra of the group with both unitary and symplectic symmetry ($\mathbf{USp}(2\mathbf{p}, \mathbf{C})$ can also be denoted $\mathbf{U}(\mathbf{p}, \mathbf{Q})$ where \mathbf{Q} is the field of quaternions). A matrix in the algebra $\mathbf{USp}(2\mathbf{p}, \mathbf{C})$ can be written as

$$X = \begin{pmatrix} A & B \\ -B^\dagger & -A^R \end{pmatrix} \quad (76)$$

where $A^\dagger = -A$, $B^R = B$, and the superscript R denotes reflection in the minor diagonal.

This splitting of the algebra is caused by the involution $I_{2,1}$ acting on the representation:

$$I_{2,1} X I_{2,1}^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} & t^3 & t^2 \\ -t^3 & & t^1 \\ -t^2 & -t^1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} & t^3 & -t^2 \\ -t^3 & & -t^1 \\ t^2 & t^1 & \end{pmatrix} \quad (79)$$

and it splits it into $\mathbf{SO}(3) = \mathbf{K} \oplus \mathbf{P} = \mathbf{SO}(2) \oplus \mathbf{SO}(3)/\mathbf{SO}(2)$. Exponentiating, as we saw in subsection 2.3, the coset representative is a point on the 2-sphere

$$M = \begin{pmatrix} \cdot & \cdot & t^2 \frac{\sin \sqrt{(t^1)^2 + (t^2)^2}}{\sqrt{(t^1)^2 + (t^2)^2}} \\ \cdot & \cdot & t^1 \frac{\sin \sqrt{(t^1)^2 + (t^2)^2}}{\sqrt{(t^1)^2 + (t^2)^2}} \\ \cdot & \cdot & \cos \sqrt{(t^1)^2 + (t^2)^2} \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & x \\ \cdot & \cdot & y \\ \cdot & \cdot & z \end{pmatrix}; \quad x^2 + y^2 + z^2 = 1 \quad (80)$$

By the Weyl unitary trick we now get the non-compact real form $\mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P}$: $\mathbf{SO}(2, 1) = \mathbf{SO}(2) \oplus \mathbf{SO}(2, 1)/\mathbf{SO}(2)$. This algebra is represented by

$$\begin{pmatrix} & t^3 & it^2 \\ -t^3 & & it^1 \\ -it^2 & -it^1 & \end{pmatrix} \begin{pmatrix} & t^3 \\ -t^3 & \end{pmatrix} \oplus \begin{pmatrix} & it^2 \\ -it^2 & -it^1 & it^1 \end{pmatrix} \quad (81)$$

and after exponentiation of the coset generators

$$M = \begin{pmatrix} \cdot & \cdot & it^2 \frac{\sinh \sqrt{(t^1)^2 + (t^2)^2}}{\sqrt{(t^1)^2 + (t^2)^2}} \\ \cdot & \cdot & it^1 \frac{\sinh \sqrt{(t^1)^2 + (t^2)^2}}{\sqrt{(t^1)^2 + (t^2)^2}} \\ \cdot & \cdot & \cosh \sqrt{(t^1)^2 + (t^2)^2} \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & ix \\ \cdot & \cdot & iy \\ \cdot & \cdot & z \end{pmatrix}; \quad (ix)^2 + (iy)^2 + z^2 = 1 \quad (82)$$

The surface in \mathbf{R}^3 consisting of points (x, y, z) satisfying this equation is the hyperboloid H^2 . Similarly, we get the isomorphic space $SO(1, 2)/SO(2)$ by applying $I_{1,2}$: $\mathbf{SO}(1, 2) = \tilde{\mathbf{K}} \oplus i\tilde{\mathbf{P}} = \mathbf{SO}(2) \oplus \mathbf{SO}(1, 2)/\mathbf{SO}(2)$ and in terms of the algebra

$$\tilde{X} = \frac{1}{2} \begin{pmatrix} & t^1 \\ -t^1 & \end{pmatrix} \oplus \frac{1}{2} \begin{pmatrix} -it^3 & -it^2 \\ it^3 & it^2 \end{pmatrix} \quad (83)$$

5 The classification of symmetric spaces

In this section we introduce the curvature tensor and the sectional curvature of a symmetric space, and we extend the family of symmetric spaces to include also flat or Euclidean-type spaces. These are identified with the subspace \mathbf{P} of the Lie algebra itself, and the group that acts on it is a semidirect product of the subgroup K and the subspace \mathbf{P} . As we will learn, to each compact subgroup K corresponds a triplet of symmetric spaces with positive, zero and negative curvature. The classification of these symmetric spaces is in exact correspondence with the new classification of random matrix models to be discussed in [3]. These spaces exhaust the Cartan classification and have a definite metric. They are listed in Table 1 together with some of their properties.

In paragraph 5.2 we introduce restricted root systems. In the same way as a Lie algebra corresponds to a given root system, each symmetric space corresponds to a restricted root system. These root systems are of primary importance in the physical applications to be discussed in our forthcoming paper [3]. The restricted root system can be of an entirely different type from the root system inherited from the complex extension algebra, and its rank may be different. We work out a specific example of a restricted root system as an illustration. In spite of their importance, we have not been able to find any explicit reference in the literature that explains how to obtain the restricted root systems. Instead, we found that they are often referred to in tables and in mathematical texts without explicitly mentioning that they are restricted, which could easily lead to confusion with the inherited root systems. In reference [12] the root system that is associated to each symmetric space is the one inherited from the complex extension algebra, whereas for example in Table B1 of reference [1] and in [19] the restricted root systems are listed.

There are also symmetric spaces with an indefinite metric, so called pseudo-Riemannian spaces, corresponding to a maximal *non-compact* subgroup H . For completeness, we will briefly discuss how these are obtained as real forms of symmetric spaces corresponding to compact symmetric subgroups. This does not require any new tools than the ones we have already introduced, namely the involutive automorphisms.

5.1 The curvature tensor and triplicity

Suppose that \mathbf{K} is a maximal compact subalgebra of the non-compact algebra \mathbf{G}^* in the Cartan decomposition $\mathbf{G}^* = \mathbf{K} \oplus i\mathbf{P}$, where $i\mathbf{P}$ is a complementary subspace. \mathbf{K} and \mathbf{P} (alternatively \mathbf{K} and $i\mathbf{P}$) satisfy eq. (36):

$$[\mathbf{K}, \mathbf{K}] \subset \mathbf{K}, \quad [\mathbf{K}, \mathbf{P}] \subset \mathbf{P}, \quad [\mathbf{P}, \mathbf{P}] \subset \mathbf{K} \quad (84)$$

\mathbf{K} is called a symmetric subalgebra and the coset spaces $\exp(\mathbf{P}) \simeq G/K$ and $\exp(i\mathbf{P}) \simeq G^*/K$ are globally symmetric Riemannian spaces. Globally symmetric means that every point on the manifold can be moved to any other point by a particular group operation (we discussed this in paragraph 2.3; for a rigorous definition of globally symmetric spaces see Helgason [11], paragraph IV.3). In the same way, the metric can be defined in any point of the manifold by moving the metric at the origin to this point, using a group operation (cf. eq. (51) in paragraph 3.4). The Killing form restricted to the tangent spaces \mathbf{P} and $i\mathbf{P}$ at any point in the coset manifold has a definite sign. The manifold is then called ‘‘Riemannian’’. The metric can be taken to be either plus or minus the Killing form so that it is always positive definite (cf. paragraph 3.4).

A *curvature tensor* with components R_{ijkl}^n can be defined on the manifold G/K or G^*/K in the usual way [11, 13]. It is a function of the metric tensor and its derivatives. It was proved for instance in [11], Ch. IV, that the components of the curvature tensor at the origin of a globally symmetric coset manifold is given by the expression

$$R_{ijkl}^n X_n = [X_i, [X_j, X_k]] = C_{im}^n C_{jk}^m X_n \quad (85)$$

where $\{X_i\}$ is a basis for the Lie algebra. The sectional curvature at a point p is equal to

$$\mathcal{K} = g([X, Y], [X, Y]) \quad (86)$$

where g is an arbitrary symmetric and nondegenerate metric (such a metric is also called a pseudo-Riemannian structure, or simply a Riemannian structure if it has a definite sign) on the tangent space at p , invariant under the action of the group elements. In (86), $g(X_i, X_j) \equiv g_{ij}$ and $\{X, Y\}$ is an orthonormal basis for a two-dimensional subspace S of the tangent space at the point p (assuming it has dimension ≥ 2). The sectional curvature is equal to the gaussian curvature on a 2-dimensional manifold. If the manifold has dimension ≥ 2 , (86) gives the sectional curvature along the section S .

Eqs. (85) and (86), together with eq. (84) show that the curvature of the spaces G/K and G^*/K has a definite and opposite sign ([11], par. V.3). Thus, we see that if G is a compact semisimple group, to the same subgroup K there corresponds a positive curvature space $P \simeq G/K$ and a dual negative curvature space $P^* \simeq G^*/K$. The reason for this is exactly the same as the reason why the sign changes for the components of the metric corresponding to the generators in $i\mathbf{P}$ as we go to the dual space \mathbf{P} . We remind the reader that the sign of the metric can be chosen positive or negative for a compact space. The issue here is that the sign changes in going from G^*/K to G/K .

Example: We can use the example of $SU(2)$ in paragraph 4.1 to see that the sectional curvature is the opposite for the two spaces G/K and G^*/K . If we take $\{X, Y\} = \{\Sigma_3, \Sigma_1\}$ as the basis in the space $i\mathbf{P}$ and $\{\tilde{\Sigma}_3, \tilde{\Sigma}_1\}$ ($\tilde{\Sigma}_i \equiv i\Sigma_i$) as the basis in the space \mathbf{P} , we see by comparing the signs of the entries of the metrics we computed in eqs. (67) and (69) that the sectional curvature \mathcal{K} at the origin has the opposite sign for the two spaces $SO(2, 1)/SO(2)$ and $SO(3)/SO(2)$.

Actually, there is also a zero-curvature symmetric space $X^0 = G^0/K$ related to $X^+ = G/K$ and $X^- = G^*/K$, so that we can speak of a *triple* of symmetric spaces related to the *same* symmetric subgroup K . The zero-curvature spaces were discussed in [1] and in Ch. V of Helgason's book [11], where they are referred to as "symmetric spaces of the euclidean type". That their curvature is zero was proved in Theorem 3.1 of [11], Ch. V.

The flat symmetric space X^0 can be identified with the subspace \mathbf{P} of the algebra. The group G^0 is a semidirect product of the subgroup K and the invariant subspace \mathbf{P} of the algebra, and its elements $g = (k, a)$ act on the elements of X^0 in the following way:

$$g(x) = kx + a, \quad k \in K, \quad x, a \in X^0 \quad (87)$$

if the x 's are vectors, and

$$g(x) = kxk^{-1} + a, \quad k \in K, \quad x, a \in X^0 \quad (88)$$

if the x 's are matrices. We will see one example of each below.

The elements of the algebra \mathbf{P} now define an *abelian additive group*, and X^0 is a vector space with euclidean geometry. In the above scenario, the subspace \mathbf{P} contains only the operators of the Cartan subalgebra and no others: $\mathbf{P} = \mathbf{H}_0$, so that \mathbf{P} is a subalgebra of \mathbf{G}^0 . The algebra $\mathbf{G}^0 = \mathbf{K} \oplus \mathbf{P}$ belongs to a non-semisimple group G^0 , since it has an abelian ideal \mathbf{P} : $[\mathbf{K}, \mathbf{K}] \subset \mathbf{K}$, $[\mathbf{K}, \mathbf{P}] \subset \mathbf{P}$, $[\mathbf{P}, \mathbf{P}] = 0$. Note that \mathbf{K} and \mathbf{P} still satisfy the

commutation relations (84). In this case the coset space X^0 is flat, since by (84), $R_{ijk}^n = 0$ for all the elements $X \in \mathbf{P}$. Eq. (85) is valid for any space with a Riemannian structure. Indeed, it is easy to see from eqs. (85), (86) that $R_{ijk}^n = \mathcal{K} = 0$ if the generators are abelian. Even though the non-semisimple algebras have a degenerate metric, it is trivial to find a non-degenerate metric on the symmetric space X^0 that can be used in (86) to find that the sectional curvature at any point is zero. For example, as we pass from the sphere to the plane, the metric becomes degenerate in the limit as $[L_1, L_2] \sim L_3 \rightarrow [P_1, P_2] = 0$ (see the example below). Obviously, we do not inherit this degenerate metric from the tangent space on \mathbf{R}^2 like in the case of the sphere, but the usual metric for \mathbf{R}^2 , $g_{ij} = \delta_{ij}$ provides the Riemannian structure on the plane.

Examples: An example of a flat symmetric space is E_2/K , where $G^0 = E_2$ is the euclidean group of motions of the plane \mathbf{R}^2 : $g(x) = kx + a$, $g = (k, a) \in G^0$ where $k \in K = SO(2)$ and $a \in \mathbf{R}^2$. The generators of this group are translations $P_1, P_2 \in \mathbf{H}_0 = \mathbf{P}$ and a rotation $J \in \mathbf{K}$ satisfying $[P_1, P_2] = 0$, $[J, P_i] = -\epsilon^{ij} P_j$, $[J, J] = 0$, in agreement with eq. (84) defining a symmetric subgroup. The abelian algebra of translations $\sum_{i=1}^2 t^i P_i$, $t^i \in \mathbf{R}$, is isomorphic to the plane \mathbf{R}^2 , and can be identified with it.

The commutation relations for E_2 are a kind of limiting case of the commutation relations for $\mathbf{SO}(3) \sim \mathbf{SU}(2)$ and $\mathbf{SO}(2, 1)$. If in the limit of infinite radius of the sphere S^2 we identify $\tilde{\Sigma}_1$ with P_1 , $\tilde{\Sigma}_2$ with P_2 , and $\tilde{\Sigma}_3$ with J , we see that the commutation relations resemble the ones described in eq. (66) and (68) – we only have to set $[\tilde{\Sigma}_1, \tilde{\Sigma}_2] = 0$, which amounts to setting $C_{12}^3 = -C_{21}^3 \rightarrow 0$. From here we get the degenerate metric of the non-semisimple algebra \mathbf{E}_2 :

$$g_{ij} = \begin{pmatrix} -1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad (89)$$

where the only nonzero element is g_{33} . This is to be confronted with eqs. (67) and (69) which are the metrics for $\mathbf{SO}(2, 1)$ and $\mathbf{SO}(3)$. This is an example of contraction of an algebra.

An example of a triplet $\{X^+, X^0, X^-\}$ corresponding to the same subgroup $K = SO(n)$ is:

1) $X^+ = SU(n, C)/SO(n)$, the set of symmetric unitary matrices with unit determinant; it is the space $\exp(\mathbf{P})$ where \mathbf{P} are real, symmetric and traceless $n \times n$ matrices. (Cf. the example in subsection 3.1.)

2) X^0 is the set \mathbf{P} of real, symmetric and traceless $n \times n$ matrices and the non-semisimple

group G^0 is the group whose action is defined by $g(x) = kxk^{-1} + a$, $g = (k, a) \in G^0$ where $k \in K = SO(n)$ and $x, a \in X^0$. The involutive automorphism maps $g = (k, a) \in G^0$ into $g' = (k, -a)$.

3) $X^- = SL(n, R)/SO(n)$ is the set of real, positive, symmetric matrices with unit determinant; it is the space $\exp(i\mathbf{P})$ where \mathbf{P} are real, symmetric and traceless $n \times n$ matrices.

We remark that the zero-curvature symmetric spaces correspond to the integration manifolds of many known matrix models with physical applications.

The pairs of dual symmetric spaces of positive and negative curvature listed in each row of Table 1 originate in the same complex extension algebra [12] with a given root lattice. This “inherited” root lattice is listed in the first column of the table. In our example in paragraph 4.2 this was the root lattice of the complex algebra $\mathbf{G}^{\mathbf{C}} = \mathbf{SL}(\mathbf{n}, \mathbf{C})$. The same root lattice A_{n-1} characterizes the real forms of $\mathbf{SL}(\mathbf{n}, \mathbf{C})$: as we saw in the example these are the algebras $\mathbf{SU}(\mathbf{n}, \mathbf{C})$, $\mathbf{SL}(\mathbf{n}, \mathbf{R})$, $\mathbf{SU}(\mathbf{p}, \mathbf{q}; \mathbf{C})$ and $\mathbf{SU}^*(2\mathbf{n})$, and we have seen how to construct them using involutive automorphisms.

However, also listed in Table 1 is the *restricted root system* corresponding to each symmetric space. This root system may be different from the one inherited from the complex extension algebra. Below, we will define the restricted root system and see an explicit example of one such system. While the original root lattice characterizes the complex extension algebra and its real forms, the restricted root lattice characterizes a particular symmetric space originating from one of its real forms. The root lattices of the classical simple algebras are the infinite sequences A_n, B_n, C_n, D_n , where the index n denotes the rank of the corresponding group. The *root multiplicities* m_o, m_l, m_s listed in Table 1 (where the subscripts refer to ordinary, long and short roots, respectively) are characteristic of the restricted root lattices. In general, in the root lattice of a simple algebra (or in the graphical representation of any irreducible representation), the roots (weights) may be degenerate and thus have a multiplicity greater than 1. This happens if the same weight $\mu = (\mu_1, \dots, \mu_r)$ corresponds to different states in the representation. In that case one can arrive at that particular weight using different sets of lowering operators $E_{-\alpha}$ on the highest weight of the representation. Indeed, we saw in the example of $SU(3, C)$ in subsection 2.5, that the roots can have a multiplicity different from 1. The same is true for the restricted roots.

The sets of simple roots of the classical root systems (briefly listed in subsection 2.5) have been obtained for example in [12, 17]. In the canonical basis in \mathbf{R}^n , the roots of type $\{\pm e_i \pm e_j, i \neq j\}$ are ordinary while the roots $\{\pm 2e_i\}$ are long and the roots $\{\pm e_i\}$ are short. Only a few sets of root multiplicities are compatible with the strict properties characterizing root lattices in general.

5.2 Restricted root systems

The restricted root systems play an important role in connection with matrix models and integrable Calogero–Sutherland models. We will discuss this in detail in [3]. In this subsection we will explain how restricted root systems are obtained and how they are related to a given symmetric space.⁸

As we have repeatedly seen in the examples using the compact algebra $\mathbf{SU}(\mathbf{n}, \mathbf{C})$ (in particular in subsection 4.2), the algebra $\mathbf{SU}(\mathbf{p}, \mathbf{q}; \mathbf{C})$ ($p+q = n$) is a non-compact real form of the former. This means they share the same rank- $(n-1)$ root system A_{n-1} . However, to the symmetric space $SU(p, q; \mathbf{C})/(SU(p) \otimes SU(q) \otimes U(1))$ one can associate another rank- r' root system, where $r' = \min(p, q)$ is the rank of the symmetric space. For some symmetric spaces, it is the same as the root system inherited from the complex extension algebra (see Table 1 for a list of the restricted root systems), but this need not be the case. For example, the restricted root system is, in the case of $SU(p, q; \mathbf{C})/(SU(p) \otimes SU(q) \otimes U(1))$, $BC_{r'}$. When it is the same and when it is different, as well as why the rank can change, will be obvious from the example we will give below.

In general the restricted root system will be different from the original, inherited root system if the Cartan subalgebra lies in \mathbf{K} . The procedure to find the restricted root system is then to define an *alternative Cartan subalgebra* that lies partly (or entirely) in \mathbf{P} (or $i\mathbf{P}$).

To achieve this, we first look for a different representation of the original Cartan subalgebra, that gives the same root lattice as the original one (i.e., A_{n-1} for the $\mathbf{SU}(\mathbf{p}, \mathbf{q}; \mathbf{C})$ algebra). In general, this root lattice is an automorphism of the original root lattice of the same kind, obtained by a permutation of the roots. Unless we find this new representation, we will not be able to find a new, alternative Cartan subalgebra that lies partly in the subspace \mathbf{P} .

Once this has been done, we take a maximal abelian subalgebra of \mathbf{P} (the number of generators in it will be equal to the rank r' of the symmetric space G/K or G^*/K) and find the generators in \mathbf{K} that commute with it. These generators will be among the ones that are in the new representation of the original Cartan subalgebra. These commuting generators now form our new, alternative Cartan subalgebra that lies partly in \mathbf{P} , partly in \mathbf{K} . Let's call it \mathbf{A}_0 .

The new root system is defined with respect to the part of the maximal abelian subalgebra that lies in \mathbf{P} . Therefore its rank is normally smaller than the rank of the root system

⁸The author is indebted to Prof. Simon Salamon for explaining to her how the restricted root systems are obtained.

inherited from the complex extension. We can define raising and lowering operators E'_α in the *whole* algebra \mathbf{G} that satisfy

$$[X'_i, E'_\alpha] = \alpha'_i E'_\alpha \quad (X'_i \in \mathbf{A}_0 \cap \mathbf{P}) \quad (90)$$

The roots α'_i define the restricted root system.

Example: Let's now look at a specific example. We will start with the by now familiar algebra $\mathbf{SU}(3, \mathbf{C})$. As before, we use the convention of regarding the T_i 's as the generators, without the i in front (recall that the algebra consists of elements of the form $\sum_a t^a X_a = i \sum_a t^a T_a$; cf. the footnote in conjunction with eq. (25)). In subsection 2.5 we explicitly constructed its root lattice A_2 . Let's write down the generators again:

$$\begin{aligned} T_1 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T_2 &= \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T_3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ T_4 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & T_5 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & T_6 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ T_7 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & T_8 &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \quad (91)$$

The splitting of the $\mathbf{SU}(3, \mathbf{C})$ algebra in terms of the subspaces \mathbf{K} and \mathbf{P} was given in eq. (37):

$$\mathbf{K} = \{iT_2, iT_5, iT_7\}, \quad \mathbf{P} = \{iT_1, iT_3, iT_4, iT_6, iT_8\} \quad (92)$$

The Cartan subalgebra is $\{iT_3, iT_8\}$. The raising and lowering operators were given in (30) in terms of T_i :

$$\begin{aligned}
E_{\pm(1,0)} &= \frac{1}{\sqrt{2}}(T_1 \pm iT_2) \\
E_{\pm(\frac{1}{2}, \frac{\sqrt{3}}{2})} &= \frac{1}{\sqrt{2}}(T_4 \pm iT_5) \\
E_{\pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})} &= \frac{1}{\sqrt{2}}(T_6 \pm iT_7)
\end{aligned} \tag{93}$$

Now let us construct the Cartan decomposition of $\mathbf{G}'^* = \mathbf{K}' \oplus i\mathbf{P}' = \mathbf{SU}(2, 1; \mathbf{C})$. We know from paragraph 4.2 that \mathbf{K}' and \mathbf{P}' are given by matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \in \mathbf{K}', \quad \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} \in \mathbf{P}' \tag{94}$$

where A and C are antihermitean and $\text{tr}A + \text{tr}C = 0$. Combining the generators to form this kind of block-structures (or alternatively, using the involution $\sigma_2 = I_{2,1}$) we need to take linear combinations of the X_i 's, with real coefficients, and we then see that the subspaces \mathbf{K}' and $i\mathbf{P}'$ are spanned by

$$\begin{aligned}
\mathbf{K}' &= \left\{ \frac{i}{2} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}, \frac{i}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & -2 \end{pmatrix} \right\} \\
&= \{iT_1, iT_2, iT_3, iT_8\} \\
i\mathbf{P}' &= \left\{ \frac{1}{2} \begin{pmatrix} & 1 & \\ & 0 & \\ 1 & 0 & \end{pmatrix}, \frac{i}{2} \begin{pmatrix} & -1 & \\ & 0 & \\ 1 & 0 & \end{pmatrix}, \frac{1}{2} \begin{pmatrix} & 0 & \\ & 1 & \\ 0 & 1 & \end{pmatrix}, \frac{i}{2} \begin{pmatrix} & 0 & \\ & -1 & \\ 0 & 1 & \end{pmatrix} \right\} \\
&= \{T_4, T_5, T_6, T_7\}
\end{aligned} \tag{95}$$

where the block-structure is evidenced by leaving blank the remaining zero entries. \mathbf{K}' spans the algebra of the symmetric subgroup $SU(2) \otimes U(1)$ and $i\mathbf{P}'$ spans the complementary subspace corresponding to the symmetric space $SU(2, 1)/(SU(2) \otimes U(1))$. $i\mathbf{P}'$ is spanned by matrices of the form

$$\begin{pmatrix} 0 & \tilde{B} \\ \tilde{B}^\dagger & 0 \end{pmatrix} \tag{96}$$

We see that the Cartan subalgebra $i\mathbf{H}_0 = \{iT_3, iT_8\}$ lies entirely in \mathbf{K}' . It is easy to see that by using the alternative representation

$$T'_3 = \frac{1}{2} \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad T'_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix} \quad (97)$$

of the Cartan subalgebra (note that this is a valid representation of $\mathbf{SU}(\mathbf{3}, \mathbf{C})$ generators) while the other T_i 's are unchanged, we still get the same root lattice A_2 . The eigenvectors under the adjoint representation, the E_α 's, are still given by eq. (93). However, their eigenvalues (roots) are permuted under the new adjoint representation of the Cartan subalgebra, so that they no longer correspond to the root subscripts in (93).

Now we choose the alternative Cartan subalgebra to consist of the generators T_4, T'_8 :

$$\mathbf{A}_0 = \{T_4, T'_8\}, \quad [T_4, T'_8] = 0, \quad iT_4 \in \mathbf{P}', \quad iT'_8 \in \mathbf{K}' \quad (98)$$

(Note that unless we first take a new representation of the original Cartan subalgebra, we are not able to find the alternative Cartan subalgebra that lies partly in \mathbf{P}' .) The restricted root system is now about to be revealed. We define raising and lowering operators E'_α in the whole algebra according to

$$E'_{\pm 1} \sim (T_5 \pm iT_3), \quad E'_{\pm \frac{1}{2}} \sim (T_6 \pm iT_2), \quad \tilde{E}'_{\pm \frac{1}{2}} \sim (T_7 \pm iT_1) \quad (99)$$

The $\pm\alpha$ subscripts are the eigenvalues of $T_4 \in i\mathbf{P}'$ in the adjoint representation:

$$[T_4, E'_{\pm 1}] = \pm E'_{\pm 1}, \quad [T_4, E'_{\pm \frac{1}{2}}] = \pm \frac{1}{2} E'_{\pm \frac{1}{2}}, \quad [T_4, \tilde{E}'_{\pm \frac{1}{2}}] = \pm \frac{1}{2} \tilde{E}'_{\pm \frac{1}{2}} \quad (100)$$

These roots form a one-dimensional root system of type BC_1 . We see that the multiplicity of the long roots is 1 and the multiplicity of the short roots is $2 = 2(p - q)$. This result is general (cf. Table 1). If we had ordinary roots, their multiplicity would be 2, but for this low-dimensional group we can have only 3 pairs of roots. Note that we can rescale the lengths of all the roots together by rescaling the operator T_4 in (100), but their characters as long and short roots can not change. The root system BC_1 is with respect to the part of the Cartan subalgebra lying in $i\mathbf{P}'$ only, thus it is called restricted.

Table 1. Irreducible symmetric spaces of positive and negative curvature originating in simple Lie groups. In the third and fourth columns the symmetric spaces G/K and G^*/K are listed for all the entries except the simple Lie groups themselves, for which the symmetric spaces G and G^C/G are listed. Note that there are also zero curvature spaces corresponding to non-semisimple groups and isomorphic to the subspace \mathbf{P} of the algebra, when \mathbf{P} is an abelian invariant subalgebra. These are not listed in the table, but can be constructed as explained in subsection 5.1. The root multiplicities listed pertain to the *restricted* root systems of the pairs of dual symmetric spaces with positive and negative curvature.

Root space	Restricted root space	Cartan class	G/K (G)	G^*/K (G^C/G)	m_o	m_l	m_s
A_{N-1}	A_{N-1}	A	$SU(N)$	$\frac{SU(N,C)}{SU(N)}$	2	0	0
	A_{N-1}	AI	$\frac{SU(N)}{SO(N)}$	$\frac{SU(N,R)}{SO(N)}$	1	0	0
	A_{N-1}	AII	$\frac{SU(2N)}{USp(2N)}$	$\frac{SU^*(2N)}{USp(2N)}$	4	0	0
	BC_q ($p>q$)	AIII	$\frac{SU(p+q)}{SU(p) \times SU(q) \times U(1)}$	$\frac{SU(p,q)}{SU(p) \times SU(q) \times U(1)}$	2	1	$2(p-q)$
	C_q ($p=q$)						
B_N	B_N	B	$SO(2N+1)$	$\frac{SO(2N+1,C)}{SO(2N+1)}$	2	0	2
C_N	C_N	C	$USp(2N)$	$\frac{Sp(2N,C)}{USp(2N)}$	2	2	0
	C_N	CI	$\frac{USp(2N)}{SU(N) \times U(1)}$	$\frac{Sp(2N,R)}{SU(N) \times U(1)}$	1	1	0
	BC_q ($p>q$)	CII	$\frac{USp(2p+2q)}{USp(2p) \times USp(2q)}$	$\frac{USp(2p,2q)}{USp(2p) \times USp(2q)}$	4	3	$4(p-q)$
	C_q ($p=q$)						
D_N	D_N	D	$SO(2N)$	$\frac{SO(2N,C)}{SO(2N)}$	2	0	0
	C_N	DIII-even	$\frac{SO(4N)}{SU(2N) \times U(1)}$	$\frac{SO^*(4N)}{SU(2N) \times U(1)}$	4	1	0
	BC_N	DIII-odd	$\frac{SO(4N+2)}{SU(2N+1) \times U(1)}$	$\frac{SO^*(4N+2)}{SU(2N+1) \times U(1)}$	4	1	4
B_N ($p+q=2N+1$)	B_q ($p>q$)	BDI	$\frac{SO(p+q)}{SO(p) \times SO(q)}$	$\frac{SO(p,q)}{SO(p) \times SO(q)}$	1	0	$p-q$
D_N ($p+q=2N$)	D_q ($p=q$)						

5.3 Real forms of symmetric spaces

Involutive automorphisms were used to split the algebra \mathbf{G} into orthogonal subspaces to obtain the real forms \mathbf{G} , \mathbf{G}^* , $\mathbf{G}^{/*}$... of a complex extension algebra \mathbf{G}^C . By re-applying the same involutive automorphisms to the spaces \mathbf{K} , \mathbf{P} , and $i\mathbf{P}$, these spaces with a definite metric tensor can in turn be split into subspaces with eigenvalue +1 and -1 under this new involutive automorphism τ . Thus,

$$\begin{aligned}
\sigma : \mathbf{G} &\rightarrow \mathbf{K} \oplus \mathbf{P}, & \mathbf{G}^* &= \mathbf{K} \oplus i\mathbf{P} \\
\tau : \mathbf{K} &\rightarrow \mathbf{K}_1 \oplus \mathbf{K}_2, & \mathbf{H} &= \mathbf{K}_1 \oplus i\mathbf{K}_2 \\
\tau : \mathbf{P} &\rightarrow \mathbf{P}_1 \oplus \mathbf{P}_2, & \mathbf{M} &= \mathbf{P}_1 \oplus i\mathbf{P}_2 \\
\tau : i\mathbf{P} &\rightarrow i\mathbf{P}_1 \oplus i\mathbf{P}_2, & i\mathbf{M} &= i\mathbf{P}_1 \oplus \mathbf{P}_2
\end{aligned} \tag{101}$$

As we already know, K is a compact subgroup, and $\exp(\mathbf{P})$ and $\exp(i\mathbf{P})$ define symmetric spaces with a *definite* metric (Riemannian spaces). In the same way, H is a non-compact subgroup, $\exp(\mathbf{M})$ and its dual space $\exp(i\mathbf{M})$ define symmetric spaces with an *indefinite* metric. These are pseudo-Riemannian symmetric coset spaces of a non-compact group by a maximal non-compact subgroup⁹. The original algebra \mathbf{G} is thereby split into four components $\mathbf{K}_1, \mathbf{K}_2, \mathbf{P}_1, \mathbf{P}_2$, depending on their eigenvalues ($++$, $+-$, $-+$, $--$) under the two successive automorphisms σ, τ . By applying all the possible σ 's and all the possible τ 's, or by replacing either σ or τ by the involutive automorphism $\sigma\tau = \tau\sigma$, we obtain all the possible *real forms of the symmetric spaces* associated with the compact algebra \mathbf{G} .

Example: The complex algebra $\mathbf{SO}(\mathbf{3}, \mathbf{C})$ has a root system of type B_n . Its compact real form is $\mathbf{SO}(\mathbf{3}, \mathbf{R})$, and its only non-compact real form is $\mathbf{SO}(\mathbf{p}, \mathbf{q}; \mathbf{R}) \simeq \mathbf{SO}(\mathbf{q}, \mathbf{p}; \mathbf{R})$ ($p + q = 3$), obtained by applying the involution $\sigma = I_{p,q}$ ($I_{q,p}$) to $\mathbf{SO}(\mathbf{3}, \mathbf{R})$. In paragraph 4.2 we constructed two Riemannian symmetric spaces associated with the algebra $\mathbf{SO}(\mathbf{3})$, the sphere $SO(3)/SO(2)$ and the double-sheeted hyperboloid $SO(2, 1)/SO(2)$. The Killing form has a definite opposite sign for the two spaces.

The single-sheeted hyperboloid, described by the equation $-x^2 + y^2 + z^2 = 1$ in \mathbf{R}^3 , corresponds to the pseudo-Riemannian symmetric space $SO(2, 1)/SO(1, 1)$ associated with the same algebra. It is obtained by applying two consecutive involutive automorphisms $\sigma = I_{2,1}, \tau = I_{1,2}$ to the algebra $\mathbf{G} = \mathbf{SO}(\mathbf{3}, \mathbf{R})$. Like in eq. (81), $I_{2,1}$ and the Weyl unitary trick transforms \mathbf{G} into \mathbf{G}^* . Let's now apply $I_{1,2}$ to \mathbf{G}^* :

⁹Note that not all the theorems governing symmetric spaces corresponding to maximal compact subgroups apply to the case at hand. A prime example is the decomposition involving radial coordinates in subsection 3.3. We will not discuss the symmetric spaces involving maximal non-compact subgroups in any detail in this paper.

$$\begin{aligned}
I_{1,2}XI_{1,2}^{-1} &= \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} & t^3 & it^2 \\ -t^3 & & it^1 \\ -it^2 & -it^1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} & -t^3 & -it^2 \\ t^3 & & it^1 \\ it^2 & -it^1 & \end{pmatrix} = \frac{1}{2} \begin{pmatrix} & -t^3 & \\ t^3 & & \end{pmatrix} \oplus \frac{1}{2} \begin{pmatrix} & -it^2 & \\ it^2 & -it^1 & it^1 \end{pmatrix} \\
&= (\mathbf{K}_1 \oplus \mathbf{K}_2) \oplus i(\mathbf{P}_1 \oplus \mathbf{P}_2)
\end{aligned} \tag{102}$$

where in this example, \mathbf{K}_1 is empty. The spaces $\mathbf{K}_1, \mathbf{K}_2, \mathbf{P}_1, \mathbf{P}_2$ consist of the generators in \mathbf{G} with the following combinations of eigenvalues under the two successive involutions, $\sigma\tau$:

$$\mathbf{K}_1 : ++ \quad \mathbf{K}_2 : +- \quad \mathbf{P}_1 : + \quad \mathbf{P}_2 : -- \tag{103}$$

Thus we see that \mathbf{K}_1 is empty and the others are spanned by

$$\mathbf{K}_2 = \left\{ \frac{1}{2} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\}, \quad \mathbf{P}_1 = \left\{ \frac{1}{2} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\}, \quad \mathbf{P}_2 = \left\{ \frac{1}{2} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\} \tag{104}$$

The new symmetric space is obtained by doing the Weyl unitary trick on the split spaces $(\mathbf{K}_1 \oplus \mathbf{K}_2)$ and $(\mathbf{P}_1 \oplus \mathbf{P}_2)$:

$$\begin{aligned}
\mathbf{H} &= \mathbf{K}_1 \oplus i\mathbf{K}_2 = \frac{1}{2} \begin{pmatrix} & it^3 \\ -it^3 & \end{pmatrix} \\
\mathbf{M} &= \mathbf{P}_1 \oplus i\mathbf{P}_2 = \frac{1}{2} \begin{pmatrix} & it^2 \\ -it^2 & -t^1 \end{pmatrix}
\end{aligned} \tag{105}$$

The second involution τ (plus the Weyl trick) gives rise to a non-compact subgroup $H=SO(1,1)$ and to the symmetric space $M \sim \exp\mathbf{M}$ and its dual $M^* \sim \exp(i\mathbf{M})$. The coset $M \sim SO(2,1)/SO(1,1)$ is represented by

$$\exp \mathbf{M} = \begin{pmatrix} \cdot & \cdot & ix \\ \cdot & \cdot & y \\ -ix & -y & z \end{pmatrix}; \quad (ix)^2 + y^2 + z^2 = 1 \quad (106)$$

The real forms of the simple Lie groups do not include all the possible Riemannian symmetric coset spaces. For example, the compact Lie group G is itself such a space, and so is its dual G^C/G (here the algebra $\mathbf{G}^C = \mathbf{G}^* \oplus i\mathbf{G}^*$ is the complex extension of all the real forms \mathbf{G}^*). By starting with a compact algebra \mathbf{G} and applying to it all the combinations of the two involutive automorphisms σ, τ , we construct, in the way just described, all the remaining pseudo-Riemannian symmetric spaces associated to the corresponding root system. A complete list of these spaces can be found in Table 9.7 of reference [12].

Note that all the properties of the Lie algebra \mathbf{G} (Killing form, rank, and so on) can be transferred to the vector subspaces $\mathbf{P}, i\mathbf{P}$ [12]. The only difference is that the subspaces are not closed under commutation.

6 Operators on symmetric spaces

The differential operator uniquely determined by the simplest Casimir operator on a symmetric space (and especially its radial part) plays an important role both in mathematics and in the physical applications of symmetric spaces. Its eigenfunctions provide a complete basis for the expansion of an arbitrary square-integrable function on the symmetric space, and are therefore important in their own right. Their importance in the applications to be discussed in [3] is evident when considering that the radial part of the Laplace–Beltrami operator on an underlying symmetric space determines the dynamics of the transfer matrix eigenvalues of the DMPK equation in the theoretical description of quantum wires, and maps onto the Hamiltonians of integrable Calogero–Sutherland models. Here we will define some concepts related to the Laplace–Beltrami operator and discuss its eigenfunctions.

6.1 Casimir operators

Let \mathbf{G} be a semisimple rank- r Lie algebra. A *Casimir operator* (invariant operator) C_k ($k = 1, \dots, r$) associated with the algebra \mathbf{G} is a homogeneous polynomial operator that satisfies

$$[C_k, X_i] = 0 \quad (107)$$

for all $X_i \in \mathbf{G}$. The simplest (quadratic) Casimir operator associated to the adjoint representation of the algebra \mathbf{G} is given by

$$C = g^{ij} X_i X_j \quad (108)$$

where g^{ij} is the inverse of the metric tensor defined in (49) and the generators X_i are in the adjoint representation. More generally, it can be defined for any representation ρ of \mathbf{G} by

$$C_\rho = g_\rho^{ij} \rho(X_i) \rho(X_j) \quad (109)$$

where g_ρ^{ij} is the inverse of the metric (50) for the representation ρ (cf. subsection 3.4). The Casimir operators lie in the enveloping algebra obtained by embedding \mathbf{G} in the associative algebra defined by the relations

$$X(YZ) = (XY)Z \quad [X, Y] = XY - YX \quad (110)$$

(note that in general, XY makes no sense in the algebra \mathbf{G}).

The number of functionally independent Casimir operators is equal to the rank r of the group. Other Casimir operators can be formed by taking polynomials of the independent Casimir operators C_k ($k = 1, \dots, r$). Since the Casimir operators commute with all the elements in \mathbf{G} , they make up the center of the associative algebra (110).

Note that Casimir operators are defined for *semisimple* algebras, where the metric tensor has an inverse. This does not prevent one from finding operators that commute with all the generators of non-semisimple algebras. For example, for the euclidean group E_3 of rotations $\{J_1, J_2, J_3\}$ and translations $\{P_1, P_2, P_3\}$, $\mathbf{P}^2 = \sum P_i P_i$ and $\mathbf{P} \cdot \mathbf{J} = \sum P_i J_i$ commute with all the generators. Also these analogous operators commuting with all the group elements associated to semisimple groups are often referred to as Casimir operators.

All the independent Casimir operators of the algebra \mathbf{G} can be obtained as follows. Suppose ρ is an n -dimensional representation of the rank- r Lie algebra \mathbf{G} . The *secular equation* for the algebra \mathbf{G} is defined as the eigenvalue equation

$$\det \left(\sum_{i=1}^{\dim \mathbf{G}} t^i \rho(X_i) - \lambda I_n \right) = \sum_{k=0}^n (-\lambda)^{n-k} \varphi_k(t^i) = 0 \quad (111)$$

where the $\varphi_k(t^i)$ are functions of the real coordinates t^i . In general, they will not all be functionally independent (for example, $\varphi_0(t^i)$ is a constant). There will be r functionally independent coefficients $\varphi_k(t^i)$ multiplying the powers of $-\lambda$ [12]. When writing down the secular equation, it is easiest to take a low-dimensional representation. By making the substitution $t^i \rightarrow X_i$ in the functionally independent coefficients, they become the functionally independent Casimir operators of the algebra \mathbf{G} :

$$\varphi_k(t^i) \xrightarrow{t^i \rightarrow X_i} C_l(X_i) \quad (112)$$

Example: The generators L_1, L_2, L_3 of the $\mathbf{SO}(3)$ algebra were given explicitly in the adjoint representation in equations (5), (6) in subsection 2.3. The secular equation for this algebra is then

$$\det(\mathbf{t} \cdot \mathbf{L} - \lambda I_3) = \begin{vmatrix} -\lambda & t^2/2 & t^2/2 \\ -t^3/2 & -\lambda & t^1/2 \\ -t^2/2 & -t^1/2 & -\lambda \end{vmatrix} = (-\lambda)^3 + (-\lambda) \frac{1}{4} \mathbf{t}^2 = 0 \quad (113)$$

The equation has one functionally independent coefficient, which is proportional to the trace of the matrix $(\mathbf{t} \cdot \mathbf{L})^2$. It equals $\varphi_1(\mathbf{t}) = \frac{1}{4} \mathbf{t}^2$. The rank of $SO(3)$ is 1 and the only Casimir operator is

$$C_1 \sim \mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2 \quad (114)$$

It is obtained by the substitution $t^i \rightarrow L_i$ in $\varphi_1(\mathbf{t})$. The Casimir operator can also be obtained from eq. (108) by using the metric $g_{ij} = -\frac{1}{2} \delta_{ij}$ for $\mathbf{SO}(3)$ given in the example in subsection 3.4. We know from elementary quantum mechanics that

$$[\mathbf{L}^2, L_1] = [\mathbf{L}^2, L_2] = [\mathbf{L}^2, L_3] = 0 \quad (115)$$

is an immediate consequence of the commutation relations, so we see that this operator indeed commutes with all the generators. Even though the commutation relations are

not the same in polar coordinates or after a general coordinate transformation, (115) will nevertheless be true.

Example: $SU(3)$ is a rank-2 group and therefore its characteristic equation will have two independent coefficients. If we denote a general $\mathbf{SU}(3)$ matrix (a_{ij}) we get the characteristic equation

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix} = (-\lambda)^3 + (-\lambda)^2(a_{11} + a_{22} + a_{33})$$

$$+ (-\lambda)(a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13})$$

$$+ (a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{32}a_{21} - a_{31}a_{22})) = 0$$
(116)

The term proportional to $(-\lambda)^2$ vanishes, because the trace of any matrix in the $\mathbf{SU}(3)$ algebra is zero. The two independent coefficients are then $\varphi_2(a_{ij})$ and $\varphi_3(a_{ij})$. Substituting the values in terms of the coordinates t^i of the algebra $\sum_i t^i T_i$ for the a_{ij} (for example, $a_{11} = t^3 + \frac{1}{\sqrt{3}}t^8$, $a_{12} = t^1 + it^2$, etc.), we see that the expression for $\varphi_2(t^i)$ becomes

$$\varphi_2(t^i) = \sum_{i=1}^8 (t^i)^2$$
(117)

and therefore the substitution (117) gives the first Casimir operator

$$C_1 = H_1^2 + H_2^2 + \sum_{\alpha} (E_{\alpha}E_{-\alpha} + E_{-\alpha}E_{\alpha}) = \sum_{i=1}^8 T_i^2$$
(118)

as expected. Making the same substitution in $\varphi_3(t^i)$ gives the second Casimir operator for $SU(3)$, which has a more complicated form.

6.2 Laplace operators

The Casimir operators can be expressed as differential operators in the local coordinates on the symmetric space. This is due to the fact that each infinitesimal generator $X_{\alpha} \in \mathbf{G}$

is a contravariant vector field on the group manifold. An element in the Lie algebra can be written

$$X = \sum_{\alpha} X^{\alpha}(x) X_{\alpha} \equiv \sum_{\alpha} X^{\alpha}(x) \frac{\partial}{\partial x^{\alpha}} \quad (119)$$

where x^{α} are local coordinates [11, 16] (for example, $L_1 = (\mathbf{r} \times \mathbf{P})_1 = x^2 \partial_3 - x^3 \partial_2$). That the generators transform as lower index objects follows from the commutation relations.

Example: As an example we take the group $SO(3)$. Under a rotation $R = R(t^1, t^2, t^3) = \exp(\sum t^k L_k)$, the vector $\mathbf{x} = x^i \hat{e}_i \in \mathbf{R}^3$ transforms as

$$\mathbf{x} \xrightarrow{R} \mathbf{x}' = x'^i \hat{e}'_i \quad (120)$$

where the transformation laws for the components and the natural basis vectors are

$$x'^i = R^i_j x^j, \quad \hat{e}'_i = \hat{e}_j R^j_i \quad (121)$$

and $R^{-1} = R^T$. The one-parameter subgroups of $SO(3)$ are rotations

$$R(t^n) = \exp(t^n L_n), \quad (n = 1, 2, 3) \quad (122)$$

(no summation) where L_n are $SO(3)$ generators. It is easy to show using the commutation relations for L_n (given after eq. (7)) that under infinitesimal rotations the L_n transform like the lower index objects \hat{e}_i :

$$R L_i R^{-1} = L_j R^j_i \quad (123)$$

Expressed in local coordinates as differential operators, the Casimirs are called *Laplace operators*. In analogy with the Laplacian in \mathbf{R}^n ,

$$\mathbf{P}^2 = \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x^{i2}} \quad (124)$$

which is invariant under the group E_n of rigid motions (isometries) of \mathbf{R}^n , the Laplace operators on (pseudo-)Riemannian manifolds are invariant under the group of isometries of the manifold. The isometry group of the symmetric space $P \simeq G/K$ is G , since G acts transitively on this space and preserves the metric, so the Laplace operators are invariant under the group operations $g \in G$.

The number of independent Laplace operators on a Riemannian symmetric coset space is equal to the rank of the space. As we defined in paragraph 3.2, the rank of a symmetric space is the maximal number of mutually commuting generators H_i in the subspace \mathbf{P} (cf. also subsection 5.2). If $X_\alpha, X_\beta, \dots \in \mathbf{K}$ and $X_i, X_j, \dots \in \mathbf{P}$, it is also equal to the number of functionally independent solutions to the equation

$$\det \left(\sum_{k=1}^{\dim \mathbf{P}} t^k \rho(X_k) - \lambda I_n \right) = \sum_{l=0}^n (-\lambda)^n \varphi_l(t^k) = 0 \quad (125)$$

where now in the determinant we sum over all $X_k \in \mathbf{P}$. This is equivalent to setting the coordinates t^γ for all the $X_\gamma \in \mathbf{K}$ equal to zero in the secular equation. In the example in the preceding paragraph, the rank of the symmetric space $SO(3)/SO(2)$ (the 2-sphere) is 1, which in this case is also the rank of the group $SO(3)$.

The *Laplace–Beltrami operator* on a symmetric space is the special second order Laplace operator defined (when acting on a function (0-form) f) as

$$\Delta_B f = g^{ij} D_i D_j f = g^{ij} (\partial_i \partial_j - \Gamma_{ij}^k \partial_k) f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x^j} f, \quad g \equiv \det g_{ij} \quad (126)$$

Here D_i denotes the covariant derivative on the symmetric space. It is defined in the usual way [11, 13, 15], for example it acts on the components x^j of a contravariant vector field in the following way:

$$D_i x^j = \partial_i x^j + \Gamma_{ki}^j x^k \quad (127)$$

where Γ_{ki}^j are Christoffel symbols (connection coefficients). The last term represents the change in x^j due to the curvature of the space. We remind the reader that on a Riemannian manifold, the Γ_{ki}^j are expressible in terms of the metric tensor, hence the formula in eq. (126).

Example: Let's calculate the Laplace–Beltrami operator on the symmetric space $SO(3)/SO(2)$ in polar coordinates using (126) and the metric at the point (θ, ϕ) given in the second example of subsection 3.4:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix} \quad (128)$$

Substituting in the formula and computing derivatives we obtain the Laplace–Beltrami operator on the sphere of radius 1:

$$\Delta_B = \partial_\theta^2 + \cot\theta \partial_\theta + \sin^{-2}\theta \partial_\phi^2 \quad (129)$$

Of course this operator is exactly \mathbf{L}^2 . We can check this by computing $L_x = y\partial_z - z\partial_y$, $L_y = z\partial_x - x\partial_z$, and $L_z = x\partial_y - y\partial_x$ in spherical coordinates (setting $r = 1$) and then forming the operator $L_x^2 + L_y^2 + L_z^2$, remembering that all the operators have to act also on anything coming after the expression for each L_i^2 . We find that \mathbf{L}^2 in spherical coordinates, expressed as a differential operator, is exactly the Laplace–Beltrami operator.

In general, a Laplace–Beltrami operator can be split into a radial part Δ'_B and a transversal part. The radial part acts on geodesics orthogonal to some submanifold S , typically a sphere centered at the origin [20].

Example: For the usual Laplace–Beltrami operator in \mathbf{R}^3 expressed in spherical coordinates,

$$\Delta_B = \partial_r^2 + 2r^{-1}\partial_r + r^{-2} \left(\partial_\theta^2 + \cot\theta \partial_\theta + \sin^{-2}\theta \partial_\phi^2 \right) \quad (130)$$

the first two terms

$$\Delta'_B = \partial_r^2 + 2r^{-1}\partial_r \quad (131)$$

constitute the radial part with respect to a sphere centered at the origin and the expression in parenthesis multiplied by r^{-2} is the transversal part. The transversal part is equal to the projection of Δ_B on the sphere of radius r and equals the Laplace–Beltrami operator on the sphere, given for $r = 1$ in eq. (129). This is a general result. For any Riemannian manifold V and an arbitrary submanifold S , the projection on S of the Laplace–Beltrami

above formulas take the form (cf. eq. (16))

$$H = \theta H_1 = \theta \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad h = e^{i\theta H_1} = \begin{pmatrix} 1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{pmatrix} \quad (135)$$

The radial coordinate is $q = (q^1) = \theta$.

There is a general theory for the radial parts of Laplace–Beltrami operators [20]. It is of interest to consider the radial part of the Laplace–Beltrami operator on a manifold V with respect to a submanifold W of V that is transversal to the orbit of an element $w \in W$ under the action of a subgroup of the isometry group of V . Of special interest to us is the case in which the manifold is a symmetric space G/K and the Lie subgroup is K .

The Jacobian $J^{(j)} = \sqrt{|g|}$ (where g is the metric tensor at an arbitrary point of the symmetric space) of the transformation to radial coordinates takes the form

$$\begin{aligned} J^{(0)}(q) &= \prod_{\alpha \in R^+} (q^\alpha)^{m_\alpha} \\ J^{(-)}(q) &= \prod_{\alpha \in R^+} (\sinh(q^\alpha))^{m_\alpha} \\ J^{(+)}(q) &= \prod_{\alpha \in R^+} (\sin(q^\alpha))^{m_\alpha} \end{aligned} \quad (136)$$

for the various types of symmetric spaces with zero, negative and positive curvature, respectively (see [20], Ch. I, par. 5). In these equations the products denoted $\prod_{\alpha \in R^+}$ are over all the positive roots of the restricted root lattice and m_α is the multiplicity of the root α . The multiplicities m_α were listed in Table 1.

Example: On the hyperboloid H^2 with metric

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \theta \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^{-2} \theta \end{pmatrix} \quad (137)$$

equations (132,136) give the radial part of Δ_B for H^2 . The radial coordinate is θ so we get, in agreement with (136)

$$J^{(-)} = \sqrt{|g|} = \sinh \theta, \quad \Delta'_B = \frac{1}{\sinh \theta} \partial_\theta \sinh \theta \partial_\theta = (\partial_\theta^2 + \coth \theta \partial_\theta) \quad (138)$$

In the same way we can also easily derive the equations (131, 139) using (132,136). In particular, we immediately get the radial part of the Laplace–Beltrami operator acting on the two–sphere S^2 transversally to a one–sphere S^1 around the north pole:

$$\Delta'_B = (\partial_\theta^2 + \cot\theta \partial_\theta) \quad (139)$$

6.3 Zonal spherical functions

The properties of the so called zonal spherical functions are important for the research results to be discussed in [3]. Since there is a natural mapping from the Hamiltonians of integrable Calogero–Sutherland systems onto the Laplace–Beltrami operators of the underlying symmetric spaces, these eigenfunctions play an important role also in the physics of the integrable systems. Regarding the DMPK equation for a quantum wire, the known asymptotic expressions for these eigenfunctions allow one to solve this equation in general or in the asymptotic regime, because of the simple mapping from the DMPK evolution operator to the radial part of the Laplace–Beltrami operator. For an example of their use see [9, 10, 22].

It is known that when ρ is an irreducible representation of an algebra, then the Casimir operator $C_{k,\rho}$ is a multiple of the identity operator [16, 18] (Schur’s lemma). This means that it has eigenvalues and eigenfunctions. Since the Casimir operators (and consequently the Laplace operators) form a commutative algebra, they have common eigenfunctions. There exists an extensive theory regarding invariant differential operators and their eigenfunctions [20]. Of particular interest are the differential operators on a group G or on a symmetric space G/K that are left–invariant under the group G and right–invariant under a maximal compact subgroup K . Suppose the smooth complex–valued function $\phi_\lambda(x)$ is an eigenfunction of such an invariant differential operator D on the symmetric space G/K :

$$D\phi_\lambda(x) = \gamma_D(\lambda)\phi_\lambda(x) \quad (140)$$

Here the eigenfunction is labelled by the parameter λ and $\gamma_D(\lambda)$ is the eigenvalue. If in addition $\phi_\lambda(kxk') = \phi_\lambda(x)$ ($x \in G/K$, $k \in K$) and $\phi_\lambda(e) = 1$ (e =identity element), the function ϕ_λ is called *spherical*. A spherical function satisfies [20]

$$\int_K \phi_\lambda(xky) dk = \phi_\lambda(x)\phi_\lambda(y) \quad (141)$$

where dk is the normalized Haar measure on the subgroup K . We will see examples of this formula below.

The common eigenfunctions of the Laplace operators are invariant under the subgroup K . They are termed *zonal spherical functions*. Because of the bi-invariance under K , these functions depend only on the radial coordinates h :

$$\phi_\lambda(x) = \phi_\lambda(h) \quad (142)$$

Example: Let's study for a moment the eigenfunctions of the Laplace operator on $G/K = SO(3)/SO(2)$. We know from quantum mechanics that the eigenfunctions of \mathbf{L}^2 are the associated Legendre polynomials $P_l(\cos\theta)$, and $-l(l+1)$ is the eigenvalue under \mathbf{L}^2 (our definition of \mathbf{L} differs by a factor of i from the definition common in quantum mechanics):

$$\mathbf{L}^2 P_l(\cos\theta) = -l(l+1)P_l(\cos\theta) \quad (143)$$

where $\cos\theta$ is the z -coordinate of the point $P = (x, y, z)$ on the sphere of radius 1. In spherical coordinates, $P = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$. As we can see, the eigenfunctions are functions of the radial coordinate θ only. The subgroup that keeps the north pole fixed is $K = SO(2)$ and its algebra contains the operator $L_z = \partial_\phi$. Indeed, $P_l(\cos\theta)$ is unchanged if the point P is rotated around the z -axis.

In terms of Euler angles, a general $SO(3)$ -rotation takes the form

$$R(\alpha, \beta, \gamma) = g(\alpha)k(\beta)h(\gamma) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix} \quad (144)$$

where g and h are rotations around the y axis by the angles α and γ respectively, and k is a rotation around the z axis by the angle β . Under such a rotation, the north pole $(0, 0, 1)$ goes into $(-\cos\alpha \cos\beta \sin\gamma - \sin\alpha \cos\gamma, -\sin\beta \sin\gamma, -\sin\alpha \cos\beta \sin\gamma + \cos\alpha \cos\gamma)$. This means that eq. (141) takes the form

$$\frac{1}{2\pi} \int_0^{2\pi} P_l(-\sin\alpha \cos\beta \sin\gamma + \cos\alpha \cos\gamma) d\beta = P_l(\cos\alpha)P_l(\cos\gamma) \quad (145)$$

Example: For the symmetric space $G/K = E_2/SO(2)$ the spherical functions are the plane waves:

$$\psi(r) = e^{ikr} \quad (146)$$

where k is a complex number. If g, h denote translations in the x -direction by a distance b, a respectively, and k is a rotation around the origin of magnitude ϕ , then the transformation $g(b)k(\phi)h(a)$ moves the point $x \in \mathbf{R}^2$ by a distance $\sqrt{a^2 + b^2 + 2ab\cos\phi}$. Therefore we obtain from (141)

$$\frac{1}{2\pi} \int_0^{2\pi} \psi\left(\sqrt{a^2 + b^2 + 2ab\cos\phi}\right) d\phi = \psi(a)\psi(b) \quad (147)$$

Defining the Jacobians (136) as in reference [1] in terms of a parameter a ,

$$\begin{aligned} J^{(0)}(q) &= \prod_{\alpha \in R^+} (q^\alpha)^{m_\alpha} \\ J^{(-)}(q) &= \prod_{\alpha \in R^+} (a^{-1} \sinh(aq^\alpha))^{m_\alpha} \\ J^{(+)}(q) &= \prod_{\alpha \in R^+} (a^{-1} \sin(aq^\alpha))^{m_\alpha} \end{aligned} \quad (148)$$

it is not hard to see that the various spherical functions listed above are related to each other by the simple transformations

$$\begin{aligned} \phi_\lambda^{(0)}(q) &= \lim_{a \rightarrow 0} \phi_\lambda^{(-)}(q) \\ \phi_\lambda^{(+)}(q) &= \phi_\lambda^{(-)}(q)|_{a \rightarrow ia} \end{aligned} \quad (149)$$

There exist integral representations of spherical functions for the various types of spaces G/K [1, 20]. We will list an integral representation only of $\phi_\lambda^{(-)}(q)$ below, recalling that formulas for the other types of spherical functions can be obtained by (149). If $\phi_\lambda^{(-)}(x)$ is spherical and h is the spherical radial part of x ,

$$\phi_\lambda^{(-)}(x) = \phi_\lambda^{(-)}(h) = \int_K e^{(i\lambda - \rho)H(kx)} dk \quad (150)$$

In (150) λ is a complex-valued linear function on the maximal abelian subalgebra \mathbf{H}'_0 of $i\mathbf{P}$ and ρ is the function defined by

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} m_\alpha \alpha \quad (151)$$

In eq. (150) they act on the unique element $H(kx) \in \mathbf{H}'_0$ such that $kx = ne^{H(kx)}k'$ in the Iwasawa decomposition. It was shown by Harish-Chandra [21] that two functions $\phi_\lambda^{(-)}(x)$ and $\phi_\nu^{(-)}(x)$ are identical if and only if $\lambda = s\nu$, where s denotes a Weyl reflection. The Weyl group is the group of reflections in hyperplanes orthogonal to the roots and was defined in subsection 2.5, eq. (34).

The eigenvalues of the radial part of the Laplace-Beltrami operator corresponding to the eigenfunctions on zero, negative and positive curvature symmetric spaces are given by the following equations (see [1] and [20], Ch. IV, par. 5).

$$\begin{aligned} \Delta'_B \phi_\lambda^{(0)} &= -\lambda^2 \phi_\lambda^{(0)} \\ \Delta'_B \phi_\lambda^{(-)} &= \left(\frac{\lambda^2}{a^2} - \rho^2 \right) \phi_\lambda^{(-)} \\ \Delta'_B \phi_\lambda^{(+)} &= \left(-\frac{\lambda^2}{a^2} + \rho^2 \right) \phi_\lambda^{(+)} \end{aligned} \quad (152)$$

where ρ is the function in (151). (To avoid confusion, note that the eigenvalue l in eq. (143) is not equal to λ . In fact, $\lambda = l + 1/2$.)

Example: Take the symmetric space $SO(3)/SO(2)$. From Table 1 we see that this space has $p - q = 2 - 1 = 1$ short root of length 1. Then

$$\rho^2 = \left(\frac{1}{2} \right)^2 \cdot 1^2 \cdot |\alpha|^2 = \frac{1}{4} \quad (153)$$

and setting $a = 1$, the eigenvalue is $-\lambda^2 + 1/4 = -l(l + 1)$.

6.4 The analog of Fourier transforms on symmetric spaces

Much of the material presented in this subsection is taken from the book by Wu–Ki Tung [23].

A continuous smooth (C^∞) spherical function f is said to be *elementary* if it is an eigenfunction of any differential operator that is invariant under left translations by G and right translations by K . Thus the eigenfunctions of the Laplace operators are elementary. The elementary spherical functions are related to irreducible representation functions for the group G . The irreducible representation functions are the matrix elements of the group elements g in the representation ρ . Let's clarify this statement by an example.

Example: The angular momentum basis for $SO(3)$ is defined by

$$\begin{aligned} \mathbf{L}^2|lm\rangle &= l(l+1)|lm\rangle \\ L_3|lm\rangle &= m|lm\rangle \\ L_\pm|lm\rangle &= \sqrt{l(l+1) - m(m\pm 1)}|lm\rangle \end{aligned} \quad (154)$$

where l labels the representation. The irreducible representation functions are, in the angular momentum basis $|lm\rangle$, the matrix elements $D^l(R)_m^{m'}$ such that

$$R|lm\rangle = |lm'\rangle D^l(R)_m^{m'} \quad (155)$$

where $R = \exp(\mathbf{t} \cdot \mathbf{L})$ is a general $SO(3)$ rotation. It is known that for $R \in SO(3)$, if α, β, γ are the Euler angles of the rotation $R = R(\alpha, \beta, \gamma)$, these matrix elements take the form

$$D^l(R)_m^{m'} = D^l(\alpha, \beta, \gamma)_m^{m'} = e^{-i\alpha m'} d^l(\beta)_m^{m'} e^{-i\gamma m}, \quad d^l(\beta)_m^{m'} \equiv \langle lm'|e^{-i\beta L_2}|lm\rangle \quad (156)$$

The associated Legendre functions $P_l^m(\cos\theta)$ and the special functions $Y_l^m(\theta, \phi)$ called spherical harmonics are essentially this kind of matrix elements:

$$\begin{aligned} P_l^m(\cos\theta) &= (-1)^m \sqrt{\frac{(l+m)!}{(l-m)!}} d^l(\theta)_0^m \\ Y_l^m(\theta, \phi) &= \sqrt{\frac{2l+1}{4\pi}} [D^l(\phi, \theta, 0)_0^m]^* \end{aligned} \quad (157)$$

The irreducible representation functions $D^l(R)_{m'}^{m'}$ satisfy orthogonality and completeness relations. In fact, they form a complete basis in the space of square integrable functions defined on the group manifold. This is the Peter–Weyl theorem. From here the corresponding theorems follow for the special functions of mathematical physics.

Example: For $SO(3)$ the orthonormality condition reads

$$(2l + 1) \int d\tau D_l^\dagger(R)_n^m D^l(R)_{m'}^{n'} = \delta_l^{l'} \delta_n^{n'} \delta_m^{m'} \quad D_l^\dagger(R)_n^m \equiv [D^l(R)_m^n]^* \quad (158)$$

where $R = R(\alpha, \beta, \gamma)$ is an $SO(3)$ rotation expressed in Euler angles and $d\tau$ is the invariant group integration measure normalized to unity, $d\tau = d\alpha d(\cos\beta) d\gamma / 8\pi^2$. That the irreducible representation functions form a complete basis for the square-integrable functions on the $SO(3)$ group manifold can be expressed as

$$f(R) = \sum_{lmn} f_{lm}^n D^l(R)_n^m \quad (159)$$

where $f(R)$ is square-integrable. Using (158) we obtain

$$f_{lm}^n = (2l + 1) \int d\tau D_l^\dagger(R)_n^m f(R) \quad (160)$$

If $R(\alpha, \beta, \gamma) = R(\phi, \theta, 0)$ we get the special case of the spherical harmonics on the unit sphere (setting $\sqrt{4\pi/(2l + 1)} f_{lm}^n \equiv \tilde{f}_{lm}$):

$$f(\theta, \phi) = \sum_{lm} \tilde{f}_{lm} Y_{lm}(\theta, \phi) \quad (161)$$

$$\tilde{f}_{lm} = \int f(\theta, \phi) Y_{lm}^*(\theta, \phi) d(\cos\theta) d\phi \quad (162)$$

and further, for $R(\alpha, \beta, \gamma) = R(0, \theta, 0)$ we get the completeness relation for the associated Legendre polynomials $P_l(\cos\theta) = Y_{l0} \sqrt{4\pi/(2l + 1)}$

$$f(\theta) = \sum_l f_l P_l(\cos\theta) \quad (163)$$

$$f_l = \frac{(2l+1)}{2} \int f(\theta) P_l^*(\cos\theta) d(\cos\theta) \quad (164)$$

where $f_l = f_{l0}^0$. These are analogous to Fourier transforms. In the above example, we considered a symmetric space with positive curvature. For a space with zero or negative curvature, we have an integral instead of a sum in (163):

$$f(q) = \int \tilde{f}(\lambda) \phi_\lambda^{(j)}(q) d\mu(\lambda) \quad (165)$$

$$\tilde{f}(\lambda) = \int f(q) [\phi_\lambda^{(j)}(q)]^* J^{(j)}(q) dq \quad (166)$$

where q are canonical radial coordinates. The integration measure $d\mu(\lambda)$ was determined by Harish–Chandra to be well–defined and proportional to $w^2|c(\lambda)|^{-2}d\lambda$ where $c(\lambda)$ is a known function whose inverse is analytic [21] (see also [1, 20]) and w is the order of the Weyl group (the number of distinct Weyl reflections). $J^{(j)}(q) dq$ is the invariant measure on the space of radial coordinates. In equation (165) the arbitrary square–integrable function $f(q)$ is expressed in terms of the complete set of basis functions $\phi_\lambda^{(j)}(q)$.

One can show [20, 21] that the dimension of the space of eigenfunctions of Δ'_B is less than or equal to w . It is a remarkable fact that the eigenfunctions of the radial part of the Laplace–Beltrami operator Δ'_B have the property of being eigenfunctions of the radial part of *any* left–invariant differential operator on the symmetric space as well ([20], Ch. IV).

The following asymptotic expression for the zonal spherical functions on spaces of zero and negative curvature holds for large values of $|h|$ [1, 21]:

$$\phi_\lambda(h) \sim \sum_{s \in W} c(s\lambda) e^{(is\lambda - \rho)(H)} \quad (167)$$

where $h = e^H$ is a spherical coordinate, H is an element of the maximal abelian subalgebra, λ is a complex–valued linear function on the maximal abelian subalgebra, and the function ρ was defined in eq. (151).

7 Discussion

In this review we have introduced the reader to some of the most fundamental concepts in the theory of symmetric spaces. We have tried to keep the discussion as simple as possible without assuming any previous familiarity of the reader with symmetric spaces. The review should be particularly accessible to physicists. In the hope of addressing a wider audience, we have almost completely avoided using concepts from differential geometry, and we have presented the subject mostly from an algebraic point of view. In addition we have inserted a large number of simple examples in the text, that will hopefully help the reader visualize the ideas.

Since our aim in the forthcoming paper [3] will be to introduce our readers to the application of symmetric spaces in physical integrable systems and random matrix models, we have chosen the background material presented here with this in mind. Therefore we have put emphasis not only on fundamental issues but on subjects that will be relevant in these applications as well.

In particular we have discussed symmetric spaces of positive, zero and negative curvature that will be relevant for matrix models of the circular, gaussian, and transfer matrix type, respectively. In [3] we will define and discuss various types of random matrix ensembles and their applications to various physical problems, and we will associate them to the corresponding symmetric spaces in Table 1. For the reader unfamiliar with random matrix theories, we will mention here that they are used for a wide range of applications in theoretical physics. They are successfully employed in describing universal spectral properties of systems with a large number of degrees of freedom, where it is difficult or impossible to solve the physical problem exactly. In typical applications, large matrices with given symmetry properties and randomly distributed elements substitute a physical operator, for example a Hamiltonian, scattering matrix, transfer matrix, or Dirac operator. The symmetry properties of the physical operator determine which ensemble of random matrices to use. Historically, large random matrices were first employed by Wigner and Dyson to describe the energy levels in complex nuclei, where they modelled the Hamiltonian of the system. A major area of application is in the description of the infrared limit of gauge theories, where an integration over an appropriate random matrix ensemble replaces the integration over gauge field configurations in the partition function. Random matrix theories are also applied in the theoretical description of mesoscopic systems, where they may be used to model the properties of scattering and transfer matrices in disordered wires and quantum dots.

As mentioned in the introduction, one of the applications of symmetric spaces in the random matrix theory of quantum transport lies in the identification of the DMPK operator with a simple transformation of the Laplace–Beltrami operator on the symmetric space

defining the random matrix universality class. This connection was discussed by Hüffmann in [24], and more recently it was exploited in [9] in solving the DMPK equation. The DMPK equation describes the evolution of the distribution of the set of eigenvalues of the transfer matrix with an increasing length of the quantum wire. The identification of the matrix eigenvalues (also called radial coordinates) with the points in the coset manifold results in the observation that only the radial part of any operator will influence their dynamics. To enable the reader to appreciate these issues, we have introduced our readers to the concepts of radial coordinates, Laplace–Beltrami operators, and discussed the properties of zonal spherical functions.

In [3] we will discuss the relationship between the restricted root lattices of symmetric spaces and integrable Calogero–Sutherland models describing interacting many–particle systems in one dimension. Olshanetsky and Perelomov [1] showed that the dynamics of these systems are related to free diffusion on a symmetric space. This relationship is due to the fact that the Hamiltonians of integrable Calogero–Sutherland models map onto the radial parts of the Laplace–Beltrami operators of the underlying symmetric spaces. It is a beautiful fact that the Calogero–Sutherland model becomes exactly integrable for values of the coupling constants determined by the restricted roots. We may call these values the root values. Since the DMPK equation can be mapped onto a Schrödinger–like equation in imaginary time, featuring a Calogero–Sutherland Hamiltonian with root values of the coupling constants, this fact can be used to extract information on the DMPK equation of a quantum wire [9, 10]. This will be described in more detail in [3]. In our future publication we will also mention various types of potentials for the Calogero–Sutherland models and show that the Weierstrass \mathcal{P} –function summarizes three such potentials in various limits. These limiting potentials correspond to particles interacting on a circle, hyperbola, or line, respectively. This reflects the triplicity of the symmetric spaces in terms of their curvature.

Finally, having discussed the applications of symmetric spaces in connection with integrable models and quantum transport problems, and the emerging classification of random matrix ensembles, in [3] we plan to indicate some possible new directions of research, hoping to have stimulated new interest in this intriguing but little known research field.

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