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1 Introduction

It is often thought that quantum groups provide the key to *q*-deforming the basic structures of physics from the point of view of non-commutative geometry. If one considered a classical algebra of observables and quantised it relative to some Poisson bracket, one might obtain a quantum group. The underlying semiclassical theory is the theory of Poisson-Lie groups – see Reyman’s lectures and Reshetikhin’s lectures on classical inverse scattering. But this is only part of the story. Our goal in these lectures is to explain that the fundamental concept needed for the full structure of even the simplest *q*-deformed spaces, such as the quantum plane, is not so much a quantum group as one of the more exotic objects called *braided groups*. These were introduced by the author in 1989[1] and subsequently developed in the course of 40 or so papers into a systematic theory of *braided geometry*. Quantum groups play a background role in this theory as the *quantum symmetry* or covariance of the geometry, but the spaces themselves tend to be braided ones.

My intention is to provide here a pedagogical introduction to this theory of braided geometry. Braided groups provide a new beginning for the theory of *q*-deformation and can be developed along-side quantum groups without requiring much experience of them. Instead, some experience with Grassmann algebras or supersymmetry will be quite helpful although not essential. We try to cover here only *q*-deformed or braided versions of \mathbb{R}^n , where the theory is fairly complete. This includes important examples such as *q*-Euclidean and *q*-Minkowski space. Only when this linear theory is thoroughly understood could one reasonably expect to move on to define *q*-manifolds etc. For some first steps in quantum geometry, see [2]. Braided Yang-Mills theory on a general braided manifold is not yet understood.

We begin with the concept itself of a braided group. This is a new concept. On the one hand we replace old ideas from the theory of superspaces by similar ones with braid statistics in place of Bose-Fermi ones. This makes it easy for the reader to get the idea of braided groups. On the other hand the true meaning and abstract definition of braided groups involves writing its algebraic structure diagrammatically as a joining of strings (the product) or a splitting of strings (the coproduct or coaddition). All information flows along these strings which can form braids and knots. Each braid crossing Ψ corresponds to a *q*-factor or more generally, to an R-matrix. This is much more fun and more systematic than trying to introduce *q* or an R-matrix by guesswork or by other ad-hoc means, which is the usual approach to *q*-deforming physics. We will not see too much of this diagrammatic side here, since we will try for a more hands-on and less abstract treatment. One can see [3] for the diagrammatic theory as well as for a review of braided groups up to about mid 1992. Section 2 below provides the briefest of introductions. In addition, there are two introductory papers [4][5] in conference proceedings, which cover the braided-groups programme since then. The present work is based in part on Chapter 10 of my forthcoming book[6].

1.1 Why *q*-deform?

There are several reasons to want to *q*-deform the basic structures of physics in the first place. We outline some of them here.

- To begin with it is simply a fact that many of our usual concepts of geometry are a special $q = 1$ case of something more general which works just as well, i.e. mathematically we

can q -deform and have no particular reason to limit ourselves to $q = 1$ in every physical situation.

- The $q \neq 1$ world seems to be less singular than the $q = 1$ world: perhaps some of the infinities we encounter in quantum field theory are really poles in $\frac{1}{q-1}$ and appear singular because we used $q = 1$ geometry in the bare theory. This has two points of view.

(a) It may be that the real world is only $q = 1$ and that expressing infinities in this way as poles is a mathematical tool of q -regularisation[7]. Even so it is useful because q -deformation is elegant and (in the braided approach) systematic. We will see that one of the themes of the q -deformed world is that q -deformed quantities bear the same mathematical relationships with each other as in the undeformed case. So we do not do serious damage to the mathematical structure as is done in more physical but brutal regularisation methods such momentum cut-off. Also, we do not have ad-hoc problems like what to do with the ϵ tensor as in dimensional regularisation. In this context it is fitting that q is dimensionless and ‘orthogonal to physics’.

(b) It may be that really $q \neq 1$ as a crude model of quantum or other corrections to our usual concept of geometry. In this case q could be an exponential of the ratio of masses in our system to the Planck mass, for example. Quantum groups do have explicit connections with Planck-scale physics. We do not cover this here, but see [8] where this connection was introduced for the first time.

- Some physical models are harder to q -deform than others. The principle of q -deformability or continuity of physics at $q = 1$ may help to single out some Lagrangians as more natural than others. Some physical Lagrangians may be based for example on accidental isomorphisms at $q = 1$ in the various families of Lie groups. Such degeneracies tend to be removed by q -deformation.
- q -deformation and quantum or braided geometry in general unifies concepts. Thus ideas which at $q = 1$ are quite different, may in fact be isomorphic as soon as $q \neq 1$. In particular, the concept in physics of covariance or symmetry is one and the same as the concept of statistics or grading (as in supersymmetry) when both are expressed in the language of Hopf algebras[3].
- Related to this, there are possible some very spectacular ‘self-duality’ unifications of particular algebras. Thus the enveloping algebra of $SU(2) \times U(1)$ becomes isomorphic to the co-ordinates x_μ of q -Minkowski space when both are q -deformed in a natural way within braided geometry[9][5].

The reader should bear in mind all of these ideas as well as any others she or he can think of. We will see the ones above realised to some extent below.

1.2 What is braided geometry?

Keeping in mind the above ideas, how can we develop a systematic and universal approach to q -deforming structures in physics? Braided geometry claims to do this. Here we explain the key idea behind it and where it may be that more fashionable ideas such as non-commutative geometry went wrong.

The point is that in our experience in quantum physics there are in fact *two* kinds of non-commutativity which we encounter. The first of these I propose to call *inner noncommutativity* or *noncommutativity of the first kind* because it is a property within a quantum system or algebra. It is the kind that we encounter when we start with a classical algebra of observables and quantise it by making it non-commutative. It is customary to make an analogy with this process of quantisation by considering algebras in which there is a parameter $q \neq 1$ analogous to $\hbar \neq 0$. In mathematical terms, an algebra is regarded as like functions on a manifold, but all

geometrical constructions are developed in such a way that the algebra need not be commutative and hence need not really be the algebra of functions on any space. It could, for example, be an algebra arising by quantisation, but this is not a prerequisite. In this usual formulation of *noncommutative geometry* the tensor product of algebras (corresponding to direct product of the manifolds) is the usual one in which the factors commute. It is the algebras themselves which become noncommutative.

The idea of *braided geometry* is to associate q not with quantisation but rather with a different *outer noncommutativity* that can exist between independent systems. This is *noncommutativity of the second kind* and is encountered in physics when we consider fermions: independent fermionic systems anticommute rather than commute. So the idea is to consider q as a generalisation of the -1 factor for fermions. In mathematical terms it is the notion of \otimes product between algebras which we will q -deform and not directly the algebras themselves. These, as far as we are concerned, can remain classical or ‘commutative’ albeit in a deformed sense appropriate to the noncommutative tensor product.

This is conceptually quite a different role for q than its usual picture as quantisation. It turns out to be the key if one wants to q -deform not one algebra but an entire universe of structures: lines, planes, matrices, differentials etc in a systematic and mutually consistent way. The reason is that we can use the systematic machinery of braided categories to deform the entire category of vector spaces with its usual \otimes to a *braided category* with tensor product \otimes_q . Most constructions in physics and many in mathematics take place in the category of vector spaces, so by deforming the category itself we carry over all our favourite constructions without any further effort. The reader should not be afraid of the term ‘category’ here. It just means a collection of objects of some specified type. The outer non-commutativity is manifested in the construction (due to the author) of the *braided tensor product algebra structure* $B \otimes C$ of two braided algebras B, C . The tensor product is physically the joint system and contains B, C as subalgebras. But the notion of braided-independence or *braid statistics* means that the two factors do not mutually commute as they would in a usual tensor product. The concept here is obviously quite general and is not tied to a single parameter q : its role can be played by a general matrix or collection of matrices R obeying suitable braid relations. So we develop in fact a braided theory of R -deformation. The standard R -matrices depend on a single parameter q but the reader can just as easily put in multi-parameter or non-standard R -matrices into our formalism.

It should be clear by now that this new approach to deformation is quite independent of, or orthogonal to, the usual role of quantum groups and non-commutative geometry. Quantum groups play no very direct role in braided geometry and moreover, the fundamental concepts here did *not* arise in the context of Quantum Inverse Scattering where quantum groups arose. The point of contact is covariance, which we come to at the end of our studies. Our starting point, which is that of a braided group, is due to the author [10][11][12] and came out of experience with fermionic systems and supersymmetry. As well as being important to keep the history straight (now that these ideas have become popular for physicists) it is also important mathematically because the two kinds of non-commutativity here are not at all mutually exclusive. They are orthogonal in the sense that one can just as well have *quantum braided groups* in which both ideas are present. We will not emphasise this here, but see [13][14][15][16] and the appendix.

Let us note also that our point of view on q does not preclude the possibility that other physical effects may induce these braid statistics. We have discussed various physical reasons to consider $q \neq 1$ in the previous subsection. The fact is that any of these lead us to q -deform geometry and in this q -deformed world the usual spin-statistics theorem fails. Braid statistics are allowed and indeed are a general feature of q -deformation.

2 Diagrammatic definition of a braided group

I would like to begin with a lightening sketch of the abstract definition of a braided group. This is not essential for the later sections, so the reader who wants to learn the definition by experience

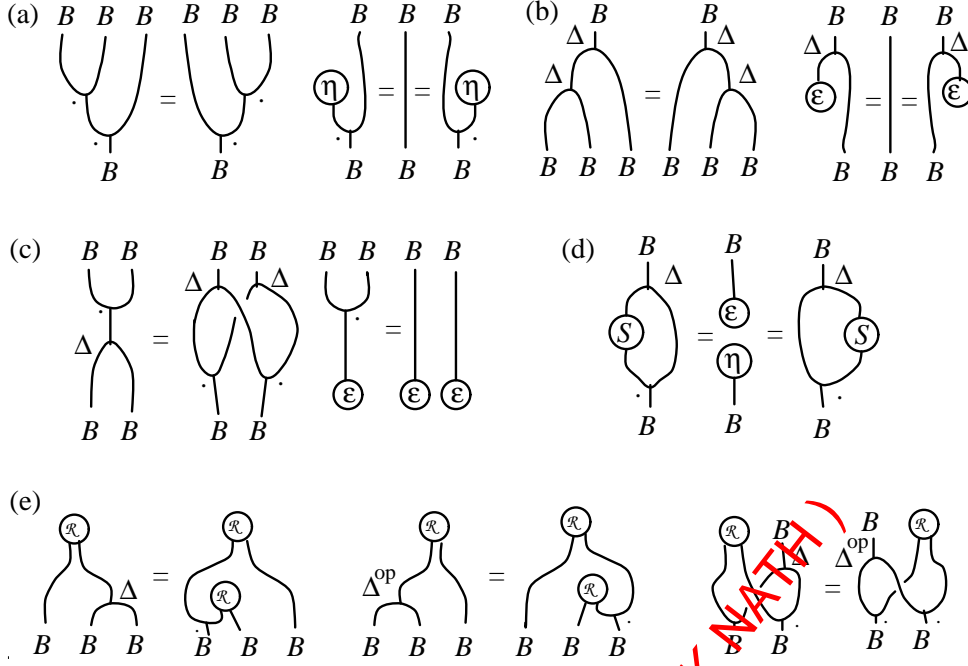


Figure 1: Axioms of a braided group showing (a) associativity and unit (b) coassociativity and counit (c) braided homomorphism property (d) antipode (e) quasitriangular structure

should proceed directly to the next section where we see lots of examples. Even so, it is useful to know that there is a firm mathematical foundation to this concept [17][13] and this is what we outline here. For much more detail on this topic, see [3].

The axioms of a braided group B are summarised in parts (a) – (d) of Figure 1 in a diagrammatic notation. We write morphisms or maps pointing downwards. There is a product $\cdot = \Psi$ which should be associative and have a unit η as we see in part (a). This is a braided algebra. The axiom for the unit says that grafting it on via the product map does not change anything. In addition we should have a coproduct $\Delta = \wedge$ which should be coassociative, and a counit ϵ . This is shown in part (b), which is just part (a) up-side-down. This is a braided coalgebra. These two structures should be compatible in the sense that Δ, ϵ are braided-multiplicative as shown in part (c). In concrete terms this means

$$\Delta(ab) = (\Delta a)(\Delta b), \quad (a \otimes c)(b \otimes d) = a\Psi(c \otimes b)d \quad (1)$$

which says that Δ is a homomorphism from B to the braided tensor product algebra $B \otimes B$. The braid crossing here corresponds to an operator $\Psi = \times$ obeying the braid relations. We can pull nodes through such braid crossings as if they are on strings in a three-dimensional space. This space is not physical space but an abstract space in which braided mathematics is written. Sometimes we also have an antipode or ‘inversion map’ obeying the axioms in part (d). It turns out that all the elementary group theory that the reader is familiar with can be developed in this diagrammatic setting, including representations or modules, adjoint actions, cross products etc. For example, Figure 2 shows the proof of the property

$$S \circ \cdot = \cdot \circ \Psi_{B,B} \circ (S \otimes S), \quad \Delta \circ S = (S \otimes S) \circ \Psi_{B,B} \circ \Delta \quad (2)$$

which we will need later. The proof grafts on two loops involving S , knowing that they are

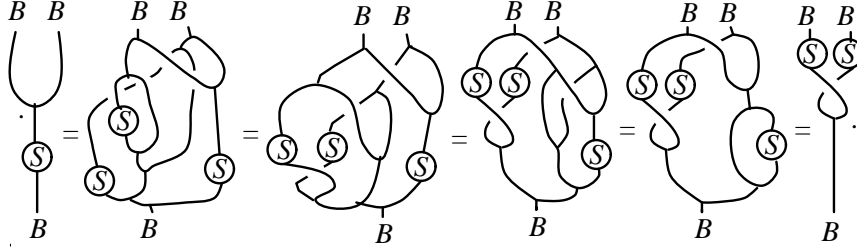


Figure 2: Diagrammatic proof of braided antihomomorphism property of S

trivial from Figure 1(d). After some reorganisation using parts (a)–(b), we use (c) and then (d) again for the final result. For the second half of (2) just turn this volume up-side-down and read Figure 2 again. On the more esoteric side, we sometimes also have a braided universal-R-matrix or quasitriangular structure shown in Figure 1(e).

The simplest example of such a braided group is the *braided line*. This is just the polynomials in a single variable x endowed with

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad Sx = -x, \quad \Psi(x^m \otimes x^n) = q^{mn} x^n \otimes x^m.$$

The first three look on the generator x the same as the usual definitions for functions in one variable. The coproduct corresponds in this usual case to addition on the underlying copy of \mathbb{R} for which x is the linear co-ordinate function. The new ingredient is the braiding Ψ and means for example that

$$\Delta x^m = \sum_{r=0}^m \begin{bmatrix} m \\ r \end{bmatrix} q; x^r \otimes x^{m-r}, \quad Sx^m = q^{\frac{m(m+1)}{2}} (-x)^m.$$

We see here the origin in braided geometry of the q -integers and q -binomials

$$[m; q] = \frac{1 - q^m}{1 - q}, \quad \begin{bmatrix} m \\ r \end{bmatrix} q = \frac{[m; q]!}{[r; q]! [m - r; q]!}$$

which are familiar when working with q -deformations. It turns out that many formulae in q -deformed analysis, such as differentiation, integration etc. are immediately recovered once one takes the braided point of view. In this example q is arbitrary but non-zero. If we take $q^2 = 1$ we can consistently add the relation $x^2 = 0$ which gives us the usual Grassmann algebra in one variable, i.e. the super-line. If we take $q^n = 1$ we can consistently add the relation $x^n = 0$ and arrive at the *anyonic line*[14][18][16].

The next simplest example is the braided plane B generated by x, y with[19]

$$\begin{aligned} yx &= qxy, & \Delta x &= x \otimes 1 + 1 \otimes x, & \Delta y &= y \otimes 1 + 1 \otimes y \\ \epsilon x &= \epsilon y = 0, & Sx &= -x, & Sy &= -y \\ \Psi(x \otimes x) &= q^2 x \otimes x, & \Psi(x \otimes y) &= qy \otimes x, & \Psi(y \otimes y) &= q^2 y \otimes y \\ \Psi(y \otimes x) &= qx \otimes y + (q^2 - 1)y \otimes x \end{aligned}$$

The algebra here is sometimes called the ‘quantum plane’; the new part is the coproduct Δ and the braiding Ψ . The latter is the same one that leads to the Jones knot polynomial or the quantum group $SU_q(2)$ in another context. It is a nice exercise for the reader to verify that Δ is indeed an algebra homomorphism using the braided tensor product (1). Again, this seems

innocent enough but has the result that we generalise to 2-dimensions all the familiar ideas from one-dimensional q -analysis. We will see this quite generally in the next section for n -dimensions and general braidings. This is one of the successes of the theory of braided groups.

The coproducts Δ in these examples are linear on the generators. They could better be called *coaddition*. All the interesting coadditions I know are braided ones. If they were not braided, they would have to be cocommutative and hence correspond essentially to ordinary Lie algebras and not quantum groups. This is why we need braided groups as the foundation of braided geometry. There are also plenty of other more complicated braided groups, including a canonical one for every strict quantum group by a transmutation construction[12]. In this way the theory of braided groups contains braided versions of the quantum groups $U_q(g)$ for example, and is a good way of getting to grips with their geometry as well[9]. One can also make partial transmutations to obtain any number of other (quantum) braided groups which lie in between quantum groups and their completely transmuted braided group versions. The theory of transmutation is covered in the Appendix.

3 Braided coaddition

We describe in this section the basic braided groups which will be the object of our study. We begin with deformations of co-ordinates x_i or vectors v^i , i.e. versions of \mathbb{R}^n . In the braided world there are many such versions depending on the precise commutation relations of the algebra and the precise braid statistics, which we encode by matrices R', R respectively. In Sections 3.2 and 3.3 we give braided versions of \mathbb{R}^{n^2} using the same formalism on a matrix of generators.

3.1 Braided coaddition on vectors and covectors

Let R, R' be invertible matrices in $M_n \otimes M_n$. We suppose that they obey[19]

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (3)$$

$$R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}, \quad R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23} \quad (4)$$

$$(PR + 1)(PR' - 1) = 0 \quad (5)$$

$$R_{21}R' = R'_{21}R \quad (6)$$

where P is the permutation matrix. The suffices refer to the position in tensor power of M_n . Thus in (3), which is called the *Quantum Yang-Baxter Equations (QYBE)*, we have $R_{12} = R \otimes \text{id}$ and $R_{23} = \text{id} \otimes R$ etc.

It is pretty easy to solve these equations. Just start for example with a matrix R solving the QYBE. Any matrix PR necessarily obeys some minimal polynomial $\prod_i (PR - \lambda_i) = 0$ and for each nonzero eigenvalue λ_i we can just normalise R so that $\lambda_i = -1$ and take

$$R' = P + P \prod_{j \neq i} (PR - \lambda_j). \quad (7)$$

This clearly solves (4)–(6) and gives us at least one braided covector space for each nonzero eigenvalue of PR . The simplest case is when there are just two eigenvalues, which is called the *Hecke case*.

Given a solution of (3)–(6) we have the *braided-covector algebra* $V^{\sim}(R', R)$ defined by generators $1, x_i$ and relations and braided group structure[19]

$$\begin{aligned} x_i x_j &= x_b x_a R'^a{}_i{}^b{}_j, \quad i.e., \quad \mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R' \\ \Delta x_i &= x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon x_i = 0, \quad S x_i = -x_i \\ \Psi(x_i \otimes x_j) &= x_b \otimes x_a R'^a{}_i{}^b{}_j, \quad i.e., \quad \Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes \mathbf{x}_1 R \end{aligned} \quad (8)$$

extended multiplicatively with braid statistics. We use the compact notation shown on the right where bold \mathbf{x} refers to the entire covector and its numerical suffices to the position in a tensor product of indices.

Next we introduce a notation for this map Δ . It is a homomorphism from the algebra to two copies of the algebra. If we denote the generators of the first copy by $x_i \equiv x_i \otimes 1$ and the generators of the second copy by $x'_i \equiv 1 \otimes x_i$ then the assertion that Δ of the above linear form is a homomorphism is just that [19][20]

$$x''_i = x_i + x'_i, \quad i.e., \quad \mathbf{x}'' = \mathbf{x} + \mathbf{x}' \quad (9)$$

obey the same relations of $V^{\sim}(R', R)$. In other words, we can treat our noncommuting generators x_i like row vector coordinates and add them, provided we remember that in the braided tensor product they do not commute but rather obey the *braid statistics*

$$x'_i x_j = x_b x_a R^a{}_i{}^b{}_j, \quad i.e., \quad \mathbf{x}'_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}'_1 R. \quad (10)$$

This is the most compact way of working with our braided groups. We can really add them and treat them like covectors provided we have the appropriate braid statistics between independent copies. In this notation, the essential fact that the coproduct extends to products as a well-defined braided Hopf algebra is checked as

$$\begin{aligned} \mathbf{x}''_1 \mathbf{x}''_2 &= (\mathbf{x}_1 + \mathbf{x}'_1)(\mathbf{x}_2 + \mathbf{x}'_2) = \mathbf{x}_1 \mathbf{x}_2 + \mathbf{x}'_1 \mathbf{x}'_2 + \mathbf{x}_1 \mathbf{x}'_2 + \mathbf{x}'_2 \mathbf{x}_1 R \\ \mathbf{x}''_2 \mathbf{x}''_1 R' &= (\mathbf{x}_2 + \mathbf{x}'_2)(\mathbf{x}_1 + \mathbf{x}'_1) R' = \mathbf{x}_2 \mathbf{x}_1 R' + \mathbf{x}'_2 \mathbf{x}'_1 R' + \mathbf{x}_2 \mathbf{x}'_1 R' + \mathbf{x}'_1 \mathbf{x}_2 R_{21} R' \end{aligned}$$

which indeed coincide by (5). Note that there is a lot more to be checked for a braided-Hopf algebra. For example, we also have to check that Ψ likewise extends consistently to products in such a way as to be functorial with respect to the product map. Details are in [19]. But the homomorphism property is the most characteristic and the one which we stress here.

The simplest example is provided by the 1-dimensional matrices $R = (q)$, $R' = 1$, where q is arbitrary but non-zero. This is the braided line which was given more explicitly in Section 2. The braided plane also given there is likewise an example of the above:

Example 3.1 [19] *The standard quantum plane algebra $\mathbb{C}_q^{2|0}$ with relations $yx = qxy$ is a braided-covector algebra with*

$$x'x = q^2xx', \quad x'y = qyx', \quad y'y = q^2yy', \quad y'x = qxy' + (q^2 - 1)yx'$$

i.e.,

$$(x'', y'') = (x, y) + (x', y')$$

obeys the same relations provided we remember these braid statistics.

Proof We use the standard solution of the QYBE associated to the Jones knot invariant and the quantum group $SU_q(2)$ in another context, namely

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}, \quad R' = q^{-2}R$$

which we put into the above. The algebra $\mathbb{C}_q^{2|0}$ here is a well-known and much-studied one: the new features are the addition law and the braid-statistics. \square

Example 3.2 The mixed quantum plane $\mathbb{C}_q^{1|1}$ with relations $\theta^2 = 0$, $\theta x = qx\theta$ is a braided covector algebra with

$$x'x = q^2xx', \quad x'\theta = q\theta x', \quad \theta'\theta = -\theta\theta', \quad \theta'x = qx\theta' + (q^2 - 1)\theta x'$$

i.e.,

$$(x'', \theta'') = (x, \theta) + (x', \theta')$$

obeys the same relations provided we remember these braid statistics.

Proof We use the solution of the QYBE associated to the Alexander-Conway knot invariant in another context[21], namely

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad R' = q^{-2}R$$

which we put into the above. \square

Example 3.3 The usual mixed $1|1$ -superplane with relations $x\theta = \theta x$, $\theta^2 = 0$ is a braided covector algebra with

$$x'x = q^2xx', \quad x'\theta = q^2\theta x', \quad \theta'\theta = -\theta\theta', \quad \theta'x = x\theta' + (q^2 - 1)\theta x'$$

i.e.,

$$(x'', \theta'') = (x, \theta) + (x', \theta')$$

obeys the same relations provided we remember these braid statistics.

Proof We use the close cousin of the preceding R-matrix,

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^2 & q^2 - 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad R' = q^{-2}R$$

which we put into the above. This example is interesting because like the braided line in Section 2, the q -deformation enters only into the braid statistics while the algebra is the usual one. \square

Example 3.4 The fermionic quantum plane $\mathbb{C}_q^{0|2}$ with relations $\theta^2 = 0$, $\vartheta^2 = 0$ and $\vartheta\theta = -q^{-1}\theta\vartheta$ is a braided covector algebra with

$$\theta'\theta = -\theta\theta', \quad \theta'\vartheta = -q^{-1}\vartheta\theta', \quad \vartheta'\vartheta = -\vartheta\vartheta', \quad \vartheta'\theta = -q^{-1}\theta\vartheta' + (q^{-2} - 1)\vartheta\theta'$$

i.e.,

$$(\theta'', \vartheta'') = (\theta, \vartheta) + (\theta', \vartheta')$$

obeys the same relations provided we remember these braid statistics.

Proof We use

$$R = -q^{-2} \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}, \quad R' = q^2 R.$$

These are the same R-matrix as in Example 3.1 but with different normalisations. In fact, we use now for R the matrix which was $-R'$ in Example 3.1 and vice-versa. We return to this symmetry in Section 5.7. \square

These ideas work just as well for vector algebras with generators $1, v^i$ with indices up. So for the same data (3)–(6) we have also a *braided vector algebra* $V(R', R)$ defined with generators $1, v^i$ and relations

$$v^i v^j = R'^i{}_a{}^j{}_b v^b v^a, \quad i.e., \quad \mathbf{v}_1 \mathbf{v}_2 = R' \mathbf{v}_2 \mathbf{v}_1. \quad (11)$$

This has a braided addition law whereby $\mathbf{v}'' = \mathbf{v} + \mathbf{v}'$ obeys the same relations if \mathbf{v}' is a second copy with braid statistics[19]

$$v^i v^j = R^i{}_a{}^j{}_b v^b v^a, \quad i.e., \quad \mathbf{v}'_1 \mathbf{v}_2 = R \mathbf{v}_2 \mathbf{v}'_1. \quad (12)$$

More formally, it forms a braided-Hopf algebra with

$$\begin{aligned} \Delta v^i &= v^i \otimes 1 + 1 \otimes v^i, \quad \epsilon v^i = 0, \quad S v^i = -v^i \\ \Psi(v^i \otimes v^j) &= R^i{}_a{}^j{}_b v^b \otimes v^a, \quad i.e., \quad \Psi(\mathbf{v}_1 \otimes \mathbf{v}_2) = R \mathbf{v}_2 \otimes \mathbf{v}_1 \end{aligned} \quad (13)$$

extended multiplicatively with braid statistics. The proof is similar to the covector case. In the shorthand notation the key braided-homomorphism or additivity property is checked as

$$\begin{aligned} \mathbf{v}''_1 \mathbf{v}''_2 &= (\mathbf{v}_1 + \mathbf{v}'_1)(\mathbf{v}_2 + \mathbf{v}'_2) = \mathbf{v}_1 \mathbf{v}_2 + \mathbf{v}'_1 \mathbf{v}_2 + \mathbf{v}_1 \mathbf{v}'_2 + R \mathbf{v}_2 \mathbf{v}'_1 \\ R' \mathbf{v}''_2 \mathbf{v}''_1 &= R'(\mathbf{v}_2 + \mathbf{v}'_2)(\mathbf{v}_1 + \mathbf{v}'_1) = R' \mathbf{v}_2 \mathbf{v}_1 + R' \mathbf{v}'_2 \mathbf{v}_1 + R' \mathbf{v}_2 \mathbf{v}'_1 + R' R_{21} \mathbf{v}_1 \mathbf{v}'_2 \end{aligned}$$

which coincide by (5). As before, one also has to check other properties too, such as the fact that Ψ also extends consistently to products in a natural manner.

Example 3.5 The quantum plane $\mathbb{C}_q^{2|0}$ with relations $wv = q^{-1}vw$ is a braided-vector algebra with braid statistics

$$v^i v = q^2 v v^i, \quad v^i w = q v w^i + (q^2 - 1) v w^i, \quad w^i v = q v w^i, \quad w^i w = q^2 w w^i$$

i.e.

$$\begin{pmatrix} v'' \\ w'' \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} v' \\ w' \end{pmatrix}$$

obeys the same relations.

Proof We take the standard R-matrix as in Example 3.1. Again, the resulting algebra is standard. To this we now add the braiding and coaddition. \square

Similarly for the other standard examples $\mathbb{C}_q^{0|2}$, $\mathbb{C}_q^{1|1}$ etc. The possibilities are the same as for the covector case. Note that it is a mistake to think that the vectors are correlated with the fermionic normalisation and the covectors with the bosonic one: in the braided approach to such algebras we (a) have more than two types of algebra if PR has more than two eigenvalues (we will see such examples below) and (b) we have both vectors and covectors for each choice of eigenvalue or more generally for each pair R, R' obeying our matrix conditions (3)–(6).

A typical application of fermionic co-ordinates in differential geometry is as describing the properties of forms $\theta = dx$. The braided vector version of Example 3.3 could be viewed for example as the exterior algebra in 1-dimension. It comes out as

Example 3.6 The 1-dimensional exterior algebra $\Omega(\mathbb{C}_q)$ with relations $dx^2 = 0$, $dx x = q^{-2} x dx$ is a braided vector algebra with braid statistics

$$x'x = q^2 xx', \quad x'dx = q^2 dx x' + (q^2 - 1)xdx', \quad dx'x = xdx', \quad dx'dx = -dx dx'$$

i.e.,

$$\begin{pmatrix} x'' \\ dx'' \end{pmatrix} = \begin{pmatrix} x \\ dx \end{pmatrix} + \begin{pmatrix} x' \\ dx' \end{pmatrix}$$

obeys the same relations.

Proof We take the same R-matrix as in Example 3.3 but compute the corresponding vector rather than covector algebra. \square

This example transforms covariantly as a vector under a q -deformed supersymmetry quantum group which mixes x, dx , here from the right. We will use a left-covariant version of it later in Example 4.4.

3.2 Braided coaddition on matrices $A(R)$ and $\bar{A}(R)$

We have seen how to coadd vectors and covectors, an idea that was missing without the braided approach. The same problem occurs for the familiar *quantum matrices* $A(R)$ studied in [22] and elsewhere. These are defined with generators $1, t^i_j$ and relations

$$R^i_a{}^k t^a_j t^b_l = t^k_b t^i_a R^a_j{}^b l, \quad i.e., \quad R t_1 t_2 = t_2 t_1 R. \quad (14)$$

It is well-known that any algebra of this type (without any condition at all on R) forms an ordinary quantum group with

$$\Delta.t^i_j = t^i_a \otimes t^a_j, \quad \epsilon t^i_j = \delta^i_j, \quad i.e., \quad \Delta.t = t \otimes t, \quad \epsilon t = \text{id} \quad (15)$$

and (usually) without an antipode. An ordinary quantum group means we just use the trivial braiding when extending $\Delta.$ to products. It means that

$$t''^i_j = t^i_a t'^a_j, \quad [t^i_j, t'^k_l] = 0, \quad i.e., \quad t'' = t t', \quad [t_1, t'_2] = 0 \quad (16)$$

is also a realisation of the same algebra if t, t' are. This well-known coproduct $\Delta.$ corresponds to multiplication of matrices in terms of the possibly non-commuting co-ordinate functions t^i_j .

But classical matrices can also be added. So what about a corresponding coaddition law for $A(R)$? Again, this can be handled with braided geometry, at least when R solves the QYBE and obeys the Hecke condition

$$(PR - q)(PR + q^{-1}) = 0. \quad (17)$$

In this case the quantum matrices form in fact a braided covector algebra with addition law[23]

$$t''^i_j = t^i_j + t'^i_j, \quad i.e., \quad t'' = t + t' \quad (18)$$

obeying the same relations of $A(R)$ provided t' is a second copy with braid statistics

$$t'^i_j t'^k_l = R^k_b{}^i a t'^b_d t'^a_c R^c_j{}^d l, \quad i.e., \quad t'_1 t'_2 = R_{21} t'_2 t'_1 R. \quad (19)$$

More formally, $A(R)$ is a braided-Hopf algebra with

$$\Psi(t_1 \otimes t_2) = R_{21} t_2 \otimes t_1 R, \quad \Delta t = t \otimes 1 + 1 \otimes t, \quad \epsilon t = 0, \quad S t = -t. \quad (20)$$

Moreover, the coaddition Δ is compatible with the usual matrix comultiplication $\Delta.$ in the sense[23]

$$\begin{aligned} (\text{id} \otimes \cdot) \circ (\text{id} \otimes \tau \otimes \text{id})(\Delta. \otimes \Delta.) \circ \Delta &= (\Delta \otimes \text{id}) \circ \Delta. \\ (\cdot \otimes \text{id}) \circ (\text{id} \otimes \tau \otimes \text{id})(\Delta. \otimes \Delta.) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta. \end{aligned} \quad (21)$$

where τ the usual transposition map. To see this, we have to show that \mathbf{t}'' in our short-hand notation obeys the same algebra relations. This is

$$\begin{aligned} R(\mathbf{t}_1 + \mathbf{t}'_1)(\mathbf{t}_2 + \mathbf{t}'_2) &= R\mathbf{t}_1\mathbf{t}_2 + R\mathbf{t}'_1\mathbf{t}'_2 + RR_{21}\mathbf{t}_2\mathbf{t}'_1R + R\mathbf{t}_1\mathbf{t}'_2 \\ (\mathbf{t}_2 + \mathbf{t}'_2)(\mathbf{t}_1 + \mathbf{t}'_1)R &= \mathbf{t}_2\mathbf{t}_1R + \mathbf{t}'_2\mathbf{t}'_1R + R\mathbf{t}_1\mathbf{t}'_2R_{21}R + \mathbf{t}_2\mathbf{t}'_1R \end{aligned}$$

which are equal because $R_{21}R = 1 + (q - q^{-1})PR$ and $RR_{21} = 1 + (q - q^{-1})RP$ from the q -Hecke assumption. One can also check that Ψ extends consistently to products in such a way as to be functorial. Details are in [23].

We have given here a direct proof of the coaddition structure on $A(R)$. Alternatively, we can put it more explicitly in the braided covector algebra form (8)–(10) by working with the covector notation $t_I = t^{i_0}_{i_1}$ for the generators where $I = (i_0, i_1)$, $J = (j_0, j_1)$ etc are multi-indices. Then[23]

$$\begin{aligned} R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1R &\Leftrightarrow t_I t_J = t_B t_A \mathbf{R}'^A_{I^B J}; & \mathbf{R}'^I_{J^K L} &= R^{-1j_0}_{i_0} R^{l_0}_{k_0} R^{i_1}_{j_1} R^{k_1}_{l_1} \\ \mathbf{t}'_1\mathbf{t}_2 = R_{21}\mathbf{t}_2\mathbf{t}'_1R &\Leftrightarrow t'_I t_J = t_B t'_A \mathbf{R}^A_{I^B J}; & \mathbf{R}^I_{J^K L} &= R^{l_0}_{k_0} R^{j_0}_{i_0} R^{i_1}_{j_1} R^{k_1}_{l_1} \end{aligned} \quad (22)$$

puts $A(R)$ explicitly into the form of a braided covector algebra with n^2 generators. We use the bold multi-index \mathbf{R}, \mathbf{R}' matrices built from our original R . They obey the conditions (3)–(6) and also the additional (113) just in virtue of the QYBE and q -Hecke condition on R . The corresponding braided vector algebra (11)–(13) in matrix form is

$$v^I \equiv v^{i_1}_{i_0}; \quad R_{21}\mathbf{v}_1\mathbf{v}_2 = \mathbf{v}_2\mathbf{v}_1R_{21}, \quad \mathbf{v}'_1\mathbf{v}_2 = R\mathbf{v}_2\mathbf{v}'_1R_{21}. \quad (23)$$

Example 3.7 The standard quantum matrices $M_q(2)$ with generators $\mathbf{t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and relations

$$\begin{aligned} ab &= q^{-1}ba, & ac &= q^{-1}ca, & bd &= q^{-1}db, & cd &= q^{-1}dc \\ bc &= cb, & ad - da &= (q^{-1} - q)bc \end{aligned}$$

have the usual multiplication law whereby[24]

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obey the same $M_q(2)$ relations provided the second primed copy commutes with the first copy. They also have a braided addition law whereby[23]

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

also obeys the relations of $M_q(2)$ provided the second primed copy has the braid statistics

$$\begin{aligned} a'a &= q^2aa', & b'b &= q^2bb', & c'c &= q^2cc', & d'd &= q^2dd' \\ a'b &= qba', & a'c &= qca', & a'd &= da', & b'd &= qdb', & c'd &= qdc' \\ b'a &= qab' + (q^2 - 1)ba', & b'c &= cb' + (q - q^{-1})da' \\ c'a &= qac' + (q^2 - 1)ca', & c'b &= bc' + (q - q^{-1})da' \\ d'b &= qbd' + (q^2 - 1)db', & d'c &= qcd' + (q^2 - 1)dc' \\ d'a &= ad' + (q - q^{-1})(cb' + bc') + (q - q^{-1})^2da'. \end{aligned}$$

Moreover, this addition law distributes in the expected way over the multiplication.

Proof We take the standard R as in Example 3.1 but in the normalisation required for the q -Hecke condition (17), which is q^{-1} times the one shown in Example 3.1. This is not relevant to the algebra but is needed for the correct braiding. We then compute from the formulae (14)–(20). \square

We have begun with the above quantum matrices $A(R)$ because they are well-known quantum groups and probably the reader has seen them somewhere before. But they are not really the example we need for braided geometry. A more interesting algebra, which we will need in Section 7.1, is the variant $\bar{A}(R)$ studied by the author in [25]. It is defined with generators $1, x^i_j$ and relations

$$R^k_b{}_a x^a_j x^b_l = x^k_b x^i_a R^a_j{}^b_l, \quad i.e., \quad R_{21} \mathbf{x}_2 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R \quad (24)$$

and forms a braided covector algebra if R is a Hecke solution of the QYBE, with addition law and braid statistics[25]

$$\mathbf{x}'' = \mathbf{x} + \mathbf{x}'; \quad x'^i_j x^k_l = R^i{}_a{}^k{}_b x'^a_c R^c{}_j{}^d{}_l, \quad i.e., \quad \mathbf{x}'_1 \mathbf{x}_2 = R \mathbf{x}_2 \mathbf{x}'_1 R. \quad (25)$$

More formally, it forms a braided-Hopf algebra with

$$\Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = R \mathbf{x}_2 \otimes \mathbf{x}_1 R, \quad \Delta \mathbf{x} = \mathbf{x} \otimes 1 + 1 \otimes \mathbf{x}, \quad \epsilon \mathbf{x} = 0, \quad S \mathbf{x} = -\mathbf{x} \quad (26)$$

To see this we check that Δ extends to products as an algebra homomorphism to the braided tensor product algebra, i.e. that \mathbf{x}'' obeys the same relations. This is checked as

$$\begin{aligned} R_{21}(\mathbf{x}_1 + \mathbf{x}'_1)(\mathbf{x}_2 + \mathbf{x}'_2) &= R_{21} \mathbf{x}_1 \mathbf{x}_2 + R_{21} \mathbf{x}'_1 \mathbf{x}'_2 + R_{21} R \mathbf{x}_2 \mathbf{x}'_1 R + R_{21} \mathbf{x}_1 \mathbf{x}'_2 \\ (\mathbf{x}_2 + \mathbf{x}'_2)(\mathbf{x}_1 + \mathbf{x}'_1) R &= \mathbf{x}_2 \mathbf{x}_1 R + \mathbf{x}'_2 \mathbf{x}'_1 R + R_{21} \mathbf{x}_1 \mathbf{x}'_2 R_{21} R + \mathbf{x}_2 \mathbf{x}'_1 R \end{aligned}$$

which are equal by the q -Hecke assumption much as before. We also have to check that Ψ extends consistently to products of the generators in such a way as to be functorial. This reduces to the QYBE for R along the lines for $A(R)$ in [23].

The usual matrix coproduct of \mathbf{x} forms neither a quantum group nor a braided one but something in between. On the other hand, as before, we can put the coaddition explicitly into our usual braided covector form by working with the multi-index notation $x_I = x^{i_0}{}_{i_1}$. Then[25]

$$\begin{aligned} R_{21} \mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R &\Leftrightarrow x_I x_J = x_B x_A \mathbf{R}^A{}_I{}^B{}_J; \quad \mathbf{R}^I{}_J{}^K{}_L = R^{-1l_0}{}_{k_0}{}^{j_0}{}_{i_0} R^{i_1}{}_{j_1}{}^{k_1}{}_{l_1} \\ \mathbf{x}'_1 \mathbf{x}_2 = R \mathbf{x}_2 \mathbf{x}'_1 R &\Leftrightarrow x'_I x_J = x_B x'_A \mathbf{R}^A{}_I{}^B{}_J; \quad \mathbf{R}^I{}_J{}^K{}_L = R^{j_0}{}_{i_0}{}^{l_0}{}_{k_0} R^{i_1}{}_{j_1}{}^{k_1}{}_{l_1} \end{aligned} \quad (27)$$

puts $\bar{A}(R)$ into the form of a braided covector algebra with n^2 generators. Its corresponding braided vector algebra (11)–(13) in matrix form is $\bar{A}(R)$ again,

$$v^I \equiv v^{i_1}{}_{i_0}; \quad R_{21} \mathbf{v}_1 \mathbf{v}_2 = \mathbf{v}_2 \mathbf{v}_1 R, \quad \mathbf{v}'_1 \mathbf{v}_2 = R \mathbf{v}_2 \mathbf{v}'_1 R. \quad (28)$$

Example 3.8 [25] The q -Euclidean space algebra $\bar{M}_q(2)$ with generators $\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and relations

$$\begin{aligned} ba = qab, \quad ca = q^{-1}ac, \quad da = ad, \quad db = q^{-1}bd \quad dc = qcd \\ bc = cb + (q - q^{-1})ad \end{aligned}$$

has a braided addition law whereby

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

also obeys the relations of $\bar{M}_q(2)$ provided the second primed copy has the braid statistics

$$\begin{aligned}
c'c &= q^2cc', & d'd &= q^2dd', & a'a &= q^2aa', & b'b &= q^2bb' \\
c'd &= qdc', & c'a &= qac', & c'b &= bc', & d'b &= qbd', & a'b &= qba' \\
d'c &= qcd' + (q^2 - 1)dc', & d'a &= ad' + (q - q^{-1})bc' \\
a'c &= qca' + (q^2 - 1)ac', & a'd &= da' + (q - q^{-1})bc' \\
b'd &= qdb' + (q^2 - 1)bd', & b'a &= qab' + (q^2 - 1)ba' \\
b'c &= cb' + (q - q^{-1})(ad' + da') + (q - q^{-1})^2bc'.
\end{aligned}$$

Proof We take the standard Jones invariant or $SU_q(2)$ R -matrix as in Example 3.1 but in the normalisation required for the q -Hecke condition (17), which is q^{-1} times the one shown there. This is needed for the correct braiding. We then compute from the formulae (24)–(26). \square

The interpretation of this standard example $\bar{M}_q(2)$ as q -Euclidean space will be covered in Section 7.1. The general $\bar{A}(R)$ construction is however, more general. A less standard example is:

Example 3.9 The algebra $\bar{M}_q(1|1)$ with generators $\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and relations

$$\begin{aligned}
b^2 &= 0, & c^2 &= 0, & ba &= abq, & ca &= q^{-1}ac, & db &= -qbd, & dc &= -cdq^{-1} \\
da &= ad, & bc &= cb + (q - q^{-1})ad
\end{aligned}$$

has a braided addition law whereby

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

also obeys the relations of $\bar{M}_q(1|1)$ provided the second primed copy has the braid statistics

$$\begin{aligned}
a'a &= q^2aa', & b'b &= -bb', & c'c &= -cc', & d'd &= dd'q^{-2} \\
a'b &= qba', & a'c &= ca'q + (q^2 - 1)ac', & a'd &= da' + (q - q^{-1})bc' \\
b'a &= ab'q + (q^2 - 1)ba', & b'c &= cb' + (q - q^{-1})^2bc' + (q - q^{-1})(da' + ad') \\
b'd &= -q^{-1}db' + (q^{-2} - 1)bd', & c'a &= qac', & c'b &= bc', & c'd &= -q^{-1}dc' \\
d'a &= ad' + (q - q^{-1})bc', & d'b &= -q^{-1}bd', & d'c &= -q^{-1}cd' + (q^{-2} - 1)dc'
\end{aligned}$$

Proof We take the Alexander-Conway R -matrix as in Example 3.2 but in the normalisation required for the q -Hecke condition (17), which is q^{-1} times the one shown. This is needed for the correct braiding. We then compute from the formulae (24)–(26). \square

3.3 Braided coaddition on matrices $B(R)$

Next we consider the *braided matrices* $B(R)$ introduced and studied as a braided group by the author in [11][26][27]. These are defined with generators $1, u^i_j$ and relations

$$R^k_b{}^i{}_a u^a{}_c R^c{}_j{}^b{}_d u^d{}_l = u^k_b{}^i{}_c R^c{}_a{}^b{}_d u^a{}_l, \quad i.e., \quad R_{21} \mathbf{u}_1 R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1 R. \quad (29)$$

Such relations are perhaps more familiar as among the relations obeyed by the matrix generators l^+Sl^- of the quantum groups $U_q(g)$ in [22] but these have many other relations too beyond (29) and are not relevant for us now. They have been used by Zumino and others to describe the differential calculus on quantum groups; see [26] for the full story here. We are interested instead in (29) purely as a quadratic algebra with generators u^i_j and these relations, which is not in general a quantum group at all.

The main property of these braided matrices in [11], from which they take their name, is their multiplicative braided group structure. We have [11]

$$\begin{aligned} \Delta.u^i_j &= u^i_a \otimes u^a_j, \quad \epsilon u^i_j = \delta^i_j, \quad i.e., \quad \Delta.\mathbf{u} = \mathbf{u} \otimes \mathbf{u}, \quad \epsilon \mathbf{u} = \text{id} \\ \Psi.(u^i_j \otimes u^k_l) &= u^p_q \otimes u^m_n R^i_a{}^d{}_p R^{-1a}{}_m{}^q{}_b R^n{}_c{}^b{}_l \tilde{R}^c{}_j{}^k{}_d \\ i.e., \quad \Psi.(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) &= \mathbf{u}_2 R^{-1} \otimes \mathbf{u}_1 R. \end{aligned} \quad (30)$$

It means that if \mathbf{u}' is another copy of $B(R)$ then the matrix product

$$u''^i_j = u^i_a u'^a{}_j, \quad i.e., \quad \mathbf{u}'' = \mathbf{u}\mathbf{u}' \quad (31)$$

obeys the relations of $B(R)$ also provided \mathbf{u}' has the *multiplicative braided statistics*

$$R^{-1i}{}_a{}^k{}_b u'^a{}_c R^c{}_j{}^b{}_d u^d{}_l = u^k{}_b R^{-1i}{}_a{}^b{}_c u'^a{}_d R^d{}_j{}^c{}_l, \quad i.e., \quad R^{-1}\mathbf{u}'_1 R\mathbf{u}_2 = \mathbf{u}_2 R^{-1}\mathbf{u}'_1 R. \quad (32)$$

To see this, we check

$$\begin{aligned} R_{21}\mathbf{u}_1\mathbf{u}'_1 R\mathbf{u}_2\mathbf{u}'_2 &= R_{21}\mathbf{u}_1 R(R^{-1}\mathbf{u}'_1 R\mathbf{u}_2)\mathbf{u}'_2 = (R_{21}\mathbf{u}_1 R\mathbf{u}_2)R^{-1}R_{21}^{-1}(R_{21}\mathbf{u}'_1 R\mathbf{u}'_2) \\ &= \mathbf{u}_2 R_{21}(\mathbf{u}_1 R_{21}^{-1}\mathbf{u}'_2 R_{21})\mathbf{u}'_1 R = \mathbf{u}_2 R_{21} R_{21}^{-1} \mathbf{u}'_2 R_{21} \mathbf{u}_1 \mathbf{u}'_1 R = \mathbf{u}_2 \mathbf{u}'_2 R_{21} \mathbf{u}_1 \mathbf{u}'_1 R \end{aligned}$$

as required for $\Delta.$ to extend to $B(R)$ as a braided-Hopf algebra. The other details such as functoriality of $\Psi.$ can also be checked in the same explicit way[11]. This is was the first braided group construction known.

Note that we have stated $\Psi.$ implicitly. To give it explicitly (for a proper braided-group structure) we need that R is *bi-invertible* in the sense that both R^{-1} and the *second inverse* \tilde{R} exist. The latter is characterised by

$$\tilde{R}^i{}_a{}^b{}_c R^a{}_j{}^k{}_b = \delta^i_j \delta^k_l = R^i{}_a{}^b{}_l \tilde{R}^a{}_j{}^k{}_b. \quad (33)$$

If we have also that R obeys the q -Hecke condition (17) then there is also a braided-covector algebra structure, discovered by U. Meyer, with addition law $\mathbf{u}'' = \mathbf{u} + \mathbf{u}'$ and braid statistics[28]

$$R^{-1i}{}_a{}^k{}_b u'^a{}_c R^c{}_j{}^b{}_d u^d{}_l = u^k{}_b R^b{}_c{}^i{}_a u'^a{}_d R^d{}_j{}^c{}_l, \quad i.e., \quad R^{-1}\mathbf{u}'_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}'_1 R. \quad (34)$$

More formally, $B(R)$ is a braided-Hopf algebra with

$$\Psi(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) = \mathbf{u}_2 R_{21} \otimes \mathbf{u}_1 R, \quad \Delta \mathbf{u} = \mathbf{u} \otimes 1 + 1 \otimes \mathbf{u}, \quad \epsilon \mathbf{u} = 0, \quad S\mathbf{u} = -\mathbf{u}. \quad (35)$$

To see this we show as usual that Δ extends to products as an algebra homomorphism to the braided tensor product algebra, i.e. that \mathbf{u}'' obeys the same relations. This is checked as

$$\begin{aligned} R_{21}(\mathbf{u}_1 + \mathbf{u}'_1)R(\mathbf{u}_2 + \mathbf{u}'_2) &= R_{21}\mathbf{u}_1 R\mathbf{u}_2 + R_{21}\mathbf{u}'_1 R\mathbf{u}'_2 + R_{21}R\mathbf{u}_2 R_{21}\mathbf{u}'_1 R + R_{21}\mathbf{u}_1 R\mathbf{u}'_2 \\ (\mathbf{u}_2 + \mathbf{u}'_2)R_{21}(\mathbf{u}_1 + \mathbf{u}'_1)R &= \mathbf{u}_2 R_{21}\mathbf{u}_1 R + \mathbf{u}'_2 R_{21}\mathbf{u}'_1 R + R_{21}\mathbf{u}_1 R\mathbf{u}'_2 R_{21}R + \mathbf{u}_2 R_{21}\mathbf{u}'_1 R \end{aligned}$$

which are equal by the q -Hecke assumption (17). Functoriality of Ψ under the product map can also be checked explicitly by these techniques, as well as the antipode and other properties needed for a braided-Hopf algebra.

This gives a direct proof of the (braided) comultiplication and coaddition structures on $B(R)$. We can put the latter explicitly into the braided covector form (8)–(10) by working with the multi-index notation $u_I = u^{i_0}_{i_1}$ and [11][28]

$$\begin{aligned}\mathbf{R}^I J^K L &= R^{-1d}_{k_0 j_0 a} R^{k_1 b a}_{i_0} R^{i_1 c b}_{l_1} \tilde{R}^c_{j_1 l_0 d} \\ \mathbf{R}^I J^K L &= R^{j_0 a d}_{k_0} R^{-1a}_{i_0 k_1 b} R^{i_1 c b}_{l_1} \tilde{R}^c_{j_1 l_0 d} \\ \mathbf{R}^I J^K L &= R^{j_0 a d}_{k_0} R^{k_1 b a}_{i_0} R^{i_1 c b}_{l_1} \tilde{R}^c_{j_1 l_0 d}.\end{aligned}\tag{36}$$

Then we have

$$\begin{aligned}R_{21} \mathbf{u}_1 R \mathbf{u}_2 &= \mathbf{u}_2 R_{21} \mathbf{u}_1 R \Leftrightarrow u_I u_J = u_B u_A \mathbf{R}^A_{I^B J} \\ \mathbf{u}'' &= \mathbf{u} \mathbf{u}'; \quad R^{-1} \mathbf{u}'_1 R \mathbf{u}_2 = \mathbf{u}_2 R^{-1} \mathbf{u}'_1 R \Leftrightarrow u'_I u_J = u_B u'_A \mathbf{R}^A_{I^B J} \\ \mathbf{u}'' &= \mathbf{u} + \mathbf{u}'; \quad R^{-1} \mathbf{u}'_1 R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}'_1 R \Leftrightarrow u'_I u_J = u_B u'_A \mathbf{R}^A_{I^B J}.\end{aligned}\tag{37}$$

It is easy to see that \mathbf{R}' , \mathbf{R} obey the conditions (3)–(6) needed for our braided covector space as well as the supplementary ones (113) needed later for the coaddition of forms. The corresponding braided vector algebra (11)–(13) in matrix form for the relations and additive braid statistics is

$$v^I \equiv v^{i_1}_{i_0}; \quad \mathbf{v}_1 \tilde{R}_{21} \mathbf{v}_2 R_{21} = R \mathbf{v}_2 \tilde{R} \mathbf{v}_1, \quad \mathbf{v}'' = \mathbf{v} + \mathbf{v}'; \quad \mathbf{v}'_1 \tilde{R}_{21} \mathbf{v}_2 R^{-1} = R \mathbf{v}_2 \tilde{R} \mathbf{v}'_1.\tag{38}$$

A braided coaddition on the following example of a braided matrix covector space was obtained by the author in [19] but not in such a nice R-matrix form, which is due to [28].

Example 3.10 *The q -Minkowski space algebra $BM_q(2)$ with generators $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and relations [29][11]*

$$\begin{aligned}ba &= q^2 ab, \quad ca = q^{-2} ac, \quad da = ad, \quad bc = cb + (1 - q^{-2})a(d - a) \\ db &= bd + (1 - q^{-2})ab, \quad cd = dc + (1 - q^{-2})ca\end{aligned}$$

has a braided multiplication law whereby [11]

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obey the same relations of $BM_q(2)$ if the primed copy does and if we use the multiplicative braid statistics

$$\begin{aligned}a'a &= aa' + (1 - q^2)bc', \quad a'b = ba', \quad a'c = ca' + (1 - q^2)(d - a)c' \\ a'd &= da' + (1 - q^{-2})bc', \quad b'a = ab' + (1 - q^2)b(d' - a'), \quad b'b = q^2bb' \\ b'c &= q^{-2}cb' + (1 + q^2)(1 - q^{-2})^2bc' - (1 - q^{-2})(d - a)(d' - a') \\ b'd &= db' + (1 - q^{-2})b(d' - a'), \quad c'a = ac', \quad c'b = q^{-2}bc' \\ c'c &= q^2cc', \quad c'd = dc', \quad d'a = ad' + (1 - q^{-2})bc' \\ d'b &= bd', \quad d'c = cd' + (1 - q^{-2})(d - a)c', \quad d'd = dd' - q^{-2}(1 - q^{-2})bc'.\end{aligned}$$

Here $q^{-1}a + qd$ is central and bosonic [11]. At the same time we have a braided addition law whereby [28]

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obey the relations again if the primed copy does and has the additive braid statistics

$$\begin{aligned}
a'a &= q^2aa', & a'b &= ba', & b'b &= q^2bb', & c'a &= ac', & c'c &= q^2cc' \\
a'c &= ca'q^2 + (q^2 - 1)ac', & a'd &= da' + (q^2 - 1)bc' + (q - q^{-1})^2aa' \\
b'a &= (q^2 - 1)ba' + ab'q^2, & b'c &= cb' + (1 - q^{-2})(da' + ad') + (q - q^{-1})^2bc' - (2 - 3q^{-2} + q^{-4})aa' \\
b'd &= db' + (q^2 - 1)bd' + (q^{-2} - 1)ba' + (q - q^{-1})^2ab', & c'b &= bc' + (1 - q^{-2})aa' \\
c'd &= dc'q^2 + (q^2 - 1)ca', & d'a &= ad' + (q^2 - 1)bc' + (q - q^{-1})^2aa' \\
d'b &= bd'q^2 + (q^2 - 1)ab', & d'c &= cd' + (q^2 - 1)dc' + (q - q^{-1})^2ca' + (q^{-2} - 1)ac' \\
d'd &= dd'q^2 + (q^2 - 1)cb' + (q^{-2} - 1)bc' - (1 - q^{-2})^2aa'
\end{aligned}$$

So we have both multiplication and addition of these braided matrices.

Proof We use the R-matrix from Example 3.1 in the q -Hecke normalisation (as in Example 3.7), which we put into (29)–(35). The normalisation and Hecke condition do not enter at all into the multiplicative braided group structure, but are needed for the additive one. \square

The interpretation of this standard example $BM_q(2)$ as q -Minkowski space will be covered in Section 7.2. Its specific six relations were first proposed as q -Minkowski space by Carow-Watamura et. al. [29] in another context as a tensor product of two quantum planes. We will see the connection later in Section 4.1. The braided matrix $B(R)$ construction is however, more general. A less standard example is:

Example 3.11 The algebra $BM_q(1|1)$ with generators $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and relations[11]

$$\begin{aligned}
b^2 &= 0, & c^2 &= 0, & d - a & \text{central}, \\
ab &= q^{-2}ba, & ac &= q^2ca, & bc &= -q^2cb + (1 - q^2)(d - a)a
\end{aligned}$$

has a braided multiplication law whereby[11]

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obey the same relations of $BM_q(1|1)$ if the primed copy does and if we use the multiplicative braid statistics consisting of $d - a$ bosonic and

$$\begin{aligned}
a'a &= aa' + (1 - q^2)bc', & b'b &= -bb', & c'c &= -cc', & a'b &= ba' \\
b'c &= -cb' - (1 - q^2)(d - a)(d' - a'), & b'a &= ab' + (1 - q^2)b(d' - a') \\
c'b &= -bc', & a'c &= ca' + (1 - q^2)(d - a)c', & c'a &= ac'.
\end{aligned}$$

At the same time we have a braided addition law whereby

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obey the relations again if the primed copy does and if we use the additive braid statistics

$$\begin{aligned}
a'a &= q^2aa', & a'b &= ba', & a'c &= q^2ca' + (q^2 - 1)ac' \\
a'd &= da' + (q^{-2} - 1)bc' + (q - q^{-1})^2aa', & b'a &= q^2ab' + (q^2 - 1)ba', & b'b &= -bb' \\
b'c &= -q^2cb' + (q - q^{-1})^2bc' + (1 - q^2)(da' + ad') + (2q^2 - 3 + q^{-2})aa' \\
b'd &= db' + (q^{-2} - 1)bd' + (q^2 - 1)ba' + (q - q^{-1})^2ab', & c'a &= ac' \\
c'b &= -q^{-2}bc' + (1 - q^{-2})aa', & c'c &= -cc', & c'd &= q^{-2}dc' + (q^{-2} - 1)ca' \\
d'a &= ad' + (q^{-2} - 1)bc' + (q - q^{-1})^2aa', & d'b &= q^{-2}bd' + (q^{-2} - 1)ab' \\
d'c &= cd' + (q^{-2} - 1)dc' + (q - q^{-1})^2ca' + (q^2 - 1)ac' \\
d'd &= q^{-2}dd' + (q^{-2} - 1)(cb' + bc') + (q - q^{-1})^2aa'.
\end{aligned}$$

So we have both multiplication and addition of these non-standard braided matrices.

Proof We use the R-matrix from Example 3.2 in the q -Hecke normalisation (as in Example 3.9), which we put into (29)–(35). The normalisation and Hecke condition do not enter at all into the multiplicative braided group structure, only the additive one. \square

This example is supercommutative in the limit $q \rightarrow 1$ with b, c odd and a, d even. The braid statistics also become ± 1 according to the degree. Thus we recover exactly the superbialgebra $M(1|1)$ consisting of these generators and their appropriate supercommutation relations. Thus the notion of braided matrices really generalises both ordinary and supermatrices[11].

For completeness we note also that there is a similarity of the braided matrix algebra (29) with the ‘reflection equation’ of Cherednik [30] whose constant form is called ‘RE’ in [31]. This paper then went on to repeat some of the algebraic constructions in [11][26][27] in terms of RE, with (29) as a variant RE. The braided matrix algebra $B(R)$ is quite interesting from a purely homological point of view too, see[32]. Further results such as covariance of this braided matrix quadratic algebra under a background quantum group[11][27], its quantum trace central elements[27] and a Hermitian $*$ -structure[33] were likewise introduced as far as I know for the first time in the braided approach. We will cover the latter two properties in the next section and covariance in Section 6.

4 Braided linear algebra

So far we have introduced braided covectors x_i , vectors v^i and matrices u^i_j , and linear structures on the quantum matrices t^i_j as well. Now we begin to explore the relationships between these objects in analogy with the usual formulae in linear algebra. This justifies our terminology further and shows that they all fit together into a systematic generalisation of our usual concepts.

4.1 Braided linear transformations

In braided geometry all independent objects enjoy mutual braid statistics with respect to each other. We therefore have to extend the braiding Ψ to work between objects and not only for a given braided group as we have done until now. This depends in fact on the applications being made, as we have seen already in Examples 3.10 and 3.11. For the present set of applications we

take[27]

$$\begin{aligned}
\Psi(x_i \otimes x_j) &= x_n \otimes x_m R^{m_i n_j}, & \Psi(v^i \otimes v^j) &= R^i_m j_n v^n \otimes v^m \\
\Psi(x_i \otimes v^j) &= \tilde{R}^{m_i j_n} v^n \otimes x_m, & \Psi(v^i \otimes x_j) &= x_n \otimes v^m R^{-1 i}_m n_j \\
\Psi(u^i_j \otimes x_k) &= x_m \otimes u^a_b R^{-1 i}_a m_n R^b_j n_k, & \Psi(x_k \otimes u^i_j) &= u^a_b \otimes x_m \tilde{R}^n_k i_a R^m_n b_j \\
\Psi(u^i_j \otimes v^k) &= v^m \otimes u^a_b R^i_a n_m \tilde{R}^b_j k_n, & \Psi(v^k \otimes u^i_j) &= u^a_b \otimes v^m R^k_n i_a R^{-1 n}_m b_j \\
\Psi(u^i_j \otimes u^k_l) &= u^p_q \otimes u^m_n R^i_a d_p R^{-1 a}_m q_b R^n_c b_l \tilde{R}^c_j k_d
\end{aligned} \tag{39}$$

or equivalently the braid statistics[27]

$$\begin{aligned}
\mathbf{x}'_1 \mathbf{x}_2 &= \mathbf{x}_2 \mathbf{x}'_1 R, & \mathbf{v}'_1 \mathbf{v}_2 &= R \mathbf{v}_2 \mathbf{v}'_1 \\
\mathbf{x}'_1 R \mathbf{v}_2 &= \mathbf{v}_2 \mathbf{x}'_1, & \mathbf{v}'_1 \mathbf{x}_2 &= \mathbf{x}_2 R^{-1} \mathbf{v}'_1 \\
\mathbf{u}'_1 \mathbf{x}_2 &= \mathbf{x}_2 R^{-1} \mathbf{u}'_1 R, & \mathbf{x}'_1 R \mathbf{u}_2 R^{-1} &= \mathbf{u}_2 \mathbf{x}'_1 \\
R^{-1} \mathbf{u}'_1 R \mathbf{v}_2 &= \mathbf{v}_2 \mathbf{u}'_1, & \mathbf{v}'_1 \mathbf{u}_2 &= R \mathbf{u}_2 R^{-1} \mathbf{v}'_1 \\
R^{-1} \mathbf{u}'_1 R \mathbf{u}_2 &= \mathbf{u}_2 R^{-1} \mathbf{u}'_1 R
\end{aligned} \tag{40}$$

where the primes denote the second algebra in the relevant braided tensor product algebra as in (1). The systematic way to derive these braid statistics will be covered in Section 6.1 using the background quantum group covariance. For the moment we verify directly that they are suitable. They are quite natural for, but not uniquely determined by, these applications alone.

With the help of such braid statistics we show now that our braided matrices act as (braided) transformations on braided vectors and covectors, as well as on themselves. The fundamental notion here is that of a braided comodule algebra of $\mathcal{B}(R)$. On covectors for example it should be an algebra homomorphism $V \rightarrow V \otimes \mathcal{B}(R)$ to the braided tensor product algebra structure as in (1), determined now by the mutual braid statistics between braided matrices and braided covectors. This indeed works with the transformation

$$x_i \mapsto x_a \otimes u^a_i, \quad i.e., \quad \mathbf{x}'' = \mathbf{x} \mathbf{u}' \tag{41}$$

obeys the same relations as the \mathbf{x} . To see this, it is convenient to assume that PR' is given as some function of PR , as for example in (7). Since nothing depends on the precise form of the function, it is clear that this is not a strong restriction. Then we compute $\mathbf{x}_1 \mathbf{u}'_1 \mathbf{x}_2 \mathbf{u}'_2 = \mathbf{x}_1 \mathbf{x}_2 R^{-1} \mathbf{u}'_1 R \mathbf{u}'_2 = \mathbf{x}_2 \mathbf{x}_1 R' R^{-1} \mathbf{u}'_1 R \mathbf{u}'_2 = \mathbf{x}_2 \mathbf{x}_1 R_{21}^{-1} \mathbf{u}'_1 R_{21} \mathbf{u}'_1 R' = \mathbf{x}_2 \mathbf{u}'_2 \mathbf{x}_1 \mathbf{u}'_1 R'$ as required. Here we used the braid statistics from (40) between \mathbf{x} and \mathbf{u}' for the first and last equalities, and the relations for $V \otimes (R', R)$ and $B(R)$ in the form $(PR) R^{-1} \mathbf{u}'_1 R \mathbf{u}'_2 = R^{-1} \mathbf{u}'_1 R \mathbf{u}'_2 (PR)$ for the middle equalities. The $B(R)$ relations imply therefore that $R' R^{-1} \mathbf{u}'_1 R \mathbf{u}'_2 = R_{21}^{-1} \mathbf{u}'_2 R_{21} \mathbf{u}'_1 R'$, which could be verified directly if PR' is not given explicitly as a function of PR .

Example 4.1 *The braided matrices $BM_q(2)$ coact from the right on the braided plane $\mathbb{C}_q^{2|0}$ in the sense that*

$$(x'' \ y'') = (x \ y) \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obeys the quantum plane relations if x, y do and a', b', c', d' are a copy of the braided-matrices, and provided we remember the braid-statistics

$$\begin{aligned}
\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} x &= \begin{pmatrix} xa' + (1 - q^2)yc' & q^{-1}xb' + (q - q^{-1})y(a' - d') \\ qxc' & xd' + (1 - q^{-2})yc' \end{pmatrix} \\
\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} y &= y \begin{pmatrix} a' & qb' \\ q^{-1}c' & d' \end{pmatrix}
\end{aligned}$$

Proof We use (41) with the standard R-matrix, computing the relevant braid statistics from (40) and deducing the coaction of $BM_q(2)$ in Example 3.10 on the usual quantum plane in Example 3.1. One can easily confirm that the transformed x'', y'' obey the same relations $y''x'' = qx''y''$. \square

Example 4.2 The braided matrices $BM_q(1|1)$ coact from the right on the braided plane $\mathbb{C}_q^{1|1}$ in the sense that

$$(x'' \ \theta'') = (x \ \theta) \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obeys the $\mathbb{C}_q^{1|1}$ relations if x, θ do and a', b', c', d' are a copy of $BM_q(1|1)$, and provided we remember the braid-statistics

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} x = \begin{pmatrix} xa' + (1 - q^2)yc' & q^{-1}xb' + (q - q^{-1})y(a' - d') \\ qxc' & xd' + (1 - q^2)yc' \end{pmatrix}$$

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \theta = \theta \begin{pmatrix} a' & -q^{-1}b' \\ -qc' & d' \end{pmatrix}$$

Proof We use (41) and (40) with the non-standard R-matrix in Example 3.2, and deducing the coaction of $BM_q(1|1)$ in Example 3.11 on it. One can easily confirm that the transformed x'', θ'' obey the same relations $\theta''x'' = qx''\theta''$ and $\theta''^2 = 0$. \square

Similarly for $BM_q(2)$ acting on the quantum superplane $\mathbb{C}_q^{0|2}$ in Example 3.4. It has the same R-matrix in a different normalisation to that in Example 3.1, but this does not affect the braid statistics which therefore comes out analogous to Example 4.1. Note that the reader has seen quantum matrices characterised as transformations of the two types of quantum plane in Manin's lectures[24]. In our case we have non-trivial braid statistics and because of this we obtain the braided matrices $BM_q(2)$ instead. This is the general principle behind the process of transmutation as explained in the Appendix. Note also that it is only in the Hecke case that two quantum planes such as $\mathbb{C}_q^{2|0}$ and $\mathbb{C}_q^{0|2}$ are enough for this. More generally one needs to consider all quantum planes for each choice of eigenvalue of PR in (7). Braided covectors alone are enough to do the job of characterising the braided matrices as (braided) transformations in this way provided we consider the various possibilities for them.

In the same way, we have braided coactions of B on the braided vectors v^i and on the braided matrices u^i_j itself. Here B is a braided group with antipode which we must make from $B(R)$ in a way compatible with the braidings. This is possible for regular R in the sense explained in Section 6.1. The braided antipode S can be found either by hand or by the systematic transmutation technique in the Appendix. Assuming it, the relevant transformations are[27]

$$v^i \mapsto (1 \otimes Su^i_a)(v^a \otimes 1), \quad u^i_j \mapsto (1 \otimes Su^i_a)(u^a_b \otimes u^b_j) \quad \text{i.e.,} \quad \mathbf{v} \rightarrow \mathbf{u}'^{-1}\mathbf{v}, \quad \mathbf{u} \rightarrow \mathbf{u}'^{-1}\mathbf{u}\mathbf{u}' \quad (42)$$

where the expressions are written in the braided tensor product algebra $V \underline{\otimes} B$ or $B(R) \underline{\otimes} B$ respectively. In the compact notation \mathbf{u}' denotes the generator of the B factor. The second of the transformations here is the *braided adjoint action* and can be used as a foundation for a theory of braided-Lie algebras[9].

Next we show that the tensor product of a quantum covector with a quantum vector, when treated with the correct braid statistics, is a braided matrix, i.e. the map $B(R) \rightarrow V(R', R) \underline{\otimes} V^{\vee}(R', R)$ given by[27]

$$u^i_j \mapsto v^i \otimes x_j, \quad \text{i.e.,} \quad \mathbf{u} \mapsto \mathbf{v}\mathbf{x}' = \begin{pmatrix} v^1x'_1 & \cdots & v^1x'_n \\ \vdots & & \vdots \\ v^nx'_1 & \cdots & v^nx'_n \end{pmatrix} \quad (43)$$

is an algebra homomorphism provided we use the braid statistics from (40). Again, this result from [27] is most easily checked under the mild assumption that PR used for the braided covectors and vectors in Section 3 is given as some function of PR' . Then we have $R_{21}\mathbf{v}_1\mathbf{x}'_1R\mathbf{v}_2\mathbf{x}'_2 = R_{21}\mathbf{v}_1\mathbf{v}_2\mathbf{x}'_1\mathbf{x}'_2 = f(PR')P\mathbf{v}_1\mathbf{v}_2\mathbf{x}'_1\mathbf{x}'_2 = \mathbf{v}_2\mathbf{v}_1f(1)\mathbf{x}'_1\mathbf{x}'_2 = \mathbf{v}_2\mathbf{v}_1\mathbf{x}'_1\mathbf{x}'_2f(PR') = \mathbf{v}_2\mathbf{v}_1\mathbf{x}'_2\mathbf{x}'_1R = \mathbf{v}_2\mathbf{x}'_2R_{21}\mathbf{v}_1\mathbf{x}'_1R$. The first and last equalities use the braid statistics relations. The middle equalities use the defining relations in the algebras $V(R', R), V^\sim(R', R)$. Hence $\mathbf{v}\mathbf{x}'$ is a realisation of the braided matrices $B(R)$. If PR is not given explicitly as some function of PR' then the proposition typically still holds but has to be verified directly according to the form in which PR', PR are given. An example is[27]

$$BM_q(2) \rightarrow \mathbb{C}_{q^{-1}}^{2|0} \otimes \mathbb{C}_q^{2|0} \quad (44)$$

which relates the braided-matrix point of view which we will later adopt as a definition of q -Minkowski space (see Example 3.10) with an independent approach pioneered in [34][29][35].

It is also easy to see that the ‘inner product’ element

$$x'_i v^i = v^b x'_a \vartheta^a_b = \text{Tr } \mathbf{v}\mathbf{x}'\vartheta; \quad \vartheta^i_j = \tilde{R}^i_a a^a_j \quad (45)$$

in the braided tensor product algebra $V(R', R) \otimes V^\sim(R', R)$ is central and bosonic with respect to the multiplicative braid statistics[27]. This applies also to $c_1 = \text{Tr } \mathbf{u}\mathbf{u}$ as one would expect in view of (43). The trace elements in Examples 3.10 and 3.11 are of just this form. More generally one has that all the powers $c_k = \text{Tr } \mathbf{u}^k \vartheta$ are bosonic and central in $B(R)$ [27]. This means of course that

$$c_k u^i_j = u^i_j c_k, \quad u^i_j c_k = c_k u^i_j, \quad c'_k u^i_j = u^i_j c'_k \quad (46)$$

in the algebras $B(R)$ and $B(R) \otimes B(R)$ respectively. The first equation is similar to the construction of Casimirs of $U_q(g)$ in [22] in another context, while the latter two equations about the bosonic nature are new features of the braided theory in [27].

One can show (again as an application of transmutation) that the bosonic central elements in the braided matrices $B(R)$ generate a subalgebra (are closed under addition and multiplication). In this subalgebra one can expect to find interesting bosonic central elements such as the braided-determinant $\underline{\det}(\mathbf{u})$. This should be group-like in the sense

$$\Delta \underline{\det}(\mathbf{u}) = \underline{\det}(\mathbf{u}) \otimes \underline{\det}(\mathbf{u}), \quad \epsilon \underline{\det}(\mathbf{u}) = 1 \quad (47)$$

with respect to the braided comultiplication. For example, in the 2-dimensional case it means

$$\underline{\det} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = \underline{\det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underline{\det} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad (48)$$

where we use the multiplicative braid statistics as in (40). One can also expect

$$\underline{\det}(\mathbf{v}\mathbf{x}') = 0 \quad (49)$$

from our above picture of $\mathbf{v}\mathbf{x}'$ as a rank-1 braided matrix. There is also a general formula for $\underline{\det}$ using the R-matrix formula for the epsilon tensor in Section 5.7. All of this suggests a fairly complete picture of our braided vectors, covectors and matrices in terms of braided linear algebra[27].

4.2 Gluing or direct sum of braided vectors

Next we consider some finer points of linear algebra in our braided framework. The first concerns how to tensor product braided groups. The braided tensor product algebra and braided tensor product coalgebra (defined in an obvious way again using the braiding) do not in general fit together to form a braided group: one must also ‘glue together’ the braidings of the two braided groups.

Such a gluing construction has been found by the author and M. Markl in [36] for the Hecke case. Firstly, if R, S are two solutions of the QYBE obeying the Hecke condition (17) then so is [36]

$$R \oplus_q S = \begin{pmatrix} R & 0 & 0 & \\ 0 & 1 & (q - q^{-1})P & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & S \end{pmatrix}. \quad (50)$$

Here P is the permutation matrix. The dimensions of R, S need not be the same. Such a phenomenon is encountered from time to time in the R-matrix literature, see [37] where Hecke R-matrices were extensively studied, although not as far as I know as a systematic gluing operation.

Many consequences of this associative gluing operation \oplus_q are then explored in [36]. Among them, it is shown that

$$V_{\sim \lambda}(R \oplus_q S) = V_{\sim \lambda}(R) \underline{\otimes}_{\lambda} V_{\sim \lambda}(S); \quad \lambda = q \text{ or } -q^{-1}. \quad (51)$$

Recall that in the Hecke case there are two natural choices for our braided covector data, which we label by λ : we keep R, S fixed in the Hecke normalisation and take $(R\lambda, \lambda^{-1}R)$ for (R, R') in Section 3.1. Likewise $(S\lambda, \lambda^{-1}S)$ for $V_{\sim \lambda}(S)$. The $\underline{\otimes}_{\lambda}$ is the braided tensor product with respect to a braiding given by the power of the braided covector generators (the scaling dimension) much as in the braided-line example in Section 2. Explicitly, the isomorphism (51) is given by writing the generators of $V_{\sim \lambda}(R \oplus_q S)$ as (x_i, y_I) say for appropriate ranges of indices according to the dimensions of R, S . Then its relations are

$$\lambda \mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R, \quad \lambda \mathbf{y}_1 \mathbf{y}_2 = \mathbf{y}_2 \mathbf{y}_1 S, \quad y_I x_j = \lambda x_j y_I \quad i.e., \quad \mathbf{y}_1 \mathbf{x}_2 = \lambda \mathbf{x}_2 \mathbf{y}_1 \quad (52)$$

which is the right hand side of (51).

Note that our algebras are like those generated by the co-ordinate functions on row vectors. Their tensor product is therefore like the direct sum of the underlying braided vector spaces. For example,

$$\mathbb{C}_q^{n|0} \underline{\otimes} \mathbb{C}_q^{m|0} = \mathbb{C}_q^{n+m|0}, \quad \mathbb{C}_q^{0|m} \underline{\otimes}_{(-q^{-1})} \mathbb{C}_q^{0|n} = \mathbb{C}_q^{0|n+m}$$

as is clear from the well-known form of these spaces. The above gluing however, works quite generally for arbitrary R, S of Hecke type.

This gluing is an example of a more general construction of *rectangular quantum matrices* $A(R : S)$ introduced in [36]. These are defined with generators $1, x^i_I$ say and relations

$$R^i_a{}^k{}_b x^a{}_J x^b{}_L = x^k{}_B x^i{}_A S^A{}_J{}^B{}_L, \quad i.e., \quad R \mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 S. \quad (53)$$

and a braided addition law $\mathbf{x} \oplus \mathbf{x}'$ with braid statistics [36][23]

$$x'^i{}_J x^k{}_L = R^k{}_b{}^i{}_a x^b{}_D x'^a{}_C S^C{}_J{}^D{}_L, \quad i.e., \quad \mathbf{x}'_1 \mathbf{x}_2 = R_{21} \mathbf{x}_2 \mathbf{x}'_1 S. \quad (54)$$

They are not in general quantum groups but instead we have partial comultiplication maps

$$\Delta_{R,S,T} : A(R : T) \rightarrow A(R : S) \otimes A(S : T), \quad \Delta x^i_{\alpha} = x^i_I \otimes x^I_{\alpha}, \quad i.e., \quad \Delta \mathbf{x} = \mathbf{x} \otimes \mathbf{x} \quad (55)$$

for any three Hecke solutions R, S, T of the QYBE, corresponding to matrix multiplication of rectangular matrices. We associate the R-matrices R, S to the rows and columns of $A(R : S)$ and there is a comultiplication when the rows match the columns to be contracted with. There is also distributivity generalising (21) and expressing linearity of $\Delta_{R,S,T}$ with respect to the coaddition. Finally, we can regard the rectangular quantum matrices $A(R : S)$ as a braided covector space as in Section 3.1 with multi-index \mathbf{R}', \mathbf{R} now built from R, S , see [23]. The corresponding braided vector space is $A(S_{21} : R_{21})$.

It is clear that setting $R = (q)$ or $(-q^{-1})$ (the 1-dimensional Hecke R-matrices) recovers the braided covectors as $1 \times n$ rectangular quantum matrices $V_{\sim \lambda}(R) = A(\lambda : R)$. The other case

$A(R : \lambda)$ of $n \times 1$ rectangular quantum matrices recovers a left handed version $R\mathbf{v}_1\mathbf{v}_2 = \mathbf{v}_2\mathbf{v}_1\lambda$ of the braided vectors. We have given conventions in Section 3.1 in which everything is right-covariant under a background quantum group, while these left-handed vectors are left-covariant by contrast. The general $A(R : S)$ is bicovariant using the maps (55) for comultiplication from the left or right by $A(R : R) = A(R)$ and $A(S : S) = A(S)$. The diagonal case of the rectangular quantum matrices are of course the usual quantum matrices of [22]. Another example is $A(R_{21} : R) = \bar{A}(R)$ which is the variant from [25]. Finally, we have good behaviour of this construction under gluing as[36]

$$A(R : S \oplus_q T) = A(R : S)_{R\Omega} \underline{\otimes} A(R : T), \quad y^k_\alpha x^i_J = R^i_a{}^k_b x^a_J y^b_\alpha, \quad i.e., \quad \mathbf{y}_2 \mathbf{x}_1 = R \mathbf{x}_1 \mathbf{y}_2 \quad (56)$$

$$A(R \oplus_q S : T) = A(R : T)_{\Omega T} \underline{\otimes} A(S : T), \quad w^J_\delta v^i_\gamma = v^i_\alpha w^J_\beta T^\beta_\delta{}^\alpha_\gamma, \quad i.e., \quad \mathbf{w}_1 \mathbf{v}_2 = \mathbf{v}_2 \mathbf{w}_1 T \quad (57)$$

which generalises the above. Here $\mathbf{x} \in A(R : S)$, $\mathbf{y} \in A(R : T)$ which we view as $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ and $\mathbf{v} \in A(R : T)$, $\mathbf{w} \in A(S : T)$ which we view as $\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}$. The braided tensor products are defined with braidings given by R, T respectively, which we have shown as the braid statistics between the two factors in each case.

Example 4.3 We can horizontally glue two quantum plane column vectors to give a quantum matrix, or vertically glue two quantum plane row vectors to achieve the same:

$$\begin{aligned} M_q(2) &= \mathbb{C}_q^{2|0} \underline{\otimes}_{R_{gl_2}} \mathbb{C}_q^{2|0} & \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} a \\ c \end{pmatrix} \right\rangle \underline{\otimes} \left\langle \begin{pmatrix} b \\ d \end{pmatrix} \right\rangle \\ &= \mathbb{C}_q^{2|0} \underline{\otimes}_{R_{gl_2}} \mathbb{C}_q^{2|0} & i.e., & \left\langle \begin{pmatrix} a & b \end{pmatrix} \right\rangle \underline{\otimes} \left\langle \begin{pmatrix} c & d \end{pmatrix} \right\rangle \end{aligned}$$

where $R_{gl_2} = (q) \oplus_q (q)$ is the standard R-matrix as in Example 3.1 but in the Hecke normalisation. This is one way to derive its relations as quoted in Example 3.7.

Proof For the first case we use (56) with R the standard R-matrix as in Example 3.1 and 3.7. Then $A(R : q) = \mathbb{C}_q^{2|0}$ as a left-covariant column vector and $A(R : q)_{R\Omega} \underline{\otimes} A(R : q) = A(R : (q) \oplus_q (q))$. But $R = (q) \oplus_q (q)$ so the latter is just $A(R) = M_q(2)$. Likewise for the second example we have $A(q : R) = \mathbb{C}_q^{2|0}$ as a right-covariant row vector and $A(q : R)_{\Omega R} \underline{\otimes} A(q : R) = A((q) \oplus_q (q) : R) = A(R) = M_q(2)$ by (57). The explicit identification of the generators for the two cases is shown on the right, where $\langle \cdot \rangle$ denotes the algebra generated. \square

We could equally well glue n rows of the quantum planes $\mathbb{C}_q^{n|0}$ or n columns to arrive at the $n \times n$ quantum matrices $M_q(n)$. If we glue m rows or columns then we arrive at rectangular quantum matrices $A(R_{gl_m} : R_{gl_n})$ or $A(R_{gl_n} : R_{gl_m})$ etc. Such decomposition properties of the standard quantum matrices $M_q(n)$ or $SL_q(n)$ are to some extent known in other contexts, but recovered here systematically as part of a general framework. For a less standard example we could just as easily glue a row vector quantum plane $\mathbb{C}_q^{2|0}$ and a ‘fermionic’ row vector $\mathbb{C}_q^{0|2}$ say, i.e. glue Examples 3.1 and 3.4 vertically. Or we could glue horizontally two copies of the mixed quantum plane $\mathbb{C}_q^{1|1}$ in Example 3.2. Either way, the result by the gluing theorems (56)–(57) will be the ‘rectangular’ quantum matrices $A(R_{gl_{1|1}} : R_{gl_2})$ where $R_{gl_{1|1}} = (q) \oplus_q (-q^{-1})$ is the Alexander-Conway R-matrix as in Example 3.2, in the Hecke normalisation. Clearly we have some powerful gluing technology which we can use any number of ways. We content ourselves here with a concrete application of such ideas to quantum differential geometry.

Example 4.4 We can horizontally glue two copies of the (left-handed version of) the 1-dimensional quantum exterior algebra in Example 3.6 to obtain the left-handed quantum exterior algebra on the quantum plane as a rectangular quantum matrix

$$\Omega(\mathbb{C}_q^{2|0}) = \Omega(\mathbb{C}_q)_{R\Omega} \underline{\otimes} \Omega(\mathbb{C}_q), \quad i.e., \quad \left\langle \begin{pmatrix} x & y \\ dx & dy \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x \\ dx \end{pmatrix} \right\rangle \underline{\otimes} \left\langle \begin{pmatrix} y \\ dy \end{pmatrix} \right\rangle$$

where R_Ω is the R -matrix as in Examples 3.3 and 3.6, in the Hecke normalisation. The rectangular quantum matrix $\Omega(\mathbb{C}_q^{2|0}) = A(R_\Omega : R_{gl_2})$ transforms covariantly under a quantum group $M_q^\Omega(1|1) = A(R_\Omega)$ for each column and under $M_q(2)$ for each row. It has relations

$$\begin{aligned} yx &= qxy, & (dx)^2 &= 0, & (dy)^2 &= 0, & dydx &= -q^{-1}dxdy \\ dx x &= q^2x dx, & dx y &= qy dx, & dy y &= q^2y dy, & dy x &= (q^2 - 1)dx y + qxdy \end{aligned}$$

from (53) and a braided addition law from (54) whereby

$$\begin{pmatrix} x'' & y'' \\ dx'' & dy'' \end{pmatrix} = \begin{pmatrix} x & y \\ dx & dy \end{pmatrix} + \begin{pmatrix} x' & y' \\ dx' & dy' \end{pmatrix}$$

obeys the same relations of $\Omega(\mathbb{C}_q^{2|0})$ provided the primed copy has the braid statistics

$$\begin{aligned} x'x &= q^2xx', & x'y &= qyx', & y'x &= (q^2 - 1)yx' + qxy', & y'y &= q^2yy' \\ x'dx &= dx x', & x'dy &= q^{-1}dy x', & y'dx &= (1 - q^{-2})dy x' + q^{-1}dx y', & y'dy &= dy y' \\ dx'x &= (q^2 - 1)dx x' + q^2x dx', & dx'y &= (q - q^{-1})dy x' + qy dx' \\ dy'x &= (q - q^{-1})^2dy x' + (q - q^{-1})(dx y' + qy dx') + qxdy', & dy'y &= (q^2 - 1)dy y' + q^2y dy' \\ dx'dx &= -dxdx', & dx'dy &= -q^{-1}dy dx' \\ dy'dx &= (q^{-2} - 1)dy dx' - q^{-1}dxdy', & dy'dy &= -dy dy' \end{aligned}$$

Proof We take $\Omega(\mathbb{C}_q) = A(R_\Omega : q)$ which is a left-covariant vector version of Example 3.6 (it has the opposite algebra). Then $\Omega(\mathbb{C}_q)_{R_\Omega} \otimes \Omega(\mathbb{C}_q) = A(R_\Omega : q)_{R_\Omega} \otimes A(R_\Omega : q) = A(R_\Omega : R_{gl_2})$ as an example of (56). We compute its relations from (53) and obtain the algebra generated by x, y, dx, dy previously proposed as a natural covariant differential calculus for the quantum plane in [38][39] and for which a coaddition was recently proposed in [40]. By obtaining it as a rectangular quantum matrix, we know from the general theory above (without any work) not only that it is covariant under the usual $M_q(2)$ in Example 3.7 whereby

$$\begin{pmatrix} x & y \\ dx & dy \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ bosonic}$$

obeys the $\Omega(\mathbb{C}_q^{2|0})$ relations, which is the usual picture (cf. Manin's lectures), but also that it is covariant under a different quantum group $M_q^\Omega(1|1)$ with relations

$$\begin{aligned} ba &= ab, & ca &= acq^2, & db &= -bd, & dc &= -q^{-2}cd \\ cb &= bcq^2, & ad - da &= bc(1 - q^2), & b^2 &= 0, & c^2 &= 0 \end{aligned}$$

whereby

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ dx & dy \end{pmatrix}; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ bosonic}$$

obeys the same relations of $\Omega(\mathbb{C}_q^{2|0})$ as well. The quantum matrix $A(R_\Omega : R_{gl_2})$ is 'rectangular' in that its rows and columns transform under different quantum groups (i.e., have a different flavour) even though they have the same dimension. Moreover, we also know from [36] that every rectangular quantum matrix in our setting has a braided addition law. We compute the required braid-statistics at once from (54). They are the same as Examples 3.1 and 3.4 for the two rows

and (the opposite version of) Example 3.6 for each column, but include cross statistics such as $dy^i x$ also. \square

The same remarks apply in n -dimensions by iterated gluing, or indeed for the quantum plane associated to any Hecke R -matrix. We have taken so far the view of horizontal gluing (of the exterior algebras for each dimension). We can also take the vertical gluing point of view whereby we glue the position co-ordinates and the basic forms as

$$\begin{aligned} \Omega_q(R) &= V_{-q}^{\sim}(R) \otimes_{qR} V_{-q^{-1}}^{\sim}(R) = A(R_{\Omega} : R), \quad i.e., \quad \left\langle \begin{pmatrix} x_1 \cdots x_n \\ dx_1 \cdots dx_n \end{pmatrix} \right\rangle = \langle x_i \rangle \otimes_{qR} \langle dx_i \rangle \\ q\mathbf{x}_1 \mathbf{x}_2 &= \mathbf{x}_2 \mathbf{x}_1 R, \quad d\mathbf{x}_1 d\mathbf{x}_2 = -q d\mathbf{x}_2 d\mathbf{x}_1 R, \quad d\mathbf{x}_1 \mathbf{x}_2 = q \mathbf{x}_2 d\mathbf{x}_1 R \end{aligned} \quad (58)$$

which works for general Hecke R . Note that in [36] we also gave generalisations of the gluing procedure (50). One of these generalisations allows operators in the inner diagonal of (50) and it is this slightly generalised version which we use to obtain $R_{\Omega} = (q) \oplus_q (-q^{-1})$. The corresponding version of (57) involves qR in the braided tensor product. This extra factor of q is needed to ensure that our notation is consistent with $d^2 = 0$ and a usual graded Leibniz rule. We arrive at a version of the quantum exterior algebra $\Omega_q(R)$ as in [39] but obtained now as a rectangular quantum matrix. As such, its $A(R)$ covariance from the right is automatic, while at the same time we see automatically covariance from the left under $A_q^{\Omega}(1|1)$. The quantum group $M_q^{\Omega}(1|1)$ clearly has some super-like qualities and, indeed, there is a theory of superisation which converts it strictly into a super-quantum group along the lines developed by the author and M.J. Rodriguez-Plaza in [41]. It is then a q -deformation of a hidden supersymmetry in the exterior algebra of \mathbb{R}^n . Also automatic is that $\Omega_q(R)$ has a braided addition law from (54). We return to such exterior algebras from a more general and more constructive point of view in Section 5.7. The recent papers [40][42] on exterior algebras in the braided setting likewise go beyond the Hecke case covered by the gluing theory in [36]. In a somewhat different direction but also related to gluing, see [43]. One can envision many other applications also of the gluing theory, such as formulating quantum path spaces and function spaces as infinitely iterated gluings, see [23].

Finally, we can consider the quantum groups $A(R \oplus_q S)$. From the above they comes out as generated by four block matrices and block relations [36]

$$\begin{aligned} \mathbf{t} &= \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad \mathbf{a} \in A(R), \quad \mathbf{b} \in A(R : S), \quad \mathbf{c} \in A(S : R), \quad \mathbf{d} \in A(S) \\ R\mathbf{a}_1 \mathbf{b}_2 &= \mathbf{b}_2 \mathbf{a}_1, \quad \mathbf{c}_1 \mathbf{a}_2 = \mathbf{a}_2 \mathbf{c}_1 R, \quad S\mathbf{c}_1 \mathbf{d}_2 = \mathbf{d}_2 \mathbf{c}_1, \quad \mathbf{d}_1 \mathbf{b}_2 = \mathbf{b}_2 \mathbf{d}_1 S \\ \mathbf{b}_1 \mathbf{c}_2 &= \mathbf{c}_2 \mathbf{b}_1, \quad \mathbf{a}_1 \mathbf{d}_2 - \mathbf{d}_2 \mathbf{a}_1 = (q^{-1} - q)P\mathbf{c}_1 \mathbf{b}_2. \end{aligned} \quad (59)$$

This is a ‘blocked form’ of the quantum matrices $M_q(2)$ in Example 3.7, where the generators are themselves now rectangular quantum matrices as shown.

We see then that these general braided tensor product constructions provide analogues of the usual constructions whereby matrices can be multiplied and blocked into smaller ones. There is a further theory for gluing or decomposition of braided matrices which remains to be studied, connecting with the results (43)–(44) in Section 4.1. Also, it would be nice to go beyond the Hecke case as well as to use other possible ‘templates’ in (50) such as the 8-vertex R -matrix in place of the standard 6-vertex one used in [36].

4.3 Braided metric

Next we consider the situation that our braided vector and covector algebras are isomorphic. Recall that this is the true meaning of the metric in differential geometry. So to complete our picture we follow [28] and define a *braided metric* as a matrix η_{ij} such that

$$x_i = \eta_{ia} v^a, \quad v^i = x_a \eta^{ai} \quad (60)$$

is an isomorphism of braided-Hopf algebras. Here η^{ij} is the inverse transpose characterised by $\eta_{ja}\eta^{ia} = \delta_j^i = \eta_{aj}\eta^{ai}$.

There are two aspects to this definition, one for the algebra isomorphism and one for an equivalence of the braiding so that the braided tensor product algebras are also isomorphic. These immediately come out respectively as[28]

$$\eta_{ia}\eta_{jb}R'^a{}_k{}^b{}_l = R'^a{}_i{}^b{}_j\eta_{ak}\eta_{bl}, \quad \eta_{ia}\eta_{jb}R^a{}_k{}^b{}_l = R^a{}_i{}^b{}_j\eta_{ak}\eta_{bl} \quad (61)$$

We will see later in Section 6.1 that the above constructions are generally covariant under a background quantum group. It is natural to demand that η preserves this (is an intertwiner for the coaction). This implies the second of (61) and other relations too between η and R which we could take along with (61) as axioms for a covariant metric. These include[44]

$$\eta_{ka}R^i{}_j{}^a{}_l = \lambda^{-2}R^{-1i}{}_j{}^a{}_k\eta_{al}, \quad \eta_{ka}\tilde{R}^i{}_j{}^a{}_l = \lambda^2R^i{}_j{}^a{}_k\eta_{al} \quad (62)$$

$$R^a{}_j{}^k{}_l\eta_{ai} = \lambda^{-2}\eta_{ja}R^{-1a}{}_i{}^k{}_l, \quad \tilde{R}^a{}_j{}^k{}_l\eta_{ai} = \lambda^2\eta_{ja}R^a{}_i{}^k{}_l \quad (63)$$

obtained by the methods in Section 6.1. The parameter λ which shows up here is called the *quantum group normalisation constant*[27] and depends on the R-matrix. Finally, we can also require our metric to be symmetric in some sense. The natural condition is to use the same notion as the sense in which the braided covector and vector algebras are commutative, i.e.,

$$\eta_{ba}R'^a{}_i{}^b{}_j = \eta_{ij}. \quad (64)$$

The corresponding equations in terms of the inverse transposed metric η^{ij} are

$$\eta^{ia}\eta^{jb}R'^k{}_a{}^l{}_b = R'^i{}_a{}^j{}_b\eta^{ak}\eta^{bl}, \quad \eta^{ia}\eta^{jb}R^k{}_a{}^l{}_b = R'^i{}_a{}^j{}_b\eta^{ak}\eta^{bl} \quad (65)$$

$$\eta^{la}R^i{}_j{}^k{}_a = \lambda^{-2}R^{-1i}{}_j{}^l{}_a\eta^{al}, \quad \eta^{la}\tilde{R}^i{}_j{}^k{}_a = \lambda^2R^i{}_j{}^l{}_a\eta^{ak} \quad (66)$$

$$R^i{}_a{}^k{}_l\eta^{aj} = \lambda^{-2}\eta^{ia}R^{-1j}{}_a{}^k{}_l, \quad \tilde{R}^i{}_a{}^k{}_l\eta^{aj} = \lambda^2\eta^{ia}R^j{}_a{}^k{}_l \quad (67)$$

$$R'^i{}_a{}^j{}_b\eta^{ba} = \eta^{ij}. \quad (68)$$

We will see in Section 5.6 how to construct quantum metrics as an application of braided-differentiation. It can also be obtained from knowledge of the $*$ -structure. We describe this next.

4.4 Braided $*$ -structures

If we think of our algebras above as like the co-ordinate functions on a manifold or (equally well) as like a quantum system, we need to specify an operation $*$ from the algebra to itself which is like pointwise complex conjugation. It should be antilinear, square to 1 and be an antialgebra homomorphism, i.e., should make our algebras into $*$ -algebras. Such a structure is important in the non-commutative or quantum case because it determines what it means for a representation of our algebras to be ‘real’. Namely in the quantum case $*$ should map over to Hermitian conjugation, which requirement generalises the notion of a unitary representation of a group. Since we do not have either points or groups, we should specify $*$ axiomatically by these and further properties.

For a braided group the most useful further axioms (as determined by experience rather than by abstract considerations) appear to be [33]

$$(* \otimes *) \circ \Delta = \tau \circ \Delta \circ *, \quad \epsilon \circ * = \bar{} \circ \epsilon, \quad * \circ S = S \circ * \quad (69)$$

where τ denotes the usual transposition.

Whether or not our braided vectors, covectors and matrices etc have a natural $*$ -structure depends on the particular algebras. These in turn depend on the chosen R, R' matrices. We

discuss here what can be said at this general level depending on the general properties of these matrices: there may be other possibilities too for individual algebras when one looks at them by hand. There are two useful cases[33]

$$\overline{R^i_j k_l} = R^l_k j_i \quad (\text{real type I}), \quad \overline{R^i_j k_l} = R^{-1j_i l_k} \quad (\text{antireal type I}). \quad (70)$$

There are also type II cases needed to cover q -Minkowski space if we treat it as a braided covector algebra[45]. We use the same classification for R' .

It is natural to ask our metric, when it exists, to respect $*$. We require

$$\overline{\eta^{ij}} = \eta_{ji} \quad (71)$$

which is compatible with the various relations (65)–(68) when λ is real, for either the real or antireal type I case for R, R' . Then one can check that

$$x_i^* = x_a \eta^{ai} \quad (72)$$

makes $V(R', R)$ into a $*$ -braided group obeying the above axioms[45]. The braided vector algebra is isomorphic when there is a metric, so it has a corresponding operation $v^{i*} = \eta_{ia} v^a$ by this isomorphism. There is also a second $*$ -structure given on the v^i (say) by $v^{i*} = \eta_{ai} v^a$ which is needed for the duality theory in[45].

When there is no metric, it is more reasonable to think of our braided covectors and vectors as holomorphic and antiholomorphic. This is because rather than a $*$ -structure as above, there is a natural map from one to the other as $x_i^* = v^i$. The details and some non-trivial theorems in this direction are in [45]. This holomorphic situation is the one that applies to the most simple examples such as the quantum plane $\mathbb{C}_q^{2|0}$.

Next, we look at $*$ -structures for our other objects. We concentrate on R of real type I, which holds for our standard R_{gl_2} and similar R -matrices when q is real. In the case of the quantum matrices $A(R)$ it is known that these often have a ‘unitary’ $*$ -structure[46]

$$t^i_j{}^* = S t^j_i \quad (73)$$

at least when made into Hopf algebras with antipode S for the multiplicative coproduct. Usually the same formula can be used at the bialgebra level too. For example, the usual quantum matrices $M_q(2)$ in Example 3.7 have a standard $*$ -structure

$$\begin{pmatrix} a & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix} \quad (74)$$

when q is real. This is the $*$ -structure familiar for $SU_q(2)$ except that the same formula works even when we do not demand the q -determinant relation. Of course, we do not obey the antipode part of Woronowicz’ axioms for a Hopf $*$ -algebra[46], but only the properties for a $*$ -bialgebra with respect to comultiplication.

At least in the Hecke case we also have the possibility of a second approach to the $*$ -structure on $A(R)$, namely to write it as a braided covector space as in Section 3.2. Then the bigger multi-index matrices \mathbf{R}, \mathbf{R}' in (22) will also be real type I, and hence when there is a quantum metric $\eta^{I'J}$ we will have a $*$ -structure by the above. This second approach ensures that we obey the axioms (69) for the additive braided group structure.

Example 4.5 *The usual quantum matrices $M_q(2)$ in Example 3.7 have a braided covector $*$ -structure*

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} -qd & c \\ b & -q^{-1}a \end{pmatrix}$$

when q is real.

Proof We regard $M_q(2)$ as a 4-dimensional braided covector space and use the quantum metric

$$\eta^{IJ} = \begin{pmatrix} 0 & 0 & 0 & -q^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -q & 0 & 0 & 0 \end{pmatrix}.$$

One can check that it obeys the conditions in Section 4.3 with respect to \mathbf{R}, \mathbf{R}' and $\lambda = q^{-1}$, as well as the condition (71). Then we get the braided covector $*$ -structure from (72). \square

The braided covector approach works just as well on the variants $\bar{A}(R)$. The multi-index matrices \mathbf{R}, \mathbf{R}' from (27) are again of real type I and a quantum metric obeying (71) then gives us a braided covector $*$ -structure. This includes our standard example of q -Euclidean space $\bar{M}_q(2)$ from Example 3.8. We describe it further in Section 7.1. Also, there is a systematic process of twisting which turns $A(R)$ into $\bar{A}(R)$. It turns the multiplicative $*$ -structure on the matrix generators \mathbf{t} of the former into the braided covector $*$ -structure on the generators \mathbf{x} of the latter. By this general theory one could begin with the usual (74) on $M_q(2)$, take the same form for the $*$ -structure on $\bar{M}_q(2)$ and hence by (72) deduce its quantum metric η^{IJ} . This is the line taken by the author in [25].

For $*$ structures on braided matrices $B(R)$ the discussion is simple because we do not need to consider an antipode. This topic was covered in [33] where it was shown that for R of real type I, we have a braided $*$ -bialgebra of Hermitian type

$$u^i_j{}^* = u^j_i \quad (75)$$

with respect to the braided comultiplication in the axiom (69). Firstly, it is easy to see that $B(R)$ for real-type R is a $*$ -algebra. See [33] or other places where the algebra relations have been studied. Next note that $B(R) \otimes B(R)$ has a natural $*$ -structure defined not in the obvious way but by [33]

$$(b \otimes c)^* = c^* \otimes b^*, \quad \forall b, c \in B(R) \quad i.e. \quad u^i_j{}^* = u^j_i, \quad u^{li}{}^* = u^j_i$$

on the generators. We just have to check that this is compatible with the relations of the braided tensor product. Thus

$$\begin{aligned} (R^{-1i}{}_a{}^k{}_b u^a{}_c R^c{}_j{}^d u^d{}_l)^* &= u^l{}_d R^d{}_b{}^j{}_c u^c{}_a R^{-1b}{}_k{}^a{}_i \\ &= R^l{}_a{}^j{}_b u^b{}_c R^c{}_i{}^a{}_d u^d{}_k = (u^k{}_d R^{-1i}{}_c{}^d{}_a u^c{}_b R^b{}_j{}^a{}_l)^* \end{aligned}$$

as required. We used the multiplicative braid statistics in Section 3.3. After this we just have to note that the multiplicative Δ is a $*$ -algebra map on the generators and hence extends to products as a $*$ -algebra map.

As before, a second way is to think of our braided matrices as a braided covector space, as explained in Section 3.3. We have \mathbf{R}, \mathbf{R}' from (36) obeying some different properties, called *type II* in [45] but again leading to a $*$ -braided group structure on our covectors, this time with (75). For a direct check, the computation is like the above but this time with Meyer's additive braid statistics:

$$\begin{aligned} (R^{-1i}{}_a{}^k{}_b u^a{}_c R^c{}_j{}^b{}_d u^d{}_l)^* &= u^l{}_d R^d{}_b{}^j{}_c u^c{}_a R^{-1b}{}_k{}^a{}_i \\ &= R^l{}_a{}^j{}_b u^b{}_c R^c{}_i{}^a{}_d u^d{}_k = (u^k{}_d R^d{}_a{}^i{}_c u^c{}_b R^b{}_j{}^a{}_l)^* \end{aligned}$$

The braided coaddition extends to products as a $*$ -algebra homomorphism to this braided tensor product algebra. It is remarkable that the same Hermitian form (75) works for both comultiplication and coaddition, making $B(R)$ a $*$ -braided group in two ways.

Note that under coaddition, the transposition τ in (69) might be unexpected – it is not visible classically because the coaddition is then cocommutative (symmetric in its output). But

the comultiplication on the other hand is not symmetric and here a τ really is to be expected for Hermitian matrices: classically the product of two Hermitian matrices M, N is not Hermitian but rather obeys $(MN)^* = NM$.

In addition, there is a strict relation of transmutation in [33] which obtains this Hermitian $*$ -structure on $B(R)$ from the multiplicative unitary-like one on $A(R)$. There is also a theory of twisting in [25] which turns the braided covector $*$ -structure of $\bar{A}(R)$ into $B(R)$. So all three algebras are connected by general theory. The Hermitian form obviously means that the standard 2×2 braided matrices $BM_q(2)$ in Example 3.10 should be regarded as a natural q -Minkowski space. We can deduce its quantum metric η^{IJ} from this $*$ -structure and (72). It is covered in Section 7.2. The twisting relation between $\bar{M}_q(2)$ and $BM_q(2)$ becomes ‘quantum Wick rotation’ in this context[25].

Once one has a suitable $*$ -structure on a braided space, it is natural to ask about its implications for the various other constructions in braided geometry, such as those to be described below. This is covered in [45], to which we refer the interested reader. Not all problems in this direction are yet solved.

5 Braided analysis

So far we have described how to q -deform \mathbb{R}^n as a braided covector or vector space and developed the associated concepts of linear algebra. We now proceed to the first steps of ‘braided analysis’ on such spaces. We concentrate first on understanding braided differentiation from [20]. This then determines braided exponentials, braided-Gaussians, braided-integration and braided-differential forms etc., i.e. some of the remaining basic concepts for ‘analysis’ on \mathbb{R}^n .

5.1 Braided differentiation

The next stage after braided addition and linear algebra is to make an infinitesimal addition, which leads to the concept of differentiation. We can think of the coaddition (9)–(10) on our covectors x_i equally well as a braided coaction of one copy of the covectors on another, i.e. as a global translation. If we denote the generators of the coacting copy by \mathbf{a} then the content of Section 3.1 is that $\mathbf{a} + \mathbf{x}$ also obeys the relations of a braided covector provided we remember the braid statistics $\mathbf{x}_1 \mathbf{a}_2 = \mathbf{a}_2 \mathbf{x}_1 R$. We are now ready to follow the ideas of I. Newton and define differentiation $\partial^i : V^*(R', R) \rightarrow V^*(R', R)$ as an infinitesimal translation[20]

$$\partial^i f(\mathbf{x}) = (a_i^{-1}(f(\mathbf{a} + \mathbf{x}) - f(\mathbf{x})))_{\mathbf{a}=0} \equiv \text{coeff of } a_i \text{ in } f(\mathbf{a} + \mathbf{x}). \quad (76)$$

We take the linear part in a_i , which is some function of \mathbf{x} and does not depend in fact on inverting a_i or taking a limit. For example, on monomials we have

$$\begin{aligned} & \text{coeff}_{a_i}((\mathbf{a}_1 + \mathbf{x}_1)(\mathbf{a}_2 + \mathbf{x}_2) \cdots (\mathbf{a}_m + \mathbf{x}_m)) \\ &= \text{coeff}_{a_i}(\mathbf{a}_1 \mathbf{x}_2 \cdots \mathbf{x}_m + \mathbf{x}_1 \mathbf{a}_2 \mathbf{x}_3 \cdots \mathbf{x}_m + \cdots + \mathbf{x}_1 \cdots \mathbf{x}_{m-1} \mathbf{a}_m) \\ &= \text{coeff}_{a_i}(\mathbf{a}_1 \mathbf{x}_2 \cdots \mathbf{x}_m (1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1,m})) \end{aligned}$$

giving us the result[20]

$$\partial^i(\mathbf{x}_1 \cdots \mathbf{x}_m) = \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_m [m; R]_{1 \dots m}, \quad \partial^i x_{i_1} \cdots x_{i_m} = \delta^i_{j_1} x_{j_2} \cdots x_{j_m} [m; R]_{i_1 \dots i_m}^{j_1 \dots j_m} \quad (77)$$

where \mathbf{e}^i is a basis covector $(\mathbf{e}^i)_j = \delta^i_j$ and[20]

$$[m; R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1,m} \quad (78)$$

is a certain *braided integer* matrix living in the m -fold matrix tensor product of M_n .

The operators ∂^i have upper indices so one should hope that they obey the relations of the braided vectors v^i in Section 3.1. This is indeed true and means that it defines an action of $V(R', R)$ on $V^{\sim}(R', R)$ given by this differentiation. We have[20]

$$\begin{aligned}\partial^i \partial^k \mathbf{x}_1 \cdots \mathbf{x}_m &= \mathbf{e}^k_1 \mathbf{e}^i_2 \mathbf{x}_3 \cdots \mathbf{x}_m [m-1; R]_{2\dots m} [m; R]_{1\dots m} \\ R'^i{}_a{}^k{}_b \partial^b \partial^a \mathbf{x}_1 \cdots \mathbf{x}_m &= R'^i{}_a{}^k{}_b \partial^b \mathbf{e}^a_1 \mathbf{x}_2 \cdots \mathbf{x}_m [m; R]_{1\dots m} \\ &= R'^i{}_a{}^k{}_b \mathbf{e}^a_1 \mathbf{e}^b_2 \mathbf{x}_3 \cdots \mathbf{x}_m [m-1; R]_{2\dots m} [m; R]_{1\dots m} \\ &= \mathbf{e}^k_1 \mathbf{e}^i_2 \mathbf{x}_3 \cdots \mathbf{x}_m (PR')_{12} [m-1; R]_{2\dots m} [m; R]_{1\dots m}\end{aligned}$$

which are equal due to the identity

$$[m-1; R]_{2\dots m} [m; R]_{1\dots m} = (PR')_{12} [m-1; R]_{2\dots m} [m; R]_{1\dots m} \quad (79)$$

proven in [20], to which we refer for further details.

Moreover, the braided covectors are braided-covariant under this action. The concept is much like the covariance in Section 4.1: we use braid statistics. In fact we need the inverse braiding Ψ^{-1} as given between vectors and covectors in (39)–(40). As usual, it extends to products in such a way as to be compatible with the product map. This consideration translates into the *braided-Leibniz rule*[20]

$$\begin{aligned}\partial^i(ab) &= (\partial^i a)b + \Psi^{-1}(\partial^i \otimes a)b \\ \Psi^{-1}(\partial^i \otimes \mathbf{x}_1 \cdots \mathbf{x}_r) &= \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_r \mathbf{x}_{r+1} (PR)_{12} \cdots (PR)_{r,r+1} \otimes \partial_{r+1}.\end{aligned} \quad (80)$$

There is an abstract or diagrammatic derivation for this based on duality[3]. For a direct proof one has[20]

$$\begin{aligned}(\partial^i \mathbf{x}_1 \cdots \mathbf{x}_r) \mathbf{x}_{r+1} \cdots \mathbf{x}_m + \Psi^{-1}(\partial^i \otimes \mathbf{x}_1 \cdots \mathbf{x}_r) \mathbf{x}_{r+1} \cdots \mathbf{x}_m \\ = \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_r [r; R]_{1\dots r} \mathbf{x}_{r+1} \cdots \mathbf{x}_m \\ + \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_r \mathbf{x}_{r'+1} (PR)_{12} \cdots (PR)_{r,r'+1} \partial_{r'+1} \mathbf{x}_{r+1} \cdots \mathbf{x}_m \\ = \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m [r; R]_{1\dots r} \\ + \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_r \mathbf{x}_{r+1} (PR)_{12} \cdots (PR)_{r,r+1} \mathbf{x}_{r+2} \cdots \mathbf{x}_m [m-r; R]_{r+1\dots m}\end{aligned}$$

where we use (77) to evaluate the differentials. The primed $r'+1$ labels a distinct matrix space from the existing $r+1$ index. These are then identified by the \mathbf{e}_{r+1} brought down by the action of $\partial_{r'+1}$. The resulting expression coincides with $\mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m [m; R]_{1\dots m}$ due to the braided-integer identity[20]

$$[r; R]_{1\dots r} + (PR)_{12} \cdots (PR)_{r,r+1} [m-r; R]_{r+1\dots m} = [m; R]_{1\dots m}. \quad (81)$$

Another way to express this braided-Leibniz rule is in terms of commutation relations between differentiation and position operators $\hat{x}_i : V^{\sim}(R', R) \rightarrow V^{\sim}(R', R)$ acting by multiplication from the left by x_i . The braided-Leibniz rule applied to $\partial^i(x_j f(\mathbf{x}))$ gives at once that

$$\partial^i \hat{x}_j - \hat{x}_a R^a{}_j{}^i{}_b \partial^b = \delta^i{}_j, \quad i.e. \quad \partial_1 \hat{x}_2 - \hat{x}_2 R_{21} \partial_1 = \text{id} \quad (82)$$

as operators on $V^{\sim}(R', R)$. This is obviously the point of view that could be called ‘braided quantum mechanics’. Indeed, we showed in [20] that there is an abstract *braided Weyl algebra* defined by

$$\mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R', \quad \mathbf{p}_1 \mathbf{p}_2 = R' \mathbf{p}_2 \mathbf{p}_1, \quad \mathbf{p}_1 \mathbf{x}_2 - \mathbf{x}_2 R_{21} \mathbf{p}_1 = \text{id} \quad (83)$$

with (82) as an operator realisation. The construction was as a braided cross product[47] and generalised the standard $GL_q(n)$ case that had been considered before[38][48].

Example 5.1 The braided-line \mathbb{C}_q at the end of Section 2 has braided differentiation

$$\partial x^m = x^{m-1}(1 + q + \cdots + q^{m-1}) = [m; q] x^{m-1} \quad \Rightarrow \quad \partial f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

The braided-Leibniz rule is

$$\partial(x^n x^m) = (\partial x^n)x^m + q^n x^n (\partial x^m), \quad \text{i.e. } \partial(fg) = (\partial f)g + (L_q f)\partial g$$

which is just as in the case of a superderivation, but with q in the role of -1 and a \mathbb{Z} -grading in the role of \mathbb{Z}_2 -grading. The degree of ∂ here is -1 and $L_q(f)(x) = f(qx)$.

Proof We use $R = (q)$ and $R' = (1)$ so that $V^-(R', R) = \mathbb{C}[x]$ is the braided-line. The braiding is $\Psi(\partial \otimes x) = q^{-1}x \otimes \partial$. We use the inverse braiding when computing ∂x^m giving the q -integer as shown. Or we just use (77) with $R = (q)$ and hence $[m; R] = [m, q]$. It is easy enough to verify the braided-Leibniz rule here explicitly. \square

This is where the familiar q -integers come from in braided geometry and is the reason that we called (78) braided integers. There is a q each time ∂^i passes an x due to the braid statistics between them. It is also the correct point of view on q -differentiation ∂ and reproduces easily well-known formulae in the 1-dimensional case. On the other hand, the braided formalism works on any higher-dimensional braided space just as well.

Example 5.2 The quantum plane $\mathbb{C}_q^{2|0}$ as a braided covector space in Example 3.1 has braided differentiation[20]

$$\frac{\partial}{\partial x} x^n y^m = [n; q^2] x^{n-1} y^m, \quad \frac{\partial}{\partial y} x^n y^m = q^n x^n [m; q^2] y^{m-1}$$

where $[m; q^2] = \frac{q^{2m}-1}{q^2-1}$. The derivatives obey the relations

$$\frac{\partial}{\partial y} \frac{\partial}{\partial x} = q^{-1} \frac{\partial}{\partial x} \frac{\partial}{\partial y}$$

Proof We know from the theory above that the relations of the ∂^i are necessarily the braided vector ones for $\mathbb{C}_{q^{-1}}^{2|0}$ in Example 3.5. The action of the generators is $\partial^i x_j = \delta^i_j$ which we then extend using the braided-Leibniz rule (82) with the standard R_{gl_2} . This gives the above results by an easy induction. Note that the q -derivatives act like in the 1-dimensional case in each variable except that $\frac{\partial}{\partial y}$ picks up a factor q when it passes x due to some braid statistics. This example recovers partial differentiation on the quantum plane as deduced by another approach (from the exterior algebra of forms) in [38][39]. \square

At another extreme, which works in any dimension, we can let R be any invertible solution of the QYBE and $R' = P$ the permutation matrix. Then $V^-(P, R)$ is the free algebra $\langle x_i \rangle$ with no relations. This could be called the *free braided plane*[20] associated to an R-matrix. It is in a certain sense universal, with the others as quotients. The vector algebra is the free algebra $\langle v^i \rangle$ with no relations and is realised by ∂^i acting as in (77). The R-matrix is still used, in the braiding.

5.2 Braided binomial theorem

Next we would like to understand much better the braided coaddition on our braided covectors from Section 3.1, namely how it looks on products of the generators. We need q -binomial coefficients as in the 1-dimensional case in Section 2, which we have to generalise now to our

higher-dimensional R-matrix setting. This was done (by the author) following the inductive way that binomial coefficients are usually defined. We require[20]

$$\begin{aligned} \left[\begin{matrix} m \\ r \end{matrix}; R \right]_{1\dots m} &= (PR)_{r,r+1} \cdots (PR)_{m-1,m} \left[\begin{matrix} m-1 \\ r-1 \end{matrix}; R \right]_{1\dots m-1} + \left[\begin{matrix} m-1 \\ r \end{matrix}; R \right]_{1\dots m-1} \\ \left[\begin{matrix} m \\ 0 \end{matrix}; R \right] &= 1, \quad \left[\begin{matrix} m \\ r \end{matrix}; R \right] = 0 \quad \text{if } r > m \end{aligned} \quad (84)$$

where the suffices refer as usual to the matrix position in tensor powers of M_n . This defines in particular

$$\begin{aligned} \left[\begin{matrix} m \\ m \end{matrix}; R \right] &= \left[\begin{matrix} m-1 \\ m-1 \end{matrix}; R \right] = \cdots = \left[\begin{matrix} 1 \\ 1 \end{matrix}; R \right] = 1 \\ \left[\begin{matrix} m \\ 1 \end{matrix}; R \right] &= (PR)_{12} \cdots (PR)_{m-1,m} + \left[\begin{matrix} m-1 \\ 1 \end{matrix}; R \right] = \cdots = [m; R]. \end{aligned} \quad (85)$$

A similar recursion defines $\left[\begin{matrix} m \\ r \end{matrix}; R \right]_{1\dots m}$ in terms of $\left[\begin{matrix} m \\ 1 \end{matrix}; R \right]$ (which is known) and $\left[\begin{matrix} m-1 \\ r \end{matrix}; R \right]$, and similarly (in succession) up to $r = m$.

The main result from a technical point of view is to compute these braided-binomial coefficients. This is the *braided binomial theorem*[20]

$$[r; R]_{1\dots r} \left[\begin{matrix} m \\ r \end{matrix}; R \right]_{1\dots m} = \left[\begin{matrix} m-1 \\ r-1 \end{matrix}; R \right]_{2\dots m} [m; R]_{1\dots m} \quad (86)$$

or formally

$$\begin{aligned} \left[\begin{matrix} m \\ r \end{matrix}; R \right]_{1\dots m} &= [r; R]_{1\dots r}^{-1} \cdots [2; R]_{r-1,r}^{-1} [m-r+1; R]_{r\dots m} \cdots [m; R]_{1\dots m} \\ &= [r; R]_{1\dots r}^{-1} [m-r; R]_{r+1\dots m}^{-1} [m; R]_{1\dots m} \end{aligned} \quad (87)$$

where

$$[m; R]! \equiv [2; R]_{m-1,m} [3; R]_{m-2,m} \cdots [m; R]_{1\dots m} \quad (88)$$

is the *braided factorial*[20]. The proof is by induction and a series of lemmas, see [20] for details. Note that one does not really need the braided-integers or factorials here to be invertible (just as for the usual binomial coefficients). For example, the recursion relation in the theorem implies that

$$\begin{aligned} \left[\begin{matrix} m \\ m-1 \end{matrix}; R \right]_{1\dots m} &= 1 + (PR)_{m-1,m} + (PR)_{m-1,m} (PR)_{m-2,m-1} + \cdots \\ &\cdots + (PR)_{m-1,m} \cdots (PR)_{12} = [m; R]_{21} \end{aligned} \quad (89)$$

One can prove numerous other identities of this type in analogy with usual combinatoric identities.

The braided binomial theorem demonstrates the beginning of some kind of braided-number-theory or braided-combinatorics. Because it holds for any invertible solution of the QYBE, it corresponds to a novel identity in the group algebra of the braid group. Physically, it corresponds to ‘counting’ the ‘partitions’ of a box of braid-statistical particles. Mathematically, it tells us the coaddition on products[20]

$$\begin{aligned} (a_{i_1} + x_{i_1}) \cdots (a_{i_m} + x_{i_m}) &= \sum_{r=0}^{r=m} a_{j_1} \cdots a_{j_r} x_{j_{r+1}} \cdots x_{j_m} \left[\begin{matrix} m \\ r \end{matrix}; R \right]_{i_1 \dots i_m}^{j_1 \dots j_m} \\ i.e., \quad (\mathbf{a}_1 + \mathbf{x}_1) \cdots (\mathbf{a}_m + \mathbf{x}_m) &= \sum_{r=0}^{r=m} \mathbf{a}_1 \cdots \mathbf{a}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m \left[\begin{matrix} m \\ r \end{matrix}; R \right]_{1\dots m} \end{aligned} \quad (90)$$

or in more formal terms

$$\begin{aligned} \Delta(x_{i_1} \cdots x_{i_m}) &= \sum_{r=0}^{r=m} x_{j_1} \cdots x_{j_r} \otimes x_{j_{r+1}} \cdots x_{j_m} \left[\begin{matrix} m \\ r \end{matrix}; R \right]_{i_1 \dots i_m}^{j_1 \dots j_m} \\ i.e., \quad \Delta(\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_m) &= \sum_{r=0}^{r=m} \mathbf{x}_1 \cdots \mathbf{x}_r \otimes \mathbf{x}_{r+1} \cdots \mathbf{x}_m \left[\begin{matrix} m \\ r \end{matrix}; R \right]_{1\dots m} \end{aligned} \quad (91)$$

The proof is once again by induction. Suppose it is true for $m - 1$, then

$$\begin{aligned}
& (\mathbf{a}_1 + \mathbf{x}_1) \cdots (\mathbf{a}_m + \mathbf{x}_m) \\
&= (\mathbf{a}_1 + \mathbf{x}_1) \cdots (\mathbf{a}_{m-1} + \mathbf{x}_{m-1}) \mathbf{a}_m + (\mathbf{a}_1 + \mathbf{x}_1) \cdots (\mathbf{a}_{m-1} + \mathbf{x}_{m-1}) \mathbf{x}_m \\
&= \sum_{r=0}^{m-1} \mathbf{a}_1 \cdots \mathbf{a}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_{m-1} \mathbf{a}_m \begin{bmatrix} m-1 \\ r \end{bmatrix}; R \Big|_{1 \cdots m-1} + (\mathbf{a}_1 + \mathbf{x}_1) \cdots (\mathbf{a}_{m-1} + \mathbf{x}_{m-1}) \mathbf{x}_m \\
&= \sum_{r=1}^m \mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{x}_r \cdots \mathbf{x}_{m-1} \mathbf{a}_m \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}; R \Big|_{1 \cdots m-1} \\
&\quad + \sum_{r=0}^{m-1} \mathbf{a}_1 \cdots \mathbf{a}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m \begin{bmatrix} m-1 \\ r \end{bmatrix}; R \Big|_{1 \cdots m-1} \\
&= \sum_{r=1}^m \mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{a}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m (PR)_{r,r+1} \cdots (PR)_{m-1,m} \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}; R \Big|_{1 \cdots m-1} \\
&\quad + \sum_{r=0}^{m-1} \mathbf{a}_1 \cdots \mathbf{a}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m \begin{bmatrix} m-1 \\ r \end{bmatrix}; R \Big|_{1 \cdots m-1} \\
&= \sum_{r=1}^m \mathbf{a}_1 \cdots \mathbf{a}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m \begin{bmatrix} m \\ r \end{bmatrix}; R \Big|_{1 \cdots m} + \mathbf{x}_1 \cdots \mathbf{x}_m \begin{bmatrix} m-1 \\ 0 \end{bmatrix}; R \Big|_{1 \cdots m-1}
\end{aligned}$$

using the induction hypothesis and (84). The last term is also the $r = 0$ term in the desired sum, proving the result for m .

5.3 Duality of braided vectors and covectors

Next we apply the braided differentiation operators and braided binomial theorem to establish a duality pairing between vectors and covectors. This is needed to round-off our concepts of linear algebra from Sections 3 and 4. We will need it also in the next section to define the braided-exponential. Two braided groups B, C say are said to be *in duality* if there is a map $\langle \cdot, \cdot \rangle : B \otimes C \rightarrow \mathbb{C}$ such that the product in one determines the coproduct or coaddition Δ in the other, etc., according to

$$\langle ab, c \rangle = \langle a, \langle b, c_{(1)} \rangle c_{(2)} \rangle, \quad \langle a, cd \rangle = \langle a_{(1)}, \langle a_{(2)}, c \rangle d \rangle, \quad \langle Sa, c \rangle = \langle a, Sc \rangle \quad (92)$$

where $\Delta c = c_{(1)} \otimes c_{(2)}$ is the braided coproduct and S the braided-antipode. We also require the pairing with 1 to be the counit. This (92) is not the usual pairing because we do not move b past $c_{(1)}$ to evaluate on $c_{(2)}$ etc, as one would usually do. It is possible to define such a more usual pairing by using the braiding Ψ to make the transposition but the result would be equivalent to (92) via the braided antipode, so we avoid such an unnecessary complication.

There is such a duality pairing of the braided vectors and covectors given by [20]

$$\langle v^{i_m} \cdots v^{i_2} v^{i_1}, x_{j_1} x_{j_2} \cdots x_{j_r} \rangle = \delta_{m,r} ([m; R]!)_{j_1 j_2 \cdots j_m}^{i_1 i_2 \cdots i_m} \quad (93)$$

$$i.e., \quad \langle \mathbf{v}_m \cdots \mathbf{v}_2 \mathbf{v}_1, \mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_r \rangle = \delta_{m,r} [m; R]!, \quad \langle f(\mathbf{v}), g(\mathbf{x}) \rangle = \epsilon \circ f(\partial)g(\mathbf{x})$$

This linear map is manifestly well-defined because the differentiation operators ∂^i are well-defined on products of the x_i by their very construction, so the relations are respected on this side of the pairing. The relations of the vector algebra on the other side of the pairing are also respected by the result in Section 5.1 that ∂^i indeed realise the vector algebra. We have still to check that the product on one side is the braided coproduct of the other. This is

$$\begin{aligned}
& \langle \mathbf{v}_m \cdots \mathbf{v}_{r+1} \cdot \mathbf{v}_r \cdots \mathbf{v}_1, \mathbf{x}_1 \cdots \mathbf{x}_m \rangle = [m; R]! \\
& \langle \mathbf{v}_m \cdots \mathbf{v}_{r+1} \otimes \mathbf{v}_r \cdots \mathbf{v}_1, \Delta \mathbf{x}_1 \cdots \mathbf{x}_m \rangle = [r, R]!_{1 \cdots r} [m-r, R]!_{r+1 \cdots m} \begin{bmatrix} m \\ r \end{bmatrix}; R
\end{aligned}$$

where we evaluate the inner V, V^\sim first and then the remaining outer two as required in (92). The coproduct is from (91) in Section 5.2 and our two expressions coincide just by the braided-binomial theorem (84). Similarly for the coproduct of products of the v^i by an analogous computation. The duality pairing with respect to the unit/counit and antipode are clear from the form on the generators. Note that one could also turn this around and recover

$$\partial^i = (\langle v^i, \cdot \rangle \otimes \text{id}) \circ \Delta \quad (94)$$

if we start from the knowledge that our vectors and covectors are dual.

The second half of this proof is clear enough by the evident symmetry between the braided vector and braided covector constructions. To give this in detail one has to redevelop the various steps above under this symmetry. Thus we have differentiation operators $\overleftarrow{\partial}_i = \frac{\partial}{\partial v^i}$ acting from the right and defined by

$$f(\mathbf{v}) \overleftarrow{\partial}_i = f(\mathbf{v} + \mathbf{w})|_{\text{coeff of } w^i}; \quad \mathbf{w}_1 \mathbf{v}_2 = R \mathbf{v}_2 \mathbf{w}_1 \quad (95)$$

$$\mathbf{v}_m \cdots \mathbf{v}_2 \mathbf{v}_1 \overleftarrow{\partial}_i = [m; R]^{\text{op}} \mathbf{v}_m \cdots \mathbf{v}_3 \mathbf{v}_2(\mathbf{f}_1)_1 \quad (96)$$

$$[m; R]^{\text{op}} = 1 + (PR)_{12} + (PR)_{23}(PR)_{12} + \cdots + (PR)_{m-1m} \cdots (PR)_{12}$$

where \mathbf{f}_1 is a basis vector $(f_i)^j = \delta_i^j$. The matrices here are like (78) but with the opposite product of matrices. We can equally well write the pairing (93) as

$$\langle \mathbf{v}_m \cdots \mathbf{v}_2 \mathbf{v}_1, \mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_r \rangle = \delta_{m,r} [m; R]^{\text{op}}!, \quad \langle f(\mathbf{v}) g(\mathbf{x}) \rangle = \epsilon \circ f(\mathbf{v}) g(\overleftarrow{\partial})$$

where

$$[m; R]^{\text{op}}! \equiv [m; R]_{1 \dots m}^{\text{op}} \cdots [3; R]_{m-2}^{\text{op}} [2; R]_{m-1}^{\text{op}} = [m; R]! \quad (97)$$

by matrix computations similar to those in Section 5.2. The symmetry between the ∂ and the $\overleftarrow{\partial}$ points of view expresses the symmetry (with left-right reversal) in the axioms of a pairing.

This pairing between braided vectors and covectors is typically nondegenerate. This is true for $q = 1$ where it becomes the usual pairing between the functions on \mathbb{R}^n and the enveloping algebra of \mathbb{R}^n , which is by usual differentiation. Non-degeneracy corresponds to the fact that the only functions which have zero differentials are constant. But this is also a feature of the braided-differentiation for standard R -matrices at generic q and other R -matrices near to the identity.

Finally, we mention briefly a more abstract or categorical way of thinking about our pairing $\langle \cdot, \cdot \rangle$. This is as an *evaluation map*

$$\text{ev} = \langle \cdot, \cdot \rangle = \vee \quad (98)$$

making V^\sim the categorical pre-dual of V . This is the line taken in [3], to which we refer the interested reader for further details. The point is that in the finite-dimensional case, as soon as we have a proper duality or evaluation map, we also have a *coevaluation* as the canonical element $e_a \otimes f^a \in V^\sim \otimes V$, where e_a is a basis and f^a a dual basis. We next give a direct definition of the braided-exponential which, from an abstract point of view, is nothing other than $\text{exp} = \text{coev} = \wedge$ but developed as a formal powerseries rather than an element of the algebraic tensor product. This is the abstract reason that exp exists on a general braided linear space as a formal powerseries.

5.4 Braided exponentials

The direct approach to the braided exponential is of course to seek eigenfunctions of the operators ∂^i . For the moment we seek these among formal powerseries in the x_i coordinates, but in our application to braided Taylor's theorem only finitely many terms will be nonvanishing. The only

difference is that we consider both ∂ and $\overleftarrow{\partial}$. Thus we define the *braided exponential* $\exp(\mathbf{x}|\mathbf{v})$ as a formal powerseries such that

$$\begin{aligned}\partial^i \exp(\mathbf{x}|\mathbf{v}) &= \exp(\mathbf{x}|\mathbf{v})v^i, & (\epsilon \otimes \text{id}) \exp(\mathbf{x}|\mathbf{v}) &= 1 \\ \exp(\mathbf{x}|\mathbf{v})\overleftarrow{\partial}_i &= x_i \exp(\mathbf{x}|\mathbf{v}), & (\text{id} \otimes \epsilon) \exp(\mathbf{x}|\mathbf{v}) &= 1.\end{aligned}\tag{99}$$

We usually also require it to be covariant under our background quantum group, just as the constructions above were all covariant. This definition corresponds to an infinitesimal version of the rigidity axioms for ev , coev in the diagrammatic language of [3]. It is also reasonable when we consider the ‘integrability condition’ for the solution of these braided-differential equations. Classically it means to ask for the constraint imposed by commutativity of partial derivatives. Here we have

$$\begin{aligned}(\partial_1 \partial_2 - R' \partial_2 \partial_1) \exp(\mathbf{x}|\mathbf{v}) &= \partial_1 \exp(\mathbf{x}|\mathbf{v})\mathbf{v}_2 - R' \partial_2 \exp(\mathbf{x}|\mathbf{v})\mathbf{v}_1 \\ &= \exp(\mathbf{x}|\mathbf{v})(\mathbf{v}_1 \mathbf{v}_2 - R' \mathbf{v}_2 \mathbf{v}_1) = 0\end{aligned}$$

since the v^i obey the vector algebra. Similarly from the other side in terms of $\overleftarrow{\partial}$. While not a proof, this tells us that, at least generically, an \exp should exist which is an eigenfunction with respect to each input.

Example 5.3 [20] *If the braided-integers $[m; R]$ are all invertible, then \exp is given by*

$$\exp(\mathbf{x}|\mathbf{v}) = \sum_{m=0}^{\infty} \mathbf{x}_1 \cdots \mathbf{x}_m [m; R]!^{-1} \mathbf{v}_m \cdots \mathbf{v}_1.$$

Proof We see this at once from (77). Differentiation from the left brings down $[m; R]$ which reduces $[m; R]!^{-1}$ to $[m-1; R]!^{-1}$. Similarly for $\exp(\mathbf{x}|\mathbf{v})\overleftarrow{\partial}$ where we bring down $[m; R]^{\text{op}}$ on the right which reduces $([m; R]^{\text{op}})^{-1}$. We use the identity (97). This example includes the braided-line \mathbb{C}_q and higher-dimensional free quantum planes where $R' = P$ and R is generic. \square

We have to work harder in the more common case where the $[m; R]$ are not all invertible, but the strategy is just the same. We write an ansatz

$$\begin{aligned}\exp(\mathbf{x}|\mathbf{v}) &= \sum_{m=0}^{\infty} \mathbf{x}_1 \cdots \mathbf{x}_m F(m; R) \mathbf{v}_m \cdots \mathbf{v}_1 \\ \mathbf{x}_1 \cdots \mathbf{x}_m F(m; R) [m; R]! &= \mathbf{x}_1 \cdots \mathbf{x}_m, & [m; R]! F(m; R) \mathbf{v}_m \cdots \mathbf{v}_1 &= \mathbf{v}_m \cdots \mathbf{v}_1\end{aligned}\tag{100}$$

and solve for F . To solve (99) in Example 5.3 we did not really need full invertibility of the $[m; R]!$ but only the weaker form contained in this ansatz (it is weaker because $\mathbf{x}_1 \cdots \mathbf{x}_2$ and $\mathbf{v}_m \cdots \mathbf{v}_1$ involve products of which some linear combinations could be zero). We have

$$\begin{aligned}\partial^i \exp(\mathbf{x}|\mathbf{v}) &= \sum_{m=0}^{\infty} \mathbf{e}_1^i \mathbf{x}_2 \cdots \mathbf{x}_m [m; R] F(m; R) \mathbf{v}_m \cdots \mathbf{v}_1 \\ &= \sum_{m=0}^{\infty} \mathbf{e}_1^i \mathbf{x}_2 \cdots \mathbf{x}_m F(m-1; R)_{2\dots m} ([m-1; R]!)_{2\dots m} [m; R] F(m; R) \mathbf{v}_m \cdots \mathbf{v}_1 \\ &= \sum_{m=0}^{\infty} \mathbf{e}_1^i \mathbf{x}_2 \cdots \mathbf{x}_m F(m-1; R)_{2\dots m} [m; R]! F(m; R) \mathbf{v}_m \cdots \mathbf{v}_1 \\ &= \sum_{m=0}^{\infty} \mathbf{e}_1^i \mathbf{x}_2 \cdots \mathbf{x}_m F(m-1; R)_{2\dots m} \mathbf{v}_m \cdots \mathbf{v}_1 = \exp(\mathbf{x}|\mathbf{v})v^i\end{aligned}$$

and a similar computation for $\overleftarrow{\partial}$ using (97). We can also impose further conditions (that $F(m; R)$ commutes with products $\mathbf{t}_1 \cdots \mathbf{t}_m$ of quantum group generators or R -matrix conditions to this effect) if we want to ensure covariance. This is how the diagrammatic definition $\exp = \text{coev}$ translates in matrix terms. We have developed this point of view in [44] where it leads to a useful and general Fourier theory.

Example 5.4 *If R is Hecke and $R' = q^{-2}R$ then*

$$\exp(\mathbf{x}|\mathbf{v}) = \sum_{m=0}^{\infty} \frac{\mathbf{x}_1 \cdots \mathbf{x}_m \mathbf{v}_m \cdots \mathbf{v}_1}{[m; q^2]!} = \sum_{m=0}^{\infty} \frac{(\mathbf{x} \cdot \mathbf{v})^m}{[m; q^{-2}]!} \equiv e_{q^{-2}}^{\mathbf{x} \cdot \mathbf{v}}$$

Proof Since $\mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_1 \mathbf{x}_2 P R'$ and $P R' \mathbf{v}_2 \mathbf{v}_1 = \mathbf{v}_2 \mathbf{v}_1$ we know without any calculation that $\mathbf{x}_1 \cdots \mathbf{x}_m [m; R]! = \mathbf{x}_1 \cdots \mathbf{x}_m [m; q^2]!$ and $[m; R]! \mathbf{v}_m \cdots \mathbf{v}_1 = [m; q^2]! \mathbf{v}_m \cdots \mathbf{v}_1$ making it obvious that $F(m; R) = [m; q^2]!^{-1}$ solves (100). The second form [49] follows at once from $x_i(\mathbf{x} \cdot \mathbf{v}) = q^2(\mathbf{x} \cdot \mathbf{v})x_i$, which is valid in the Hecke case as an easy consequence of the braid-statistics relations (40) between vectors and covectors. This example includes the quantum planes $\mathbb{C}_q^{2|0}$ and $\mathbb{C}_q^{1|1}$ in Examples 3.1 and 3.2 and their higher-dimensional analogues. There is clearly a similar result for the other ‘fermionic-type’ quantum planes where $R' = q^2 R$. \square

We assume then that we have these eigenfunctions $\exp(\mathbf{x}|\mathbf{v})$. Since the ∂^i themselves are a realisation of the vector algebra, we are now able to formulate a *braided-Taylor’s theorem* as c.f.[20]

$$\exp(\mathbf{a}|\partial)f(\mathbf{x}) = f(\mathbf{a} + \mathbf{x}) = \Delta f(\mathbf{x}) \quad (101)$$

where we use braided addition with $\mathbf{x}_1 \mathbf{a}_2 = \mathbf{a}_2 \mathbf{x}_1 R$ as usual. This follows at once from the braided-binomial theorem in Section 5.2 as

$$\begin{aligned} & \exp(\mathbf{a}|\partial)\mathbf{x}_1 \cdots \mathbf{x}_m \\ &= \sum_{r=0}^{r=m} \mathbf{a}_{1'} \cdots \mathbf{a}_{r'} F(r; R)_{1' \dots r'} \partial_{r'} \cdots \partial_{1'} \mathbf{x}_1 \cdots \mathbf{x}_m \\ &= \sum_{r=0}^{r=m} \mathbf{a}_1 \mathbf{a}_{2'} \cdots \mathbf{a}_{r'} F(r; R)_{1 \dots r'} \partial_{r'} \cdots \partial_1 \mathbf{x}_2 \cdots \mathbf{x}_m \\ &= \sum_{r=0}^{r=m} \mathbf{a}_1 \cdots \mathbf{a}_r F(r; R)_{1 \dots r} \mathbf{x}_{r+1} \cdots \mathbf{x}_m [m-r+1; R]_{r \dots m} \cdots [m; R]_{1 \dots m} \\ &= \sum_{r=0}^{r=m} \mathbf{a}_1 \cdots \mathbf{a}_r \mathbf{x}_{r+1} \cdots \mathbf{x}_m F(r; R)[r; R]! \begin{bmatrix} m \\ r \end{bmatrix} R = (\mathbf{a}_1 + \mathbf{x}_1) \cdots (\mathbf{a}_m + \mathbf{x}_m) = \Delta(\mathbf{x}_1 \cdots \mathbf{x}_m). \end{aligned}$$

Here the $1', 2'$ etc refer to copies of M_n distinct from the copies labelled by $1 \cdots m$, but they are successively identified by the \mathbf{e}^i (which are Kronecker delta-functions) brought down by the application of ∂^i . The $\mathbf{x}_{r+1} \cdots \mathbf{x}_m$ commute to the left and (86), (91) give the result. There is clearly a corresponding form of Taylor’s theorem for $\overleftarrow{\partial}$ recovering the coaddition of braided vectors.

Finally, we can apply the braided Taylor’s theorem to exp itself and deduce its usual bicharacter properties

$$\begin{aligned} \exp(\mathbf{a} + \mathbf{x}|\mathbf{v}) &= \exp(\mathbf{x}|\mathbf{v}) \exp(\mathbf{a}(\mid)\mathbf{v}) \\ \exp(\mathbf{x}|\mathbf{v} + \mathbf{w}) &= \exp(\mathbf{x}(\mid)\mathbf{w}) \exp(\mathbf{x}|\mathbf{v}) \end{aligned} \quad (102)$$

where (\mid) denotes a space for $\exp(\mathbf{x}|\mathbf{v})$ to be inserted in each term of the exponentials. The additions here are braided ones, i.e. we use the braid statistics $\mathbf{x}_1 \mathbf{a}_2 = \mathbf{a}_2 \mathbf{x}_1 R$ and $\mathbf{w}_1 \mathbf{v}_2 = R \mathbf{v}_2 \mathbf{w}_1$

from (40). In the usual covariant case \exp is bosonic and we do not need to write the $(\ | \)$ when we work in the appropriate braided tensor product algebra.

These formulae are rather important in physics, where they are key properties of addition of plane waves in position and momentum space. On the other hand in categorical terms they just say that the product in the vector algebra corresponds to the additive coproduct in the covector algebra and vice-versa via the exponential. I.e. they are just the statement that our braided-Hopf algebras are *copaired*. We expressed duality in Section 5.3 in terms of the evaluation map $\text{ev} = \langle \ , \ \rangle$. Now we see how it looks equivalently in terms of the coevaluation map $\text{coev} = \exp$. This is the line taken in [44].

5.5 Braided Gaussians

Next we turn to the Gaussian. We proceed in the same direct way by writing down a differential equation that characterises it as a formal powerseries. The simplest (but not the only) case for which this strategy works is when there is given a covariant metric in the sense explained in Section 4.3. We concentrate on this case for simplicity. Assuming such a metric, we define the corresponding Gaussian g_η to be a formal powerseries in x_i such that[44]

$$\partial^i g_\eta = -x_a \eta^{ai} g_\eta, \quad \epsilon(g_\eta) = 1. \quad (103)$$

As before, we check that this is a good definition by checking that this equation is integrable. We have

$$\begin{aligned} (\partial^i \partial^j - R^i{}_a{}^j{}_b \partial^b \partial^a) g_\eta &= -\partial^i (x_a \eta^{aj} g_\eta) + R^i{}_a{}^j{}_b \partial^b (x_c \eta^{ca} g_\eta) \\ &= (R^i{}_a{}^j{}_b \eta^{ba} - \eta^{ij}) g_\eta + R^i{}_a{}^j{}_b x_d R^d{}_c{}^e{}_f \eta^{ca} \partial^e g_\eta - x_c R^c{}_a{}^i{}_d \partial^d \eta^{aj} g_\eta \\ &= (x_c R^c{}_a{}^i{}_d x_e \eta^{ed} \eta^{aj} - R^i{}_a{}^j{}_b x_d R^d{}_c{}^e{}_f \eta^{fe} \eta^{ca}) g_\eta \\ &= (x_c R^c{}_a{}^i{}_d x_e \eta^{ed} \eta^{aj} - R^i{}_a{}^j{}_b x_d \lambda^{-2} R^{-1d}{}_c{}^f{}_e \eta^{eb} \eta^{ca} x_f) g_\eta \\ &= (x_c R^c{}_a{}^i{}_d x_e \eta^{ed} \eta^{aj} - \eta^{ai} \eta^{bj} R^i{}_c{}^e{}_b x_d x_f \lambda^{-2} R^{-1d}{}_c{}^f{}_e) g_\eta \\ &= (x_c R^c{}_a{}^i{}_d x_e \eta^{ed} \eta^{aj} - \eta^{ai} \eta^{bj} \lambda^{-2} R^{-1e}{}_b{}^c{}_a x_d x_f R^f{}_e{}^d{}_c) g_\eta \\ &= (x_c R^c{}_a{}^i{}_d x_e \eta^{ed} \eta^{aj} - \eta^{ai} \eta^{bj} \lambda^{-2} R^{-1e}{}_b{}^c{}_a x_e x_c) g_\eta = 0 \end{aligned}$$

where we use the Gaussian equation, braided-Leibniz rule (82), (68), the Gaussian equation again, (66), (65) and (6). After that we use the relations among the x_i and (66) again to obtain zero. While not a proof, this computation suggests that g_η exists at least as a formal powerseries.

The Gaussian can be found explicitly if we suppose that η obeys some additional conditions. For example[44]

$$\eta^{ka} R^l{}_j{}^i{}_a = R^{l-1}{}_j{}^k{}_a \eta^{ai} \quad (104)$$

ensures that $\mathbf{x} \cdot \mathbf{x} = x_a x_b \eta^{ba}$ is central in the braided tensor product algebra. This is just $x_i x_a x_b \eta^{ba} = x_d x_c x_b R^i{}_c{}^d{}_a \eta^{ba} = x_d x_c x_b \eta^{ad} R^{i-1}{}_c{}^b{}_a = x_d x_a \eta^{ad} x_i$ using our assumption and the relations in the braided covector algebra. On the other hand, the condition [44]

$$R^i{}_a{}^j{}_b \eta^{ba} = q^{-2} \eta^{ij} \quad (105)$$

ensures the operator identity $\partial^i \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = (1 + q^{-2}) \hat{x}_a \eta^{ai} + \lambda^{-2} \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} \partial^i$ as a consequence of the braided Leibniz rule (82) and the quantum metric identity (67). With these constraints on the metric, we deduce at once that

$$g_\eta = e_{\lambda^{-2}}^{-[2; q^{-2}]^{-1} \mathbf{x} \cdot \mathbf{x}} \quad (106)$$

solves our differential equation. There is a similar conclusion when there are factors in (104) and (105). Concrete examples include q -Euclidean space Example 3.8 and q -Minkowski space Example 3.10. The Euclidean case also makes contact with the treatment of Gaussians on $SO_q(N)$ -covariant quantum planes in [50][51].

5.6 Braided integration

Next we use the braided-Gaussian to define translation-invariant integration. This should be a map \int from suitable functions of \mathbf{x} to \mathbb{C} . One might think that integration over \mathbb{R}^n is one thing that cannot be done algebraically since polynomials are not integrable. However, we can reduce it to algebra if we only want integrals of the form $\int f(\mathbf{x})g_\eta$ where f is a polynomial in the braided co-vector coordinates. Indeed, such integrals are done by parts under the boundary assumption

$$\int \partial^i (f(\mathbf{x})g_\eta) = 0, \quad \forall f \quad (107)$$

which expresses translation-invariance on our class of functions assuming that the Gaussian ‘vanishes at infinity’. As is well-known in physics, the ratios of such integrals with $\int g_\eta$ are well-defined and algebraically-computable objects.

There is an R-matrix algorithm to do just this in [44]. It can also be used without a quantum metric, but we concentrate on the nicer case where there is one, obeying the conditions in Section 4.3. If g_η and \int exist obeying (103) and (107), then the *Gaussian weighted integral*[44]

$$\mathcal{Z}[\mathbf{x}_1 \cdots \mathbf{x}_m] = \left(\int \mathbf{x}_1 \cdots \mathbf{x}_m g_\eta \right) \left(\int g_\eta \right)^{-1} \quad (108)$$

is a well-defined linear functional on the braided covector algebra and can be computed inductively by[44]

$$\begin{aligned} \mathcal{Z}[1] &= 1, \quad \mathcal{Z}[x_i] = 0, \quad \mathcal{Z}[x_i x_j] = \eta_a R^a{}_{i j} \lambda^2 \\ \mathcal{Z}[\mathbf{x}_1 \cdots \mathbf{x}_m] &= \sum_{i=0}^{m-2} \mathcal{Z}[\mathbf{x}_1 \cdots \mathbf{x}_i \mathbf{x}_{i+3} \cdots \mathbf{x}_m] \mathcal{Z}[\mathbf{x}_{i+1} \mathbf{x}_{i+2}] (PR)_{i+2 i+3} \cdots (PR)_{m-1 m} \lambda^{2(m-2-i)}. \end{aligned} \quad (109)$$

Indeed, assuming \int, g_η exist we compute

$$\begin{aligned} \int x_{i_1} \cdots x_{i_m} g_\eta &= - \int x_{i_1} \cdots x_{i_{m-1}} \eta_{i_m c} \partial^c g_\eta \\ &= - \int x_{i_1} \cdots x_{i_{m-2}} \eta_{i_m a} \left(-\tilde{R}^b{}_{i_{m-1} a} g_\eta + \tilde{R}^c{}_{i_{m-1} a} \partial^b (x_c g_\eta) \right) \\ &= - \int x_{i_1} \cdots x_{i_{m-2}} \left(-\lambda^2 R^b{}_{i_{m-1} a} \eta_{ab} g_\eta + \lambda^2 R^c{}_{i_{m-1} a} \eta_{ab} \partial^b (x_c g_\eta) \right) \\ &= \left(\int x_{i_1} \cdots x_{i_{m-2}} g_\eta \right) \mathcal{Z}[x_{i_{m-1}} x_{i_m}] - \left(\int x_{i_1} \cdots x_{i_{m-2}} \eta_{ab} \partial^b (x_c g_\eta) \right) \lambda^2 R^c{}_{i_{m-1} a} \end{aligned}$$

where we used the braided-Leibniz rule (in reverse) for the second equality and (61) for the third. When $m = 2$ there is no second term here as it is a total derivative, and we obtain $\mathcal{Z}[x_i x_j]$ as stated. When $m > 2$ we recognise the form shown in terms of $\mathcal{Z}[x_{i_{m-1}} x_{i_m}]$ and an expression similar to our second expression but with a two-lower power of \mathbf{x} to the left of ∂ . We now repeat the above process, each time lowering the degree in the residual term by 2, until we reach $\int \partial = 0$. This gives the iterative formula for \mathcal{Z} . One can just take it as a definition even when the Gaussian and \int themselves are not known.

It is worth noting that the quantum metric needed in these constructions can be recovered if one knows the norm element $\mathbf{x} \cdot \mathbf{x}$ in the braided covector algebra, by partial differentiation. For example, in the setting (104)–(105) we have the identity

$$\eta^{ij} = \partial^i \partial^j (1 + q^{-2})^{-1} \mathbf{x} \cdot \mathbf{x}. \quad (110)$$

The same idea applies more generally to generate R' -symmetric tensors with more indices[52].

It is clear that we have the ingredients now to do most of the constructions of classical scalar field theory in our general braided setting. Using the integral and exp from Section 5.4 we can write down a *braided Fourier transform*[44]

$$\mathcal{F}(f)(\mathbf{v}) = \int f(\mathbf{x}) \exp(\mathbf{x}|\mathbf{v}) \quad (111)$$

and prove reasonable theorems such as a convolution theorem and the theorem that turns a differential operation in position space x_i into algebraic multiplication in momentum space v^i . See [44] for details. The inverse Fourier transform takes a similar form but integrating over momentum space. Our constructions above have been symmetric under position and momentum interchange modulo a left-right reversal. So in principle we can now write down such things as the braided-Green function

$$G(\mathbf{x}) = \mathcal{F}^{-1}((\mathbf{v} \cdot \mathbf{v} - m^2)^{-1}); \quad \mathbf{v} \cdot \mathbf{v} = v^a v^b \eta_{ba} \quad (112)$$

using the quantum metric. In practice we have to introduce a Gaussian regulator before we can apply \int in the form \mathcal{Z} constructed above to compute \mathcal{F} and \mathcal{F}^{-1} . We also have to expand the propagator here and any exponentials as powerseries in monomials. A closed formula for the Green function remains to be computed, but in principle it is now constructed by the above.

5.7 Braided electromagnetism

One of the nice things about braided geometry is that it handles both q -deformed bosonic constructions and q -deformed fermionic ones equally well. We have seen this right from the start in Examples 3.1–3.6. So we can apply our braided geometrical constructions above to the q -deformed Grassmann co-ordinates $\theta_i = dx_i$ just as well as to the q -deformed bosonic x_i on which we have concentrated so far. In order to do this, we now impose the further conditions

$$R_{12}R'_{13}R'_{23} = R'_{23}R'_{13}R_{12}, \quad R_{23}R'_{13}R'_{12} = R'_{12}R'_{13}R_{23} \quad (113)$$

$$R'_{12}R'_{13}R'_{23} = R_{23}R'_{13}R'_{12} \quad (114)$$

in addition to (3)–(6) in Section 3.1. This ensures that there is a symmetry[52]

$$R \leftrightarrow -R' \quad (115)$$

in this combined system of equations and means that we are free to reverse their roles. It means in particular that we can define the *braided group of forms* $\Lambda(R', R) \equiv V^{\vee}(-R, -R')$ with generators $1, \theta_i$ say, relations

$$\theta_i \theta_j = -\theta_b \theta_a R^a{}_i{}^b{}_j, \quad i.e., \quad \theta_1 \theta_2 = -\theta_2 \theta_1 R \quad (116)$$

and braided addition law whereby $\theta''_i = \theta_i + \theta'_i$ obey the same relations provided we have the braid statistics

$$\theta'_i \theta_j = -\theta_b \theta'_a R'^a{}_i{}^b{}_j, \quad i.e., \quad \theta'_1 \theta_2 = -\theta_2 \theta'_1 R'. \quad (117)$$

The roles of relations and statistics have been reversed relative to the co-ordinates x_i in Section 3.1, and there is also a minus sign in each case. Likewise, we have the *braided group of coforms* $\Lambda^*(R', R) \equiv V(-R, -R')$ with generators $1, \phi^i$ and relations corresponding to the vector case with upper indices. When it comes to questions of covariance, we assume that the quantum group obtained from $A(R')$ coincides with that obtained from $A(R)$.

This line has been taken by the author in [52] and we outline now some of its results. Namely, differentiating $\frac{\partial}{\partial \theta_i}$ in form-space gives tensors which must be manifestly $-R$ -symmetric i.e., R -antisymmetric. For example, in nice cases the algebra of forms will have an element of top degree given by $\theta_1 \cdots \theta_n$. We then define[52]

$$\epsilon^{i_1 i_2 \cdots i_n} = \frac{\partial}{\partial \theta_{i_1}} \cdots \frac{\partial}{\partial \theta_{i_n}} \theta_1 \cdots \theta_n = ([n; -R']!)_{i_2 \cdots i_n}^{i_1} \quad (118)$$

and by the reasoning above, it will be R -antisymmetric. Likewise, an R -antisymmetric tensor with lower indices can be obtained by applying any m -th order operator built from $\frac{\partial}{\partial \theta_i}$ to $\theta_{i_1} \cdots \theta_{i_m}$. For example, we define[52]

$$\epsilon_{i_1 \cdots i_n} = \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1} \theta_{i_1} \cdots \theta_{i_n} = ([n; -R']!)_{i_1 \cdots i_n}^{12 \cdots n}. \quad (119)$$

Its total R -antisymmetry is inherited this time from antisymmetry of the θ_i coordinates in form-space.

Example 5.5 *The q -epsilon tensors on $\mathbb{C}_q^{2|0}$ are the well known ‘spinor metric’*

$$\epsilon^{ij} = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}, \quad \epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}$$

Proof We have $\epsilon^{ij} = (1 - PR')^{j_1 i_2} = \delta^j_{i_1} \delta^i_{i_2} - R'^i_{i_1} \delta^j_{i_2}$ from (118) and $\epsilon_{ij} = \delta^1_{i_1} \delta^2_{i_2} - R'^2_{i_1} \delta^1_{i_2}$ from (119). We just put in R' from Example 3.1. The epsilon tensor on $\mathbb{C}_q^{2|0}$ is of course well-known by hand[22]. On the other hand, putting in a different R' gives the braided epsilon tensor for any other 2-dimensional example such as $\mathbb{C}_q^{1|1}$ etc., just as well. \square

Such epsilon tensors form the next layer of braided geometry. One can use them in conjunction with a metric to define the Hodge $*$ -operator $\Lambda \rightarrow \Lambda$ along the usual lines[52][53]

$$(\theta_{i_1} \cdots \theta_{i_m})^* = \epsilon^{a_1 \cdots a_m b_n \cdots b_{m+1}} \eta_{a_1 i_1} \cdots \eta_{a_m i_m} \theta_{b_{m+1}} \cdots \theta_{b_n}. \quad (120)$$

There are also concrete applications to quantum group theory, such as a general R -matrix formula

$$\det(\mathbf{t}) \propto \epsilon_{i_1 \cdots i_n} t^{i_1}_{j_1} \cdots t^{i_n}_{j_n} \epsilon^{j_n \cdots j_1} \quad (121)$$

for a q -determinant in our background quantum group $A(R)$. Usually such objects are introduced by hand on a case by case basis, see e.g. [22][54][53] among others; the general formulae need the above braided geometry.

Finally, we consider both the space coordinates x_i and the forms θ_i together with the interpretation $\theta_i = dx_i$. Thus we consider the *right-handed exterior algebra*

$$\Omega_R = \Lambda \underline{\otimes} V^*, \quad \mathbf{x}_1 \mathbf{x}_2 = R' \mathbf{x}_2 \mathbf{x}_1, \quad \theta_1 \theta_2 = -R \theta_2 \theta_1, \quad \mathbf{x}_1 \theta_2 = \theta_2 \mathbf{x}_1 R \quad (122)$$

where $\underline{\otimes}$ is the braided tensor product with braid statistics as shown. We then define the components Ω_R^p of form-degree p and the exterior derivative

$$\overleftarrow{d} : \Omega_R^p \rightarrow \Omega_R^{p+1}, \quad (\theta_{i_1} \cdots \theta_{i_p} f(\mathbf{x})) \overleftarrow{d} = \theta_{i_1} \cdots \theta_{i_p} \theta_a \frac{\partial}{\partial x_a} f(\mathbf{x}). \quad (123)$$

In our conventions it obeys a right-handed \mathbb{Z}_2 -graded-Leibniz rule[52]

$$(fg) \overleftarrow{d} = f(g \overleftarrow{d}) + (-1)^p f \overleftarrow{d} g, \quad \forall f \in \Omega, g \in \Omega^p. \quad (124)$$

and gives us a differential complex when

$$\overleftarrow{d}^2 = 0, \quad i.e. \quad \theta_1 \theta_2 \partial_2 \partial_1 = 0. \quad (125)$$

The latter holds quite generally because ∂^i are symmetric and θ_i are antisymmetric. It is immediate when $PR' = f(PR)$ for some function f with $f(-1) \neq 1$ and can be verified in other cases too according to the form of R and R' , see [52] for such examples. We can equally well define a

left-handed exterior algebra $\Omega_L = V \overleftarrow{\otimes} \Lambda$ as the braided tensor product in the other order. It has exterior derivative

$$d(f(\mathbf{x})\theta_{i_1} \cdots \theta_{i_m}) = f(\mathbf{x}) \overleftarrow{\partial}_{i_1} \theta_{i_1} \cdots \theta_{i_m}, \quad \theta_1 \mathbf{x}_2 = \mathbf{x}_2 \theta_1 R \quad (126)$$

which obeys the usual left-handed Leibniz rule for exterior differentials. The proof of these properties for the right-handed exterior algebra is in [52]. For the left-handed theory we just reflect the diagram-proofs there and reverse crossings. One can also verify the Leibniz properties using properties of the braided integers along the lines in Section 5.2. Note that $x_i \overleftarrow{d} = \theta_i = dx_i$ so the two exterior algebras coincide on basic forms. The general Ω_L reduces to the rectangular quantum matrix (58) in the Hecke case.

This is the construction of the exterior differential calculus on a quantum or braided vector space coming out of braided geometry. The resulting R-matrix formulae are just

$$\begin{aligned} \Omega_R : \quad \mathbf{x}_1 \mathbf{x}_2 &= \mathbf{x}_2 \mathbf{x}_1 R', & d\mathbf{x}_1 d\mathbf{x}_2 &= -dx_2 dx_1 R, & \mathbf{x}_1 d\mathbf{x}_2 &= dx_2 \mathbf{x}_1 R \\ \Omega_L : \quad \mathbf{x}_1 \mathbf{x}_2 &= \mathbf{x}_2 \mathbf{x}_1 R', & d\mathbf{x}_1 d\mathbf{x}_2 &= -dx_2 dx_1 R, & d\mathbf{x}_1 \mathbf{x}_2 &= \mathbf{x}_2 d\mathbf{x}_1 R \end{aligned} \quad (127)$$

which are the relations usually deduced by consistency arguments[39] within the axiomatic framework of Woronowicz[55]. In the braided approach d is explicitly constructed as an operator from partial differentials, which in turn come from the braided coaddition.

We have already covered the exterior algebra on the quantum plane in Example 4.4. If we want the exterior algebra $\Omega(A(R))$ on a quantum matrix with R -Hecke, we just use the braided covector form (22) explained in Section 3.2, computing (127) with the multi-index \mathbf{R}', \mathbf{R} . The cross relations are just the same braid statistics as for coaddition so we can just as well write the exterior algebra (say the right-handed one) in the original matrix form

$$R \mathbf{t}_1 \mathbf{t}_2 = \mathbf{t}_2 \mathbf{t}_1 R, \quad \mathbf{t}_1 d\mathbf{t}_2 = R_{21} dt_2 \mathbf{t}_1 R, \quad d\mathbf{t}_1 dt_2 = -R_{21} dt_2 dt_1 R \quad (128)$$

due to Sudbery[56]. The 1-form relations are again of the same structure with a minus sign. This is the general story and we see that our braided covector point of view recovers known formulae. The same applies when we look at $\Omega(\bar{A}(R))$ and $\Omega(B(R))$ in the braided-covector form (27) and (37) respectively and use the formulae (127) with the multi-index \mathbf{R}', \mathbf{R} ,

$$\mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 \mathbf{R}', \quad d\mathbf{x}_1 d\mathbf{x}_2 = -dx_2 dx_1 \mathbf{R}, \quad \mathbf{x}_1 d\mathbf{x}_2 = dx_2 \mathbf{x}_1 \mathbf{R} \quad (129)$$

$$\mathbf{u}_1 \mathbf{u}_2 = \mathbf{u}_2 \mathbf{u}_1 \mathbf{R}', \quad d\mathbf{u}_1 d\mathbf{u}_2 = -du_2 du_1 \mathbf{R}, \quad \mathbf{u}_1 d\mathbf{u}_2 = du_2 \mathbf{u}_1 \mathbf{R} \quad (130)$$

These may obviously be rearranged as in (27) and (37) to the matrix form

$$R_{21} \mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R, \quad d\mathbf{x}_1 d\mathbf{x}_2 = -R dx_2 dx_1 R, \quad \mathbf{x}_1 d\mathbf{x}_2 = R dx_2 \mathbf{x}_1 R \quad (131)$$

$$R_{21} \mathbf{u}_1 \mathbf{R} \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1 R, \quad R^{-1} d\mathbf{u}_1 R d\mathbf{u}_2 = -du_2 R_{21} du_1 R, \quad R^{-1} \mathbf{u}_1 R d\mathbf{u}_2 = du_2 R_{21} \mathbf{u}_1 R. \quad (132)$$

Some authors[57] prefer the second form for $\Omega(B(R))$ but let us stress that they are just the Wess-Zumino construction (127) in the matrix notation and not mathematically new once the required matrices \mathbf{R}', \mathbf{R} in (36) had been introduced by the author and U. Meyer in [11],[28]. Some interesting new results about $\Omega(B(R))$ from the point of view of braided groups are in [40][42].

Example 5.6 *The (right handed) quantum exterior algebra $\Omega(M_q(2))$ of $M_q(2)$ as a 4-dimensional braided covector space is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as in Example 3.7 and $\begin{pmatrix} da & db \\ dc & dd \end{pmatrix}$ with relations[58]*

$$dada = 0, \quad dbdb = 0, \quad dc dc = 0, \quad dddd = 0$$

$$dad b = -qdbda, \quad dadc = -qdcda, \quad dadd = -ddda, \quad dbdd = -qdbdb, \quad dcdd = -qdddc$$

$$dbdc = -dcdb + (q - q^{-1})dadd$$

and cross relations given by the braid statistics in Example 3.7. So

$$ada = q^2 da a, \quad bdb = q^2 db b, \quad adb = qdb a, \quad bda = qda b + (q^2 - 1)db a, \quad \text{etc.}$$

Proof We do not need to make any fresh computations: the cross relations between the 1-forms and co-ordinates are read off from the braid statistics for coaddition displayed in Example 3.7. The relations among the 1-forms are of just the same form with an extra minus sign, forcing a finite-dimensional algebra as shown. So we recover a known example computed in [58], cf.[55], now from our general braided approach. \square

We also have the q -epsilon tensor and the other constructions above on $M_q(2)$ regarded as a braided covector space. Now we can put all our constructions and ideas together to see that we have the basic formulae for the theory of electromagnetism in our setting of general braided vector and covector spaces. Actions such as

$$\text{YM} = \int F(A)^* F(A), \quad \text{CS} = \int \text{Ad}A \tag{133}$$

are now defined by the above as elements of \mathbb{C} . Here $A = A^i(\mathbf{x})dx_i$ is the gauge potential treated as a 1-form and $F = dA$ is its curvature. We use the Hodge $*$ operator from (120) and the integral from Section 5.6, understood on an n -form to be \int computed on the coefficient of the top form $\theta_1 \cdots \theta_n$. We can of course also write down the Maxwell equations $(d(F^*))^* = J$ etc, without trying to compute the action itself. The case of these equations on q -Minkowski space has been studied recently in [53]. The latter also studied q -scalar electrodynamics with some interesting results. A different (spinorial) approach to wave equations in the q -Minkowski example is in [59] and may relate to the above for spin 1. Braided geometry provides however, the only systematic R-matrix approach that works quite generally.

6 Covariance

We have mentioned from time to time that all our constructions are covariant under a ‘background quantum group’. This is an automatic or inherent feature of the whole braided approach, as we shall see in this section. To explain it requires rather more familiarity with advanced aspects of quantum group theory[60, Sec. 3] which is why we have left it till the end. Covariance is the reason that we can write the constructions of braided geometry as braid diagrams (see Section 2) and is therefore rather a deep feature. It ensures for example that the product map Υ can be pulled through a braid crossing, as explained in Section 2. Likewise invariance of $\exp = \frown$ makes it bosonic so that we can treat it like a free node:

$$\text{etc.} \tag{134}$$

all hold as a consequence of this hidden quantum group symmetry in our constructions. This is the role of quantum groups in braided geometry.

The conditions (4) in Section 3.1 essentially ensure such a covariance for our braided covector and vector algebras x_i and v^i . The covariance is like the transformation properties in Section 4.1 but without a braiding between the coacting quantum group and the vector or covector: the quantum group is treated bosonically,

$$x_i \mapsto x_a \otimes t^a_i, \quad v^i \mapsto v^a \otimes St^i_a, \quad \text{i.e.,} \quad \mathbf{x} \mapsto \mathbf{x}\mathbf{t}, \quad [\mathbf{x}_1, \mathbf{t}_2] = 0, \quad \mathbf{v} \mapsto \mathbf{t}^{-1}\mathbf{v}, \quad [\mathbf{v}_1, \mathbf{t}_2] = 0. \tag{135}$$

Here t^i_j is the quantum group generator obtained from the quantum matrix $A(R)$. The transformation of \mathbf{v} is a right coaction but written on the left as a matrix action. (There are also

left-covariant vectors which do not need the antipode or ‘inversion’ operation and corresponding left-covariant covectors using it). This formulation works fine in practice although strictly speaking, to be sure of covariance from only the conditions (4) requires us to work with the full symmetry quantum group of the system as obtained by Tannaka-Krein reconstruction[19] from the \mathbf{R} -matrix. In practice we usually already know the quantum group which we are going to get by the reconstruction and the generators t^i_j by which we are going to describe it. In this case it is easier to check the covariance directly by adopting the covariance condition

$$R'\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_1\mathbf{t}_2R' \quad (136)$$

in place of (4). Then clearly we have $\mathbf{x}_1\mathbf{t}_1\mathbf{x}_2\mathbf{t}_2 = \mathbf{x}_1\mathbf{x}_2\mathbf{t}_1\mathbf{t}_2 = \mathbf{x}_2\mathbf{x}_1R'\mathbf{t}_1\mathbf{t}_2 = \mathbf{x}_2\mathbf{x}_1\mathbf{t}_2\mathbf{t}_1R' = \mathbf{x}_2\mathbf{t}_2\mathbf{x}_1\mathbf{t}_1R'$ so that the transformed x_i obey the same relations. Similarly for the v^i . The conditions (4) are recovered automatically by applying the fundamental and conjugate-fundamental representations ρ^\pm to (136).

Once we have ensured covariance of our vector and covector algebras, the covariance of the additive braid statistics and the braided coaddition is ensured by the QYBE or braid relations (3). So all the structure maps of our braided covector and vector spaces are covariant (intertwiners for the quantum group action). Next, all our constructions based on diagrams will remain covariant because we work in a braided category[3]. The category is the category of objects covariant under our chosen quantum group, so it is really inherent. Sometimes an algebra may be in more than one braided category at the same time. Finally, when we add special data, such as the quantum metric η in Section 4.3, we will have to add conditions on it to keep in our categorical setting. Again, the easiest way is to demand it directly in terms of \mathbf{t} as

$$\eta_{ab}t^a_i t^b_j = \eta_{ij}, \quad \text{or,} \quad t^i_a t^j_b \eta^{ab} = \eta^{ji} \quad (137)$$

which indeed imply (62)–(63) etc. We will see this below. Likewise, the differentiation and duality pairing in Section 5 are automatically covariant, as will be the integration and Hodge $*$ operator etc., when the metric is covariant. If the exponential is defined abstractly as coevaluation, it will also be covariant. But if we try to construct it by means of an ansatz solving (99) we need also to ensure covariance by demanding

$$\mathbf{t}_1 \cdots \mathbf{t}_m F(m; R) = F(m; R) \mathbf{t}_1 \cdots \mathbf{t}_m. \quad (138)$$

The algebras $A(R)$, $\bar{A}(R)$ and $B(R)$ in Section 3 are examples of braided covector algebras so they are covariant by the above under a quantum group t^I_J obtained from $A(\mathbf{R})$. As usual, we can also write their covariances in matrix terms. For $\bar{A}(R)$ this is the right coaction[25]

$$\mathbf{x} \mapsto \mathbf{s}^{-1}\mathbf{x}\mathbf{t}, \quad R\mathbf{s}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{s}_1R, \quad R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1R, \quad [\mathbf{s}_1, \mathbf{t}_2] = [\mathbf{s}_1, \mathbf{x}_2] = [\mathbf{t}_1, \mathbf{x}_2] = 0 \quad (139)$$

where the quantum group is obtained from $A(R) \otimes A(R)$ with generators \mathbf{s}, \mathbf{t} respectively. This is easily checked[25] and is connected with the multi-index braided-covector point of view by

$$t^I_J \mapsto S s^{j_0}_{i_0} \otimes t^{i_1}_{j_1} \quad (140)$$

and \mathbf{R} from (27). The matrix form of covariance of $A(R)$ is similar but more natural in a bicovariant (i.e. left- and right-covariant) setting as in Section 4.2. That is why we prefer $\bar{A}(R)$ for a simpler right-covariant (or left-covariant) theory. This quantum symmetry becomes the q -orthogonal group in our $\bar{M}_q(2)$ example in Section 7.1.

For $B(R)$ the covariance is under a more complicated quantum group obtained from the double cross product $A(R) \bowtie A(R)$ with generators \mathbf{s}, \mathbf{t} and transformation as before but the non-trivial cross relations[61][34]

$$\mathbf{u} \mapsto \mathbf{s}^{-1}\mathbf{u}\mathbf{t}, \quad R\mathbf{s}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{s}_1R, \quad R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1R, \quad R\mathbf{t}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{t}_1R, \quad [\mathbf{s}_1, \mathbf{u}_2] = [\mathbf{t}_1, \mathbf{u}_2] = 0. \quad (141)$$

The same formula (140) connects it with the braided covector description \mathbf{t}^I_J but with \mathbf{R} now from (36). This bigger symmetry becomes the q -Lorentz group in the case of $BM_q(2)$ in Section 7.2. There is also a homomorphism $A(R) \bowtie A(R) \rightarrow A(R)$ given by multiplication, with the result that $B(R)$ is covariant under [11][27]

$$u^i_j \mapsto u^a_b \otimes (St^i_a)t^b_j, \quad i.e., \quad \mathbf{u} \rightarrow \mathbf{t}^{-1}\mathbf{u}\mathbf{t} \quad (142)$$

which puts our braided matrices u^i_j into the same category as the x_i, v^i above. This is useful if we want to do braided linear algebra involving all three objects as in Section 4.1. The additive coproduct (35) and its braiding have the larger symmetry (141) while the diagonal case (142) is the symmetry of the multiplicative coproduct (30) and its braiding as introduced by the author in [11][27]. Note that because we have covariance built in from the start in the concept of braided groups, it is surprising to see the covariance of some of these algebras presented sometimes subsequently as new results.

6.1 Induced braiding

Here we explain how the background quantum group symmetry mentioned above leads to the braid statistics and functoriality properties such as (134). We will be brief because this is well covered in the author's original papers on this topic in which braided groups were introduced [12][17][11][27][19] and also reviewed in [3].

The main concept needed, introduced by the author in [62][10][17][27] and in an earlier form in [60, Sec. 3] is that of a *dual-quasitriangular structure* or 'universal R-matrix functional' $\mathcal{R} : A \otimes A \rightarrow \mathbb{C}$ on a Hopf algebra or quantum group A . It is characterised by axioms which are the dual of those of Drinfeld [63], namely

$$\begin{aligned} \mathcal{R}(a \otimes bc) &= \mathcal{R}(a_{(1)} \otimes c)\mathcal{R}(a_{(2)} \otimes b), & \mathcal{R}(ab \otimes c) &= \mathcal{R}(a \otimes c_{(1)})\mathcal{R}(b \otimes c_{(2)}) \\ b_{(1)}a_{(1)}\mathcal{R}(a_{(2)} \otimes b_{(2)}) &= \mathcal{R}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)} \end{aligned} \quad (143)$$

for all $a, b, c \in A$ and $\Delta a = a_{(1)} \otimes a_{(2)}$ the coproduct. The main theorem we need is the result due to the author in [60, Sec. 3] that the quantum matrices $A(R)$ indeed have such a universal R-matrix functional such that

$$\mathcal{R}(t^i_j \otimes t^k_l) = R^i_j{}^k_l, \quad i.e., \quad \mathcal{R}(\mathbf{t}_1 \otimes \mathbf{t}_2) = R \quad (144)$$

on the matrix generators. This is where we need that R obeys the QYBE, not to ensure that $A(R)$ is a bialgebra as often mistakenly written! The proof is in [60, Sec. 3] as well as in the more modern form in [64] and Chapter 4 of my forthcoming book.

The reason we need this \mathcal{R} is that in this case any two vector spaces V, W say on which the quantum group A coacts have an intrinsic braiding [62][10][17][27]

$$\Psi_{V,W}(v \otimes w) = w^{(1)} \otimes v^{(1)} \mathcal{R}(v^{(2)} \otimes w^{(2)}) \quad (145)$$

where $v \mapsto v^{(1)} \otimes v^{(2)}$ denotes the coaction $V \rightarrow V \otimes A$, etc. We have not explained the term coaction very formally here, but (135)–(142) are typical examples; see [60] for more details. Moreover, which is the *fundamental lemma for the theory of braided groups*, if B, C are two algebras on which the quantum group A coacts covariantly (the coaction is an algebra homomorphism) then the algebra $B \otimes C$ defined with Ψ as in (1) is again an algebra on which the quantum group coacts covariantly [12][11][17]. This is the reason that the braided tensor product construction \otimes was introduced (by the author).

We can apply these ideas directly to our braided covectors if we are content to work with the quantum symmetry as $A(R)$ without necessarily an antipode. Then (144) and (145) immediately gives the braiding in Section 3.1 and 4.1 between braided covectors. If we have an inverse \mathbf{t}^{-1}

then we can also do the braided vectors, braided matrices and indeed all the braidings between them using the relevant coaction put into (145). For example,

$$\begin{aligned}
\Psi(x_i \otimes v^j) &= v^b \otimes x_a \mathcal{R}(t^a_i \otimes St^j_b) = v^b \otimes x_a \tilde{R}^a_{i^j_b} \\
\Psi(u^i_j \otimes x_k) &= x_b \otimes u^m_n \mathcal{R}((St^i_m)t^n_j \otimes t^b_k) = x_b \otimes u^m_n \mathcal{R}^{-1}(t^i_m \otimes t^b_c) \mathcal{R}(t^n_j \otimes t^c_k) \\
\Psi(u^i_j \otimes u^k_l) &= u^c_d \otimes u^a_b \mathcal{R}((St^i_a)t^b_j \otimes (St^k_c)t^d_l) \\
&= u^c_d \otimes u^a_b \mathcal{R}(St^i_a \otimes (St^e_c)t^d_f) \mathcal{R}(t^b_j \otimes (St^k_e)t^f_l) \\
&= u^c_d \otimes u^a_b \mathcal{R}(St^i_m \otimes St^e_c) \mathcal{R}(St^m_a \otimes t^d_f) \mathcal{R}(t^n_j \otimes (St^k_e) \mathcal{R}(t^b_n \otimes t^f_l)
\end{aligned}$$

and so on. We use (143)–(144). The first of these ensures that when there is an antipode, it obeys

$$\mathcal{R}(St_1 \otimes t_2) = R^{-1}, \quad \mathcal{R}(t_1 \otimes St_2) = \tilde{R}, \quad (146)$$

which we also use. This is how the various mutual braidings (39) and their corresponding braid statistics (40) were obtained in [27]. We see that *quantum group covariance induces braid statistics on whatever the quantum group acts on*. This is the fundamental reason that braided geometry and not more conventional non-commutative geometry is the correct concept of geometry for which quantum groups are the generalised symmetries[12][11]. Note also that the same algebra could be covariant under two different quantum groups, resulting possibly in two different braid statistics: these are not therefore intrinsic properties of the algebra (like fermions or bosons) but depend on which quantum group covariance we are interested in. Some constructions may be covariant under one quantum group using its induced statistics and other constructions under another. The addition and multiplication of braided matrices $B(R)$ are a case in point.

There is one technicality which we have glossed over until now. Namely, while (144) works in any normalisation, i.e. on any $A(R)$ with R invertible, our later assumption that we can adjoin t^{-1} may not. This is because one typically obtains an antipode by quotienting by determinant-like and other relations (such as cutting $M_q(2)$ down to $SL_q(2)$ by the constraint $ad - q^{-1}bc = 1$). Such relations may well not be compatible with \mathcal{R} . For most R -matrices there is no problem provided R is normalised correctly (the *quantum group normalisation*) when defining \mathcal{R} in (144). These are called *regular*[27]. All the R -matrices the reader is likely to encounter are regular in this sense, although not usually presented in the quantum group normalisation. In this paper we have used either the braided covector normalisation needed for (5) in Section 3.1 or the Hecke normalisation needed for (17) in Sections 3.2, 3.3 and 4.2. Therefore we suppose that λR is the quantum group normalisation, where λ is called the *quantum group normalisation constant*[27], and use this λR in place of R in (144), (146).

Keeping this in mind we can derive correctly the covariance identities for the quantum metric, such as (62)–(63) etc. in Section 4.3. For example,

$$\eta_{ab} \mathcal{R}(t^i_c \otimes t^b_l) \mathcal{R}(t^c_j \otimes t^a_k) = \eta_{ab} \mathcal{R}(t^i_j \otimes t^a_k t^b_l) = \mathcal{R}(t^i_j \otimes 1) \eta_{kl} = \delta^i_j \eta_{kl}$$

derives (62) from the covariance condition (137). Similarly for the other quantum metric covariance conditions.

On the other hand, we really do not want a λ in the braiding $\Psi(x_i \otimes x_j)$ between braided covectors in Section 3.1. Unless the quantum group normalisation coincides with the braided covector normalisation (which requires that PR in the quantum group normalisation has an eigenvalue -1) the covariance of our algebras does not induce the correctly normalised braid statistics for coaddition. To induce the correctly normalised braid statistics we have to extend our quantum group A to the *dilatonic extension* $A[\varsigma]$ where we adjoin a new generator ς (the dilaton or dilatation element) with[19]

$$\Delta \varsigma = \varsigma \otimes \varsigma, \quad \epsilon \varsigma = 1, \quad S \varsigma = \varsigma^{-1}, \quad [A, \varsigma] = 0, \quad \mathcal{R}(\varsigma^m \otimes \varsigma^n) = \lambda^{-mn} \quad (147)$$

where we extend the \mathcal{R} as shown. For its value on mixed products we use (143). The need for a central element like ς has been realised by hand in specific constructions (see below), but we

see the need now at a fundamental level to do with the braiding[19]. The dual quasitriangular structure \mathcal{R} for the dilaton is the new and crucial ingredient coming out of braided geometry. We then let ς coact by scale transformations. For example, we extend the coactions (135) of $A = \langle \mathbf{t} \rangle$ to coactions of $A[\varsigma] = \langle \mathbf{t}, \varsigma \rangle$ acting by

$$\mathbf{x} \rightarrow \mathbf{x}\mathbf{t}\varsigma, \quad \mathbf{v} \rightarrow \varsigma^{-1}\mathbf{t}^{-1}\mathbf{v} \quad (148)$$

where we include this scale transformation. This time the covariance under the new extended quantum group induces the correctly normalised braid statistics $\Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = \mathbf{x}_1 \otimes \mathbf{x}_2 \mathcal{R}(\mathbf{t}_1 \varsigma \otimes \mathbf{t}_2 \varsigma) = \mathbf{x}_1 \otimes \mathbf{x}_2 \mathcal{R}(\varsigma \otimes \mathbf{t}_2 \varsigma) \mathcal{R}(\mathbf{t}_1 \otimes \mathbf{t}_2 \varsigma) = \mathbf{x}_1 \otimes \mathbf{x}_2 \mathcal{R}(\varsigma \otimes \varsigma) \mathcal{R}(\varsigma \otimes \mathbf{t}_1) \mathcal{R}(\mathbf{t}_1 \otimes \varsigma) \mathcal{R}(\mathbf{t}_1 \otimes \mathbf{t}_2) = \mathbf{x}_1 \otimes \mathbf{x}_2 \lambda^{-1} \lambda R = \mathbf{x}_1 \otimes \mathbf{x}_2 R$ as explained in [19]. Likewise for the vector-vector and mixed vector-covector braid statistics. This is the correct derivation of (39)–(40). The braid statistics involving \mathbf{u} in this group are not affected; one can consider that it has zero scale dimension and transforms as before in (142) with zero power of ς .

6.2 Induced Poincaré group

Most of the time in braided geometry we do not need to worry about the background quantum group symmetry: it is there and makes sure that our constructions are coherent in the way explained above. Sometimes however, the covariance of the system gets into the algebra and we see $A[\varsigma]$ directly. One of these situations is the construction of the q -Poincaré quantum group in the approach of [19]. The idea is very simple: if we consider one of our covector spaces x_i as ‘space’ then the symmetry (148) is obviously some kind of ‘extended rotation’. But because all the braided covector constructions are fully covariant under this, it is natural to make a semidirect product by this coaction. The theorem in [47][19] is that this semidirect product or ‘bosonisation’ is always an ordinary (bosonic) Hopf algebra and that its corepresentations are equivalent to the covariant representations of the original x_i . The semidirect product $A[\varsigma] \ltimes V$ is then an ‘extended Poincaré or Euclidean group’. We call it the induced Poincaré group if the braided-covector algebra is regarded as spacetime. The best proof is diagrammatic[47] and works for *any* braided group B where the braiding is induced by some quantum group covariance: just make a semidirect product by that quantum symmetry. So this is a very general phenomenon and not an ad-hoc construction.

The idea for the construction comes from the Jordan-Wigner transformation in physics where the super-representations of a fermionic or graded system are equivalent to the ordinary representations of a new algebra obtained by adjoining the degree operator. The grading corresponds in our language to a background quantum group \mathbb{Z}'_2 under which everything is covariant. It has a generator ς with $\varsigma^2 = 1$ and a non-trivial dual quasitriangular structure \mathcal{R} . This ‘hidden quantum group of supersymmetry’ is what generates the ± 1 Bose-Fermi statistics we encounter, using exactly the formalism above. We explained this in detail in [47][65] in a form where \mathbb{Z}'_2 was quasitriangular rather than dual-quasitriangular, the two being equivalent. See also the Appendix below.

The same mathematical ideas apply just as well in our geometrical example where the ‘hidden quantum group’ is much more like a q -deformation of a rotation group than a discrete one: the mathematical formalism of Hopf algebras *unifies* two quite different concepts in physics. One is the concept of supersymmetry where everything is \mathbb{Z}_2 -graded and the other is the concept of Lorentz transformation or rotational invariance where everything is covariant under a background linear co-ordinate transformation. One needs the concept of quantum groups to do this (ordinary groups will not do) but once we have that concept, these two physically different ideas are just extremes of the same phenomenon. *This unification of the concept of Lorentz transformation or rotation (or other quantum group) covariance and supersymmetry is one of the deepest reasons to be interested in quantum groups in physics.*

We content ourselves here with the formulae from [19] for how these constructions look in practice in R-matrix form. We use the braided covector algebra of Section 3.1 but call the generators momentum p_i . It lives in the category of $A[\varsigma]$ -covariant systems as explained in Section 6.1,

where $A[\varsigma]$ has generators $t^i_{j,\varsigma}$. The constant λ is such that λR is in the quantum group normalisation. Then the induced Poincaré quantum group has cross relations and (ordinary) Hopf algebra structure[19]

$$\begin{aligned} \mathbf{p}\varsigma &= \lambda^{-1}\varsigma\mathbf{p}, \quad \mathbf{p}_1\mathbf{t}_2 = \lambda\mathbf{t}_2\mathbf{p}_1R, \quad \Delta\mathbf{t} = \mathbf{t} \otimes \mathbf{t}, \quad \Delta\varsigma = \varsigma \otimes \varsigma \\ \Delta\mathbf{p} &= \mathbf{p} \otimes \mathbf{t}\varsigma + 1 \otimes \mathbf{p}, \quad \epsilon\mathbf{t} = \text{id}, \quad \epsilon\varsigma = 1, \quad \epsilon\mathbf{p} = 0 \\ S\mathbf{t} &= \mathbf{t}^{-1}, \quad S\varsigma = \varsigma^{-1}, \quad S\mathbf{p} = -\mathbf{p}\varsigma^{-1}\mathbf{t}^{-1}. \end{aligned} \tag{149}$$

Some examples of such type were first considered by hand in [66], but the braided approach in [19] provided the first systematic construction for general R, R' . Moreover, which was entirely missing before [19], it provided for a covariant action of this R-Poincaré group on another copy x_i of the braided covectors, regarded this time as space, by[19]

$$x_i \mapsto x_a \otimes t^a_{i\varsigma} + p_i, \quad \text{i.e.,} \quad \mathbf{x} \mapsto \mathbf{x}\mathbf{t}\varsigma + \mathbf{p}. \tag{150}$$

This is ensured by the general braided group theory of bosonisation in [47] which says that braided-covariant systems under any braided group become automatically ordinary covariant systems under its bosonisation. We have seen in Section 5.1 that the braided covectors x_i indeed coact covariantly on themselves by the braided coaddition or finite braided translation. Hence by this theorem the induced Poincaré coacts too. In keeping with the theme of this paper, we can give a direct proof too:

$$\begin{aligned} (\mathbf{x}_1\mathbf{t}_1\varsigma + \mathbf{p}_1)(\mathbf{x}_2\mathbf{t}_2\varsigma + \mathbf{p}_2) &= \mathbf{x}_1\mathbf{x}_2\mathbf{t}_1\mathbf{t}_2\varsigma^2 + \mathbf{p}_1\mathbf{p}_2 + \mathbf{p}_1\mathbf{x}_2\mathbf{t}_2\varsigma + \mathbf{x}_1\mathbf{t}_1\varsigma\mathbf{p}_2 \\ (\mathbf{x}_2\mathbf{t}_2\varsigma + \mathbf{p}_2)(\mathbf{x}_1\mathbf{t}_1\varsigma + \mathbf{p}_1)R' &= \mathbf{x}_2\mathbf{x}_1\mathbf{t}_2\mathbf{t}_1R'\varsigma^2 + \mathbf{p}_2\mathbf{p}_1R' + \mathbf{x}_2\mathbf{t}_2\varsigma\mathbf{p}_1R' + \mathbf{p}_2\mathbf{x}_1\mathbf{t}_1\varsigma R' \\ &= \mathbf{x}_2\mathbf{x}_1R'\mathbf{t}_1\mathbf{t}_2\varsigma^2 + \mathbf{p}_2\mathbf{p}_1R' + \mathbf{x}_1\mathbf{t}_1\varsigma\mathbf{p}_2 + \mathbf{x}_2\mathbf{t}_2\varsigma\mathbf{p}_1R \end{aligned}$$

where we use covariance (136) for the first term and the cross-relations (149), the condition (6) and the action of the usual permutation P for the last two terms. Using again the cross-relations (149) and the algebra relations of the \mathbf{x}, \mathbf{p} we see that the two expressions coincide, i.e. the transformed $\mathbf{x}\mathbf{t}\varsigma + \mathbf{p}$ obey the same braided covector algebra.

Example 6.1 The bosonisation of $\mathbb{C}_q^{2|0} = \{x, y\}$ is the ordinary Hopf algebra $GL_q(2) \bowtie \mathbb{C}_q^{2|0}$ with relations

$$\begin{aligned} \varsigma^2 &= ad - q^{-1}bc, \quad x\varsigma = q^{\frac{3}{2}}\varsigma x, \quad y\varsigma = q^{\frac{3}{2}}\varsigma y \\ x \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} q^2a & qb \\ q^2c & qd \end{pmatrix} x, \quad y \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} qay + (q^2 - 1)bx & q^2by \\ qcy + (q^2 - 1)dx & q^2dy \end{pmatrix} \end{aligned}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have the relations of $M_q(2)$ with $(ad - q^{-1}bc)^{\pm\frac{1}{2}}$ adjoined. The coproduct is

$$\Delta x = x \otimes a + y \otimes c + 1 \otimes x, \quad \Delta y = x \otimes b + y \otimes d + 1 \otimes y$$

and the matrix coproduct for the $GL_q(2)$ generators.

Proof We consider $\mathbb{C}_q^{2|0}$ a braided covector algebra with the standard R as in Example 3.1. Then $\lambda = q^{-\frac{3}{2}}$ takes us to the quantum group normalisation. The quantum group $SL_q(2)[\varsigma]$ consists of $M_q(2)$ generated by t^i_j as in Example 3.7 with a q -determinant relation added (so $SL_q(2)$ as usual) and the element ς adjoined as in (147). It is convenient to the redefine generators $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{t}\varsigma$. They have the same commutation relations as the quantum matrices $M_q(2)$ but instead of $\det_q(\mathbf{t}) = 1$ we have the relation $ad - q^{-1}bc = \varsigma^2$. So we can identify $SL_q(2)[\varsigma]$ as a version of $GL_q(2)$ consisting of $M_q(2)$ and $(ad - q^{-1}bc)^{\pm\frac{1}{2}}$ adjoined. The commutation relations

are then obtained from (149) using the explicit form of R . The coaction and corresponding coproduct are given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on (x, y) from the right as a matrix. \square

This is a more or less well-known example and in the general family in [66] and elsewhere, but demonstrates the kind of structure one gets. From [19] we know that it necessarily coacts covariantly on $\mathbb{C}_q^{2|0}$ by matrix multiplication and translation (150). Moreover, the braided approach allows us to bosonise all the other braided covector algebras in this paper just as well and obtain their natural induced ‘Poincaré’ quantum groups and their coactions.

For example, when R is Hecke, we can apply the bosonisation to the additive braided groups $A(R)$, $\bar{A}(R)$ and $B(R)$ just as well. We just use \mathbf{R}, \mathbf{R}' from (27), (22), (36) respectively. The Poincaré groups consist of adjoining the Lorentz generators $t^I{}_J$ obtained from $A(\mathbf{R})$. We can also give a spinorial or matrix version using the quantum symmetry in the matrix form, as given for $\bar{A}(R)$ and $B(R)$ in (139) and (141) respectively. In these cases we chose the quantum group normalisation constant λ such that $A(R)$ has a quotient dual-quasitriangular Hopf algebra A and from two copies of this we build $A \otimes A$ as the quantum covariance of $\bar{A}(R)$ and $A \bowtie A$ as the quantum covariance of $B(R)$. We use for the latter the \bowtie construction introduced by the author and making use of the dual universal R-matrix \mathcal{R} of A , see [19, Sec. 4]. In both cases we then adjoin ς .

The R-matrix form of the induced Poincaré quantum group $(A \otimes A)[\varsigma] \bowtie \bar{A}(R)$ then has cross relations and coproduct [25]

$$\begin{aligned} \mathbf{p}_1 \mathbf{t}_2 &= \lambda \mathbf{t}_2 \mathbf{p}_1 R, & \mathbf{p}_1 \mathbf{s}_2 &= \mathbf{s}_2 \lambda^{-1} R^{-1} \mathbf{p}_1, & \mathbf{p} \varsigma &= \lambda^{-2} \varsigma \mathbf{p} \\ \Delta p^i{}_j &= p^a{}_b \otimes (S s^i{}_a) t^b{}_j \varsigma + 1 \otimes p^i{}_j, & i.e., & & \Delta \mathbf{p} &= \mathbf{p} \otimes \mathbf{s}^{-1}(\) \mathbf{t} \varsigma + 1 \otimes \mathbf{p} \end{aligned} \quad (151)$$

where the indices of \mathbf{p} have to be inserted into the space. There is also a counit $\epsilon \mathbf{p} = 0$ and antipode. The standard R-matrix put into these formulae provides the 4-dimensional q -Euclidean space Poincaré group appropriate to the q -Euclidean space algebra in Section 7.1 below.

The R-matrix form of the induced Poincaré group $(A \bowtie A)[\varsigma] \bowtie B(R)$ has the cross-relations and coproduct [67]

$$\begin{aligned} \mathbf{p}_1 \mathbf{t}_2 &= \lambda^2 \mathbf{t}_2 R_{21} \mathbf{p}_1 R, & \mathbf{p}_1 \mathbf{s}_2 &= \mathbf{s}_2 R^{-1} \mathbf{p}_1 R, & \mathbf{p} \varsigma &= \lambda^{-2} \varsigma \mathbf{p} \\ \Delta \mathbf{p} &= \mathbf{p} \otimes \mathbf{s}^{-1}(\) \mathbf{t} \varsigma + 1 \otimes \mathbf{p} \end{aligned} \quad (152)$$

in a similar way. There is also a counit $\epsilon \mathbf{p} = 0$ and antipode. The standard R-matrix inserted into these formulae gives us the q -Minkowski space Poincaré group appropriate to the q -Minkowski algebra in Section 7.2. Of course, our constructions (151)–(152) are quite general and not limited to the standard R-matrix.

We conclude here by mentioning that also in [19] was given the corresponding Poincaré enveloping algebra-type quantum group for the general construction (149). We give it in terms of the $\mathbb{1}^\pm$ generators of $U_q(g)$ in [22]. One can also use it for non-standard quantum enveloping algebras when the universal R-matrix is known. This enveloping algebra is our ‘Lorentz or rotation’ algebra and all our constructions are covariant under its action (rather than coaction as before). First we extend $U_q(g)$ by adjoining an infinitesimal scaling central generator ξ dual to the finite dilaton ς ,

$$\Delta \xi = \xi \otimes 1 + 1 \otimes \xi, \quad \mathcal{R}_\xi = \lambda^{-\xi} \otimes \xi, \quad \langle \varsigma, \xi \rangle = 1, \quad \text{or} \quad \langle \varsigma, \lambda^\xi \rangle = \lambda \quad (153)$$

We use its *quantum line* universal R-matrix [68]. The universal R-matrix of the extension of $U_q(g)$ is its usual one [63][69] times this \mathcal{R}_ξ . We focus on braided vectors p^i (say) as momentum

generators covariant under this extension. Bosonisation proceeds using the general theorem in [47] and gives the cross relations and Hopf structure [19]

$$\begin{aligned} \lambda^\xi \mathbf{p} \lambda^{-\xi} &= \lambda^{-1} \mathbf{p}, & \mathbf{I}_1^+ \mathbf{p}_2 &= \lambda^{-1} R_{21}^{-1} \mathbf{p}_2 \mathbf{I}_1^+, & \mathbf{I}_1^- \mathbf{p}_2 &= \lambda R_{21} \mathbf{p}_2 \mathbf{I}_1^- \\ \Delta \mathbf{p} &= \mathbf{p} \otimes 1 + 1 \otimes \lambda^\xi \mathbf{p}, & \epsilon \mathbf{p} &= 0, & S \mathbf{p} &= -\lambda^{-\xi} S \mathbf{I}^- \mathbf{p} \end{aligned} \quad (154)$$

and a covariant action on braided covectors x_i by

$$\mathbf{I}_2^+ \triangleright \mathbf{x}_1 = \mathbf{x}_1 \lambda R, \quad \mathbf{I}_2^- \triangleright \mathbf{x}_1 = \mathbf{x}_1 \lambda^{-1} R_{21}^{-1}, \quad \lambda^\xi \triangleright \mathbf{x} = \lambda^{-1} \mathbf{x}, \quad p^i \triangleright x_j = \delta^i_j. \quad (155)$$

The action of p^i on products is according to the coproduct and is equivalent to a version (in some other conventions) of the braided-Leibniz rule (80) for braided derivatives. We can just as easily give the enveloping algebra form of the matrix Poincaré groups (151)–(152) by the same techniques. See [25] for the action of the \mathbf{I}^\pm in these cases. We note that there are many other applications too of the bosonisation theory. See [70][71] for recent applications in connection with the differential calculus and Koszul complex of $U_q(g)$.

7 q -Deformed spacetime

At various points in our tour of braided geometry we have mentioned that the examples $\bar{M}_q(2)$ and $BM_q(2)$ make natural q -Euclidean and q -Minkowski spaces when q is real. We just have to specialise our general $\bar{A}(R)$ and $B(R)$ constructions to the case of R given by the standard quantum plane or Jones polynomial R-matrix in the Hecke normalisation. We have seen above many results about these general algebras, all of which constitutes the *braided approach* to q -spacetime due to the author [11][27][33][19][25] and U. Meyer [28][67]. Among other general results, there is a theory of *quantum Wick rotation* introduced in [25] which strictly connects $\bar{A}(R)$ and $B(R)$ by an algebraic twisting construction. There is also a purely quantum isomorphism $B(R) \cong U(\mathcal{L})$ where \mathcal{L} is a braided-Lie algebra and $U(\mathcal{L})$ its braided enveloping algebra [9]. These are among the more advanced results of braided geometry which were not able to cover here.

Many authors have considered what algebras should be taken as the q -deformation of Euclidean and Minkowski spaces. In the Euclidean case the braided geometry approach based on $\bar{M}_q(2)$ is compatible with the $N = 4$ case of the theory of $SO_q(N)$ -covariant quantum planes in [22][50][51][72][73] and elsewhere. It has been taken quite far by Fiore, Weich and others. In the Minkowski case our approach based on the braided matrices $BM_q(2)$ is compatible with the approach of Carow-Watamura, Schlieker, Scholl, Watamura, Wess, Zumino and others, based on the idea of q -Minkowski space as a tensor product of two quantum planes, see [34][29][35] and elsewhere. The mathematical basis for this overlap is the homomorphism (43)–(44) introduced by the author in [27], though its significance was not appreciated until somewhat later when the two approaches began to converge [19]. The final details for this convergence were worked out in [67, Sec. 3] using the dual-universal R-matrix functional \mathcal{R} on the q -Lorentz group.

In the following two sections we explain cleanly and simply how some of our various braided group constructions look in the case of $\bar{M}_q(2)$ and $BM_q(2)$. Some comments about which were known before for these particular algebras and which were obtained as far as I know for the first time as examples of the braided approach are collected at the end of each section.

7.1 q -Euclidean space

The algebra $\bar{M}_q(2)$

$$\begin{aligned} ba &= qab, & ca &= q^{-1}ac, & da &= ad, & db &= q^{-1}bd & dc &= qcd \\ bc &= cb + (q - q^{-1})ad \end{aligned} \quad (156)$$

was already computed in Example 3.8. The covariance (139) is $SU_q(2) \otimes SU_q(2)$ regarded as the spinorial version of $SO_q(4)$. We can also work with $O_q(4)$ obtained from \mathbf{R} but here we concentrate on the spinorial setting.

Next, the element $ad - qcb$ is central in the algebra and invariant under $SU_q(2) \otimes SU_q(2)$. Hence we take it as a natural radius function. The corresponding metric by differentiation (110) is

$$\eta^{IJ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -q & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{\partial}{\partial x_I} \frac{\partial}{\partial x_J} (ad - qcb). \quad (157)$$

It obeys all the conditions in Section 4.3 with $\lambda = q^{-\frac{1}{2}}$, and the conditions (104)–(105) as well. The Gaussian has the simple form (106).

Next, there is a natural $*$ -structure

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix} \quad (158)$$

on this algebra, making it into a $*$ -braided group as explained in Section 4.4. It corresponds to the metric under the multi-index version of (72). We can therefore choose new variables

$$t = \frac{a-d}{2i}, \quad x = \frac{c-qb}{2}, \quad y = \frac{c+qb}{2i}, \quad z = \frac{a+d}{2} \quad (159)$$

which are self-adjoint in the sense $t^* = t, x^* = x, y^* = y, z^* = z$. They are the physical spacetime coordinates and

$$ad - qcb = \left(\frac{1+q^2}{2}\right)t^2 + x^2 + y^2 + \left(\frac{1+q^2}{2}\right)z^2 \quad (160)$$

so that the signature of the metric is the Euclidean one. This justifies our defining $\mathbb{R}_q^4 \equiv \bar{M}_q(2)$, the algebra of ‘co-ordinate functions’ on q -Euclidean space.

After this, we can routinely apply the rest of the $\bar{A}(R)$ constructions of braided geometry with this interpretation. The braided vector algebra of derivatives from (28) is

$$\begin{aligned} \frac{\partial}{\partial d} \frac{\partial}{\partial b} &= q^{-1} \frac{\partial}{\partial b} \frac{\partial}{\partial d}, & \frac{\partial}{\partial d} \frac{\partial}{\partial c} &= \frac{\partial}{\partial c} \frac{\partial}{\partial d} q, & \frac{\partial}{\partial d} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial d} \\ \frac{\partial}{\partial b} \frac{\partial}{\partial c} &= \frac{\partial}{\partial c} \frac{\partial}{\partial b} + (q - q^{-1}) \frac{\partial}{\partial a} \frac{\partial}{\partial d}, & \frac{\partial}{\partial b} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial b} q, & \frac{\partial}{\partial c} \frac{\partial}{\partial a} &= q^{-1} \frac{\partial}{\partial a} \frac{\partial}{\partial c}. \end{aligned}$$

Their Leibniz rule (82) is read off from the braiding listed in Example 3.8.

The algebra of forms from (129) is

$$\begin{aligned} dada &= 0, & dbdb &= 0, & dc dc &= 0, & dddd &= 0 \\ dbda &= -q^{-1}dadb, & dcda &= -dadcq, & dddb &= -dbddq \\ dcdb &= -dbdc, & dddc &= -q^{-1}dcdd, & ddda &= -(q - q^{-1})dbdc - dadd \end{aligned}$$

or deduced immediately from the braid statistics in Example 3.8 with an extra - sign. These relations generate a finite-dimensional algebra with the dimension at each degree being the same as in the classical case. The exterior algebra $\Omega(\mathbb{R}_q^4)$ is generated by these forms and the co-ordinates, with cross-relations again being read off from the additive braid statistics in Example 3.8 as explained in Section 5.7. So we do not list it again.

The q -epsilon tensor from (119) is

$$\begin{aligned}
\epsilon_{abcd} &= -\epsilon_{acbd} = \epsilon_{adbc} = -\epsilon_{adcb} = \epsilon_{bcad} = -\epsilon_{bcda} = 1 \\
-\epsilon_{cbad} &= \epsilon_{cbda} = -\epsilon_{dabc} = \epsilon_{dacb} = -\epsilon_{dbca} = \epsilon_{dcba} = 1 \\
\epsilon_{acdb} &= -\epsilon_{cdba} = -\epsilon_{dcab} = q, \quad -\epsilon_{abdc} = -\epsilon_{bacd} = \epsilon_{bdca} = \epsilon_{dbac} = q^{-1} \\
-\epsilon_{cadb} &= \epsilon_{cdab} = q^2, \quad \epsilon_{badc} = -\epsilon_{bdac} = q^{-2}, \quad -\epsilon_{adad} = \epsilon_{dada} = q - q^{-1}.
\end{aligned}$$

We see that there are a few unexpected entries which are zero at $q = 1$.

Finally, the Gaussian-weighted integral on degree two from Section 5.6 is $\mathcal{Z}[x_I x_J] = q^{-4} \eta_{IJ}$. Its nonzero values on degree four are

$$\begin{aligned}
\mathcal{Z}[abcd] &= -q^{-11}, \quad \mathcal{Z}[acbd] = -q^{-7} \\
\mathcal{Z}[a^2 d^2] &= q^2 \mathcal{Z}[b^2 c^2] = q^{-2} \mathcal{Z}[c^2 b^2] = \mathcal{Z}[bc^2 b] = \mathcal{Z}[cb^2 c] = q^{-10} (q^2 + 1)
\end{aligned}$$

along with the other cases implied by the relations in \mathbb{R}_q^4 . In terms of the spacetime coordinates we have

$$\begin{aligned}
\mathcal{Z}[t^2] &= \mathcal{Z}[z^2] = \frac{1}{2} q^{-4}, \quad \mathcal{Z}[x^2] = \mathcal{Z}[y^2] = \frac{[2; q^2]}{4} q^{-4} \\
\mathcal{Z}[t^4] &= \mathcal{Z}[z^4] = \frac{3[2; q^2]}{8} q^{-10}, \quad \mathcal{Z}[x^4] = \mathcal{Z}[y^4] = \frac{[3; q^2]}{8} q^{-10}
\end{aligned}$$

which shows a degree of spherical symmetry even in the noncommutative case.

Our approach to q -Euclidean space is a useful R-matrix one introduced by the author in [25] based on the formulae $R_{21} \mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R$ etc., and making contact in the above specific example with the $N = 4$ case of the approach based on $SO_q(N)$ -covariance. The general N could also be studied in our braided formalism as braided covector spaces. The differentials, epsilon tensor, Gaussian and integration in this family were known already [50][54][51][72] while our braided approach adds such things as the braided addition law, the R-matrix form (151) of the Poincaré group[25], its covariant action on the \mathbf{x} , existence of the braided-exponential and several other results, including the quantum wick rotation by twisting[25] which systematically turns the above into the following results for q -Minkowski space. The specific relations (156) are in fact isomorphic in this example with usual quantum matrices $M_q(2)$ which were already anticipated as q -Euclidean space in [29, Appendix], with $SU_q(2) \otimes SU_q(2)$ as rotations.

7.2 q -Minkowski space

Finally we follow the same constructions as in the preceding section but now for the braided matrices $B(R)$. It is ironic that the simplest example $BM_q(2)$ was the very first braided group known[11] but also turns out to be the most interesting for q -deformed physics. We have already given the algebra in Example 3.10 as

$$\begin{aligned}
ba &= q^2 ab, \quad ca = q^{-2} ac, \quad da = ad, \quad bc = cb + (1 - q^{-2})a(d - a) \\
db &= bd + (1 - q^{-2})ab, \quad cd = dc + (1 - q^{-2})ca.
\end{aligned} \tag{161}$$

It has the covariance $SU_q(2) \bowtie SU_q(2)$ from (141). Its natural $*$ -structure is not the usual tensor product one but involves also reversing the two $SU_q(2)$ factors. So these two $SU_q(2)$ are like the $SU_q(2)$ which physicists use when they work with $SO_q(1, 3)$ at the level of a complex Lie algebra.

This algebra has a nice braided-coproduct too corresponding to matrix multiplication as explained in Section 3.3 and this at once picks out the braided determinant $\underline{\det}(\mathbf{u}) = ad - q^2 cb$ as

a natural radius function. It is group-like, central and bosonic with respect to the multiplicative braid statistics as explained in (47)–(49). We obtain the quantum metric from (110) as

$$\eta^{IJ} = \frac{\partial}{\partial u_I} \frac{\partial}{\partial u_J} (ad - q^2 cb) = \begin{pmatrix} q^{-2} - 1 & 0 & 0 & 1 \\ 0 & 0 & -q^2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (162)$$

The conditions in Section 4.3 hold with $\lambda = q^{-\frac{1}{2}}$, as well as (104)–(105) so the Gaussian has the simple form (106).

There is a natural $*$ -structure for real q which is the Hermitian one

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

giving a $*$ -braided group under both braided addition and multiplication. This is from the general construction [33][45] recalled in Section 4.4. This leads to a change of variables

$$t = \frac{qd + q^{-1}a}{2}, \quad x = \frac{b+c}{2}, \quad y = \frac{b-c}{2i}, \quad z = \frac{d-a}{z} \quad (163)$$

which are self-adjoint in the sense $t^* = t$ etc, and the metric becomes

$$ad - q^2 cb = \frac{4q^2}{(q^2 + 1)^2} t^2 - q^2 x^2 - q^2 y^2 - \frac{2(q^4 + 1)q^2}{(q^2 + 1)^2} z^2 + 2q \left(\frac{q^2 - 1}{q^2 + 1} \right)^2 tz. \quad (164)$$

This has the Minkowski signature when $q = 1$ and justifies the definition $\mathbb{R}_q^{1,3} \equiv BM_q(2)$. The natural time direction t is the quantum trace element in Example 3.10 and is central and bosonic with respect to the multiplicative braid statistics.

The rest of the structure is routinely computed from the R-matrix formulae for $B(R)$. The braided vector algebra of differentiation operators from (38) is

$$\begin{aligned} \frac{\partial}{\partial d} \frac{\partial}{\partial b} &= q^{-2} \frac{\partial}{\partial b} \frac{\partial}{\partial d}, & \frac{\partial}{\partial a} \frac{\partial}{\partial c} &= \frac{\partial}{\partial c} \frac{\partial}{\partial a} q^2, & \frac{\partial}{\partial d} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial d} \\ \frac{\partial}{\partial c} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial c} + \frac{\partial}{\partial c} \frac{\partial}{\partial d} (q^2 - 1), & \frac{\partial}{\partial b} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial b} + \frac{\partial}{\partial b} \frac{\partial}{\partial d} (q^{-2} - 1) \\ \frac{\partial}{\partial b} \frac{\partial}{\partial c} &= \frac{\partial}{\partial c} \frac{\partial}{\partial b} + \frac{\partial}{\partial d} \frac{\partial}{\partial d} (q^{-2} - 1) + \frac{\partial}{\partial a} \frac{\partial}{\partial d} (q^2 - 1) \end{aligned}$$

Their Leibniz rule comes from the additive braid statistics listed in Example 3.10.

The algebra of forms from (30) is

$$\begin{aligned} dc dc &= 0, & da da &= 0, & db db &= 0, & db da &= -da db \\ dc da &= -da dc, & dc db &= -db dc, & dddd &= db c(1 - q^{-2}) \\ dddc &= -cd dd q^{-2} + da dc(1 - q^{-2}), & dddb &= -db dd q^2 - da db(q^2 - 1) \\ ddda &= -db dc(q^2 - 1) - dadd \end{aligned}$$

also obtained from the additive braid statistics (with an extra minus sign). These relations generate a finite-dimensional algebra with the dimension at each degree being the same as in the classical case. The braided exterior algebra $\Omega(\mathbb{R}_q^{1,3})$ is generated by the space-time coordinates and these forms, with relations again read off at once from the additive braid statistics in Example 3.10.

The q -epsilon tensor from (119) is

$$\begin{aligned}
\epsilon_{add} &= -\epsilon_{bcd} = -\epsilon_{dadd} = \epsilon_{dbdc} = \epsilon_{ddad} = -\epsilon_{ddda} = 1 - q^{-2} \\
-\epsilon_{adad} &= -\epsilon_{cdbd} = \epsilon_{dada} = \epsilon_{dcdb} = q^2 - 1 \\
\epsilon_{abcd} &= -\epsilon_{acbd} = \epsilon_{adbc} = -\epsilon_{adcb} = -\epsilon_{bacd} = \epsilon_{bcad} = -\epsilon_{bcda} = \epsilon_{cabd} = 1 \\
-\epsilon_{cbad} &= \epsilon_{cbda} = -\epsilon_{dabc} = \epsilon_{dacb} = \epsilon_{dbac} = -\epsilon_{dbca} = -\epsilon_{dcab} = \epsilon_{dcba} = 1 \\
\epsilon_{acdb} &= -\epsilon_{cadb} = \epsilon_{cdab} = -\epsilon_{cdba} = q^2, \quad -\epsilon_{abdc} = \epsilon_{badc} = -\epsilon_{bdac} = \epsilon_{bdca} = q^{-2}
\end{aligned}$$

and has even more nonzero elements than in the Euclidean case.

Finally, the Gaussian-weighted integral on degree two from Section 5.6 is $\mathcal{Z}[u_I u_J] = q^{-4} \eta_{IJ}$. Its nonzero values on degree four are

$$\begin{aligned}
\mathcal{Z}[abcd] &= -q^{-12}, \quad \mathcal{Z}[acbd] = -q^{-8}, \quad \mathcal{Z}[bcd^2] = -q^{-10}(1 - q^{-4}) \\
\mathcal{Z}[cbd^2] &= -\frac{1}{2}\mathcal{Z}[d^4] = -q^{-8}(1 - q^{-2})^2, \quad \mathcal{Z}[ad^3] = q^{-10}(1 + q^{-4})(1 - q^{-2}) \\
\mathcal{Z}[a^2 d^2] &= q^4 \mathcal{Z}[b^2 c^2] = \mathcal{Z}[c^2 b^2] = q^2 \mathcal{Z}[bc^2 b] = q^2 \mathcal{Z}[cb^2 c] = q^{-10}(q^2 + 1)
\end{aligned}$$

along with various other cases implied by the relations in $\mathbb{R}_q^{1,3}$. In terms of spacetime coordinates (163) we have

$$\begin{aligned}
\mathcal{Z}[t^2] &= \frac{[2; q^2]}{4} q^{-4}, \quad \mathcal{Z}[x^2] = \mathcal{Z}[y^2] = \mathcal{Z}[z^2] = -\frac{[2; q^2]}{4} q^{-6} \\
\mathcal{Z}[z^4] &= \frac{3[2; q^2]}{8} q^{-12}, \quad \mathcal{Z}[t^4] = q^4 \mathcal{Z}[x^4] = q^4 \mathcal{Z}[y^4] = \frac{[3; q^2]}{8} q^{-10}.
\end{aligned}$$

We see that the Gaussian-weighted integral \mathcal{Z} is quite similar to the Euclidean one in its values, except for the sign in the spacelike directions. We also note that the negative sign in the space-like directions agrees with the sign obtained in physics by Wick rotation to make sense of Gaussian integrals in Minkowski space.

We have developed q -Minkowski space cleanly and simply as a braided matrix $R_{21} \mathbf{u}_1 R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1 R$ in the braided approach started in [11][26][27][33]. Because of the mapping (44) our results must necessarily recover in the standard 2×2 case the pioneering work in the spinorial approach of [34][29][35] when one looks at the explicit algebra. On the other hand, our braided formulae are always in a general R -matrix form; for example the spinorial form (152) for the q -Poincaré group is a new result in [67] and can be contrasted with the explicit algebra relations found in [35] by other means. The role of $SU_q(2) \rtimes SU_q(2)$ as q -Lorentz group is due to [74][34] although its abstract structure is due to [61][19, Sec. 4], in the second of which the link with twisting of the Euclidean rotation group was made precise. The algebra of differential forms [35] is not new but developed now in a constructive way from the braided coaddition [52]. The braided approach added principally such things as the coaddition itself [28], the comultiplication [11], the duality with braided vectors, the covariant action of the q -Poincaré group on the spacetime generators [19], the existence of the braided exponential (though no very good formula for it), the braided Gaussian and associated integral [44] even for this particular example. Further results in the braided approach are in [28] where a braided covector picture of both spacetime and the Lorentz group is developed, and [53].

The braided approach also added the canonical isomorphism [9][5]

$$\mathbb{R}_q^{1,3} \equiv BM_q(2) \cong U(gl_{q,2}) \quad (165)$$

which is a ‘purely quantum’ phenomenon since at $q = 1$ the left hand side becomes the commutative co-ordinates on Minkowski space while the right hand side becomes the non-commutative

enveloping algebra of $su_2 \oplus u(1)$. This is obviously very interesting for particle physics since two fundamental ingredients of electroweak theory are unified by q -deformation. Under this quantum-geometry transformation the mass shell $ad - q^2cb = 1$ in q -Minkowski space (the braided group $BSL_q(2)$ in [11]) is isomorphic to the algebra of $U_q(sl_2)$ [26]. This also provided of course plenty of representations of the q -Minkowski space algebra by pulling back the usual finite-dimensional, q -boson or other favourite representations of the latter. A further application of braided-Lie algebras is in [75].

A Transmutation

Braided groups were first constructed (by the author) using a process of *transmutation* which turns any quantum group containing a strict subquantum group (with universal R-matrix) into a braided one[12][13][17]. We give here an introduction to this theory. As in Section 6, we require the reader to be more familiar with advanced aspects of quantum group theory[60]. Of course, it is not necessary to know this construction if one just wants to work with braided groups already given to you, which is the line we have taken above.

The idea of transmutation is that the type of an algebraic object (the kind of object it is) is to some extent a matter of choice. As an example, we could start with a bosonic object (an ordinary quantum group) and consider the collection or category of things on which it (co)acts. This category might well be equivalent to some other category, such as the category of braided-coactions of a braided group. By applying braided-Tambara-Krein reconstruction we could reconstruct that braided-group[17]. The same principle applies quite generally, whenever we have a concept of ‘representation’ powerful enough to reconstruct the object being represented. We start with one type of object, take its representations, identify that category as equivalent to another category of representations of some other type of object, and reconstruct it. I call this principle ‘transmutation’ because it changes the flavour of the object. It is a kind of Fourier transform technique for mathematical concepts.

In our example, this idea becomes the following theorem[17][25]. If $f : A \rightarrow A_1$ is a Hopf algebra homomorphism between two quantum groups, where A_1 is dual-quasitriangular in the sense that there is a universal R-matrix functional $\mathcal{Z} : A_1 \otimes A_1 \rightarrow \mathbb{C}$ obeying (143) as in Section 6, then A can be transmuted by f into a braided group B as follows. It has the same linear space and coproduct as A but a new product, antipode and braiding[17][25]

$$\begin{aligned} \underline{a} \cdot b &= a_{(2)} b_{(2)} \mathcal{Z}(f((Sa_{(1)})a_{(3)}) \otimes f((Sb_{(1)})b_{(3)})), \quad \underline{S}a = Sa_{(2)} \mathcal{Z}(f((S^2a_{(3)})Sa_{(1)}) \otimes f(a_{(4)})) \\ \Psi(a \otimes b) &= b_{(2)} \otimes a_{(2)} \mathcal{Z}(f((Sa_{(1)})a_{(3)}) \otimes f((Sb_{(1)})b_{(3)})). \end{aligned} \quad (166)$$

We are underlining the braided product and antipode here to distinguish them from the quantum group ones for A in terms of which they are given. This braided group lives in the braided category of objects covariant under A_1 , by the coaction

$$b \mapsto b_{(2)} \otimes f((Sb_{(1)})b_{(3)}) \quad (167)$$

which induces the braiding Ψ via (145).

If A has its own dual-quasitriangular structure $\mathcal{R} : A \otimes A \rightarrow \mathbb{C}$ obeying (143) then the product of B is braided-quantum-commutative in the sense

$$\begin{aligned} \mathcal{Z}(f(b_{(1)}) \otimes f(a_{(1)})) \mathcal{R}(a_{(2)} \otimes b_{(2)}) \mathcal{Z}(f(Sa_{(3)}) \otimes f(b_{(3)})) a_{(4)} \mathcal{Z}(f(a_{(5)}) \otimes f(b_{(4)})) \cdot b_{(5)} \\ = b_{(1)} \mathcal{Z}(f(b_{(2)}) \otimes f(a_{(1)})) \cdot a_{(2)} \mathcal{R}(a_{(3)} \otimes b_{(3)}) \end{aligned} \quad (168)$$

as one verifies easily from the definition of its product in (166) and the axioms (143) for \mathcal{Z} and \mathcal{R} . This expresses the theorem in [17] that B in this case is a braided-quantum group with braided-universal R-matrix functional, the above being the braided version of the second of (143).

Finally, all of this applies just as well at the bialgebra level, if A does not have an antipode. Just replace $f \circ S$ by f^{-1} , the convolution-inverse of f which we can suppose instead[17].

This is the general theory of transmutation in function algebra form. We also gave it in enveloping algebra form[13]. As an immediate example, suppose there is a quantum group homomorphism $f : A(R) \rightarrow A(Z)$ sending t^i_j to the matrix generator f^i_j say of $A(Z)$. The condition for this is $R\mathbf{f}_1\mathbf{f}_2 = \mathbf{f}_2\mathbf{f}_1R$ which is just the same as our covariance assumption (136). It implies and is essentially equivalent[19] to the matrix condition

$$R_{12}Z_{13}Z_{23} = Z_{23}Z_{13}R_{12}, \quad R_{23}Z_{13}Z_{12} = Z_{12}Z_{13}R_{23} \quad (169)$$

as explained in Section 6 when deriving (4). From the transmutation formula (166) and (144) we obtain a braided group $B(R, Z)$ with product cf.[27]

$$u^i_j = t^i_j, \quad u^i_j u^k_l = t^a_b t^d_l Z^i_c \tilde{Z}^b_j{}^k{}_c, \quad i.e., \quad \mathbf{u} = \mathbf{t}, \quad \mathbf{u}_1 Z \mathbf{u}_2 = Z \mathbf{t}_1 \mathbf{t}_2, \quad etc. \quad (170)$$

where we write the generators of $B(R, Z)$ as \mathbf{u} and use their product \cdot , while on the right we use the original product of $A(R)$. The braiding comes out as in (30) with Z in place of R and corresponding braid-statistics $Z^{-1}\mathbf{u}'_1 Z \mathbf{u}_2 = \mathbf{u}_2 Z^{-1}\mathbf{u}'_1 Z$ for braided matrix multiplication. The braided-quantum-commutativity (168) becomes the relations

$$Z_{21} R Z^{-1} \mathbf{u}_1 Z \mathbf{u}_2 = \mathbf{u}_2 Z_{21} \mathbf{u}_1 R \quad (171)$$

which is also immediate by transmutation (170) of the FRT relations of $A(R)$ as $RZ^{-1}\mathbf{u}_1 Z \mathbf{u}_2 = R\mathbf{t}_1 \mathbf{t}_2 = \mathbf{t}_2 \mathbf{t}_1 R = Z_{21}^{-1} \mathbf{u}_2 Z_{21} \mathbf{u}_1 R$. We assume in the derivation that Z is regular so that in principle one has a quantum group with antipode \mathbf{f}^{-1} , but this vanishes from the final formulae using (146). We gave these explicit formulae in [11][27] for the case $B(R) \equiv B(R, R)$, where we checked everything directly from the matrix data as a self-contained example of a braided group. This case was emphasised because the abstract braided universal R-matrix functional is the ratio of the universal R-matrix functionals of $A(R)$, $A(Z)$ and therefore trivial when $R = Z$, i.e. $B(R)$ has been made totally 'braided commutative'. In this extreme the quantum non-commutativity of $A(R)$ is fully traded for braid-statistics of $B(R)$ as a 'classical' but braided matrix. Another extreme is $B(R, \text{id}) = A(R)$. Other cases of $B(R, Z)$ are intermediate between these extremes but their derivation and the direct check that they form a braided group follows in the same way as for $B(R)$. [15],[76] seem to be the first to explicitly study such algebras $B(R, Z)$ and check them directly from (171), (169). More importantly, Hlavaty obtained some interesting examples, see [15]. One can obtain further examples in the context of quantum principle bundles[2] and quantum homogeneous spaces, where homomorphisms between two quantum groups, which are all we need, abound.

The transmutation theory tells of course not only that $B(R, Z)$ is a braided group, but also that the representations of $A(R)$ become automatically Z -braided representations of $B(R, Z)$. This is the essence of transmutation, as explained above. To see how it works in the matrix setting, transmutation tells us that whenever our braided covectors x_i from Section 3.1 are $A(R)$ -covariant under (139) the same linear map $\mathbf{x} \rightarrow \mathbf{x}\mathbf{u}$ becomes a Z -braided coaction of $B(R, Z)$. The $A(R)$ -covariance becomes through the map f an $A(Z)$ -covariance, and the Z -braiding of the x_i with anything else is induced by this, e.g., $\Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes \mathbf{x}_1 Z(f(\mathbf{t}_1) \otimes f(\mathbf{t}_2)) = \mathbf{x}_2 \otimes \mathbf{x}_1 Z$ and likewise $\Psi(\mathbf{u}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 Z^{-1} \otimes \mathbf{u}_1 Z$, being derived in the same way as we derived (39)–(40) in Section 6.1. The braided-covariance under $B(R, Z)$ has to work by the general theory above, but one can check it directly as $\mathbf{x}_1 \mathbf{u}'_1 \mathbf{x}_2 \mathbf{u}'_2 = \mathbf{x}_1 \mathbf{x}_2 Z^{-1} \mathbf{u}'_1 Z \mathbf{u}'_2 = \mathbf{x}_2 \mathbf{x}_1 R' Z^{-1} \mathbf{u}'_1 Z \mathbf{u}'_2 = \mathbf{x}_2 \mathbf{x}_1 Z_{21}^{-1} \mathbf{u}'_2 Z_{21} \mathbf{u}'_1 R' = \mathbf{x}_2 \mathbf{u}'_2 \mathbf{x}_1 \mathbf{u}'_1 R'$ where the third equality is the covariance assumption (136) in terms of \mathbf{u} . Recall that this in turn implies and is essentially implied by (4). The diagonal case $Z = R$ of this theory is how we obtained the braided coactions of $B(R)$ in [27] as described in Section 4.1 above. Also, if \mathbf{t} has an antipode then so does \mathbf{u} , which is how to obtain the braided antipode on $BSL_q(2)$ in [11] from that of $SL_q(2)$. Indeed, all constructions for quantum groups have braided parallels because of transmutation. Details of the above and some further results about $B(R, Z)$ from the point of view of transmutation will appear in [77].

To give some other simple examples of transmutation, we let G be an Abelian group, $\mathbb{C}G$ its group algebra and β a bicharacter on G , i.e. a function on $G \times G$ such that $\beta(g, \cdot)$ and $\beta(\cdot, g)$ are multiplicative for each $g \in G$. We know from [68] that any bicharacter on G defines a (dual)-quasitriangular structure $\mathcal{Z}(g, h) = \beta(g, h)$ extended linearly. So we have a dual quantum group. The idea to use bicharacters in this was introduced in [14][68] as a quasitriangular structure on the Hopf algebra $\mathbb{C}(G)$, which is equivalent. Transmutation was also covered there and we are just repeating it in the dual form for convenience of the reader. So any quantum group A mapping onto $\mathbb{C}G$ gets transmuted to a quantum braided group B with structure from (166).

Example A.1 cf[41] Let R be a matrix solution of the QYBE and let its indices $i = 1, \dots, n$ be assigned a degree $|i| \in G$, where G is an Abelian group and such that R has G -degree zero in the sense

$$R^i_j{}^k_l = 0 \quad \text{for all } |i| \cdot |k| \neq |j| \cdot |l|.$$

Then for any bicharacter β the quantum matrices $A(R)$ have a corresponding transmutation $B(R, \beta)$ with braided group structure

$$\begin{aligned} \beta(i, k)\beta(j, l)\underline{R}^i_a{}^k_b u^a_j u^b_l &= \beta(j, l)\beta(b, i)u^k_b u^i_a \underline{R}^a_j{}^b_l; & \underline{R}^i_j{}^k_l &= \underline{R}^k_j{}^i_l \beta(j, l)^{-1} \\ \beta(i, j) &\equiv \beta(|i|, |j|), & \underline{\Delta} \mathbf{u} &= \mathbf{u} \otimes \mathbf{u}, \quad \underline{\epsilon} \mathbf{u} = \text{id}, & \Psi(u^i_j \otimes u^k_l) &= u^i_l \otimes u^j_k \frac{\beta(i, k)\beta(j, l)}{\beta(j, k)\beta(i, l)}. \end{aligned}$$

Proof This is immediate as an application of transmutation. The choice of G -degree $|i|$ and the degree-zero condition give us precisely a homomorphism $A(R) \rightarrow \mathbb{C}G$ by $f(t^i_j) = \delta^i_j |i|$. It clearly maps the coproducts in the sense $(f \otimes f) \circ \Delta = \Delta \circ f$ where $\Delta g = g \otimes g$ and $\Delta \mathbf{t} = \mathbf{t} \otimes \mathbf{t}$, as required for a bialgebra homomorphism. The convolution-inverse is $f^{-1}(t^i_j) = Sf(t^i_j) = \delta^i_j |i|^{-1}$ using the group-inverse. The new product from (166) is just

$$u^i_j u^k_l = t^a_b t^c_l \mathcal{Z}(\delta^i_a |i|^{-1} \delta^b_j |j| \otimes \delta^k_c |k|^{-1}) = t^i_j t^k_l \beta(|i|^{-1} |j|, |k|^{-1}) = t^i_j t^k_l \frac{\beta(i, k)}{\beta(j, k)}. \quad (172)$$

The braided-quantum commutation relations (168) reduce to the ones stated, where we have written them in terms of a matrix \underline{R} obeying a β -version of the QYBE. The braided group lives in the category of $\mathbb{C}G$ -covariant algebras with coaction from (167)

$$u^i_j \mapsto u^a_b \otimes f^{-1}(t^i_a) f(t^b_j) = u^i_j \otimes |i|^{-1} \cdot |j| \quad (173)$$

which then induces the braiding $\Psi(u^i_j \otimes u^k_l) = u^k_l \otimes u^i_j \beta(|i|^{-1} |j|, |k|^{-1} |l|)$ as shown. We also have all the theory of comodules too, automatically. Thus our braided covectors x_i and vectors v^i from Section 3.1 have the coaction

$$x_i \mapsto x_a \otimes f(t^a_i) = x_i \otimes |i|, \quad v^i \mapsto v^a \otimes f^{-1}(t^i_a) = v^i \otimes |i|^{-1} \quad (174)$$

and with the induced braiding $\Psi(u^i_j \otimes x_k) = x_k \otimes u^i_j \beta(|i|^{-1} |j|, |k|) = x_k \otimes u^i_j \frac{\beta(j, k)}{\beta(i, k)}$ etc. Their usual $A(R)$ -covariance becomes a braided-covariance under $\mathbf{x} \rightarrow \mathbf{xu}$ etc., provided we remember these induced braid statistics. Note also that if \mathbf{t} has an antipode (if we work with a Hopf algebra obtained from $A(R)$) then the corresponding \mathbf{u} has a braided antipode (166) computed in the present setting. \square

These simplest braided-quantum groups where the braiding is given by a \mathbb{C} -number bicharacter β are called \mathbb{C} -statistical braided groups in [68]. Their construction is an elementary application of the author's transmutation theory. On the other hand, to find interesting examples and applications is rather more challenging. The very simplest example is the super case where $G = \mathbb{Z}_2$ and $\beta = \pm 1$ (this is the quantum group \mathbb{Z}'_2 which is the hidden covariance of supersymmetry[13] as mentioned in Section 6.1). The above transmutation in this case becomes

the process of *superisation* studied by the author and M.J. Rodriguez-Plaza in [41] (in an equivalent form). The next simplest is $G = \mathbb{Z}_N$ and $\beta(g, h) = e^{\frac{2\pi i gh}{N}}$ in an additive notation [14][16][77]. Here [41] and [14] use transmutation in the enveloping form while [77] uses the dual form as above. Further details of Example A.1, further results about $B(R, Z)$ and some concrete examples will appear in [77]. Another choice is $G = \mathbb{Z}^n$ or \mathbb{C}^n and $\beta(\vec{\phi}, \vec{\psi}) = e^{i(\vec{\phi}, \vec{\psi})}$ induced by an arbitrary non-degenerate bilinear form, see [68, Sec. 3] where braided groups of such \mathbb{C} -statistics were introduced and studied from a bosonisation point of view as a novel approach to quantisation of free fields. Following exactly our transmutation ideas from [11][27] there appeared recently in [78] some explicit examples $M_{q,\mu}(n)$ of the $G = \mathbb{Z}^n$ type obtained by transmutation of the usual quantum matrices $M_q(n)$. The underlying mathematics is not new (in view of the above transmutation theory) but these examples are nevertheless interesting because of a connection with statistical mechanical models in physics[78]. Finally, we note that braided-quantum groups have as rich a theory as usual quantum groups on account of their braided-universal R-matrix, as developed in [13] by diagrammatic means. Recent results using such means are in [79].

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