Chapter 1

Banach algebras

Whilst we are primarily concerned with C^*-algebras, we shall begin with a study of a more general class of algebras, namely, Banach algebras. These are of interest in their own right and, in any case, many of the concepts introduced in their analysis are needed for that of C^*-algebras. Furthermore, some feeling for the kind of behaviour that can occur in various Banach algebras helps one to appreciate how well-behaved C^*-algebras are.

Definition 1.1. A Banach algebra is a complex Banach space A together with an associative and distributive multiplication such that

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

and

$$\|ab\| \leq \|a\|\,\|b\|$$

for all $a, b \in A$, $\lambda \in \mathbb{C}$.

For any $x, x', y, y' \in A$, we have

$$\|xy - x'y'\| = \|x(y - y') + (x - x')y\| \leq \|x\|\,\|y - y'\| + \|x - x'\|\,\|y'\|$$

and so we see that multiplication is jointly continuous.

The algebra A is said to be commutative (or abelian) if $ab = ba$ for all a,b in A, and A is said to be unital if it possesses a (multiplicative) unit (—this is also called an identity). Note that if A has an identity, then it is unique: since if $\mathbb{1}$ and $\mathbb{1}'$ are units, then $\mathbb{1} = \mathbb{1}\mathbb{1}' = \mathbb{1}'$.

Example 1.2. If E is a complex Banach space, then $\mathcal{B}(E)$, the set of bounded linear operators on E is a unital Banach algebra when equipped with the usual linear structure and operator norm.

If $\mathbb{1}$ denotes the unit in the unital Banach algebra A, then $\mathbb{1} = \mathbb{1}^2$ and so we have $\|\mathbb{1}\| \leq \|\mathbb{1}\|\,\|\mathbb{1}\|$, which implies that $\|\mathbb{1}\| \geq 1$.

Lemma 1.3. *Let A be a Banach algebra with identity $\mathbb{1}$. Then there is a norm $|||\cdot|||$ on A, equivalent to the original norm, such that $(A, |||\cdot|||)$ is a unital Banach algebra with $|||\mathbb{1}||| = 1$.*

Proof. For each $x \in A$, let L_x denote the linear operator $L_x : y \mapsto xy \in A$, $y \in A$. Then if $L_x = L_{x'}$, it follows that $L_x \mathbb{1} = L_{x'} \mathbb{1}$ and so $x = x'$. Hence $x \mapsto L_x$ is an injective map from A into the set of linear operators on A. Now,

$$\|L_x y\| = \|xy\| \leq \|x\| \, \|y\| \,, \text{ for } y \in A$$

which implies that L_x is bounded, and $\|L_x\| \leq \|x\|$. Put $|||x||| = \|L_x\|$. Then we have just shown that $|||x||| \leq \|x\|$, for any $x \in A$.

On the other hand,

$$\begin{aligned} |||x||| = \|L_x\| &= \sup\{\|L_x y\| : \|y\| \leq 1\} \\ &= \sup\{\|xy\| : \|y\| \leq 1\} \\ &\geq \|xy'\|, \qquad \text{where } y' = \frac{\mathbb{1}}{\|\mathbb{1}\|} \\ &= \frac{\|x\|}{\|\mathbb{1}\|}. \end{aligned}$$

Hence, $\|x\|/\|\mathbb{1}\| \leq |||x||| \leq \|x\|$, for all $x \in A$, which shows that the two norms $|||\cdot|||$ and $\|\cdot\|$ are equivalent.

Moreover, for any $x, y \in A$,

$$\begin{aligned} |||xy||| &= \|L_{xy}\| \\ &= \|L_x L_y\| \\ &\leq \|L_x\| \, \|L_y\| \\ &= |||x||| \, |||y||| \end{aligned}$$

and so A with norm $|||\cdot|||$ is a Banach algebra. To complete the proof, we have $|||\mathbb{1}||| = \|L_{\mathbb{1}}\| = 1$. ∎

This lemma allows us to assume that the unit of a unital Banach algebra has norm 1. In fact, this is often taken as part of the definition of a unital Banach algebra. If A does not have a unit, then we can "adjoin" one as follows.

Lemma 1.4. *A Banach algebra A without a unit can be embedded into a unital Banach algebra A_I as an ideal of codimension one.*

Proof. Let $A_I = A \oplus \mathbb{C}$ as a linear space, and define a multiplication in A_I by

$$(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu).$$

It is easily checked that this is associative and distributive. Moreover, the element $(0,1)$ is a unit for this multiplication:

$$(x,\lambda)(0,1) = (x0 + x + \lambda 0, \lambda 1) = (x,\lambda) = (0,1)(x,\lambda).$$

Put $\|(x,\lambda)\| = \|x\| + |\lambda|$. Then A_I is a Banach space when equipped with this norm. Furthermore,

$$\begin{aligned}\|(x,\lambda)(y,\mu)\| &= \|(xy + \mu x + \lambda y, \lambda\mu)\| \\ &= \|xy + \mu x + \lambda y\| + |\lambda\mu| \\ &\le \|x\|\|y\| + |\mu|\|x\| + |\lambda|\| + \||\lambda||\mu| \\ &= (\|x\| + |\lambda|)(\|y\| + |\mu|) \\ &= \|(x,\lambda)\|\,\|(y,\mu)\|\,.\end{aligned}$$

Hence A_I is a Banach algebra with unit. We may identify A with the ideal $\{(x,0) : x \in A\}$ in A_I via the isometric isomorphism $x \mapsto (x,0)$. ■

We write (x,λ) as $(x,\lambda) = x + \lambda\mathbb{1} \in A_I$. (Compare this with complex numbers $a + ib \leftrightarrow (a,b)$.) Note that $\|\mathbb{1}\| = 1$. We will see, later, that an analogous result holds for C^*-algebras, but more care has to be taken regarding the norm.

Examples 1.5.

1. Consider $\mathcal{C}([0,1])$, the Banach space of continuous complex-valued functions defined on the interval $[0,1]$ equipped with the sup-norm, namely, $\|f\| = \sup_{s\in[0,1]} |f(s)|$, and with multiplication defined pointwise:

   $$(fg)(s) = f(s)\,g(s)\,, \quad \text{for } s \in [0,1].$$

   Then $\mathcal{C}([0,1])$ is a commutative unital Banach algebra; the constant function 1 is the unit element.

2. As above, but replace $[0,1]$ by any compact topological space.

3. Let D denote the closed unit disc in $\mathbb{C}$, and let A denote the set of continuous complex-valued functions on D which are analytic in the interior of D. Equip A with pointwise addition and multiplication and the norm

   $$\|f\| = \sup\{|f(z)| : z \in \partial D\}$$

   where ∂D is the boundary of D, that is, the unit circle. (That this is, indeed, a norm follows from the maximum modulus principle.) Then A is complete, and so is a (commutative) unital Banach algebra. A is called the disc algebra.

4. Let A be the Banach space $\ell^1(\mathbb{Z})$ and define xy by $(xy)_n = \sum_m x_m y_{n-m}$ for $x = (x_n)$ and $y = (y_n)$ in A. Then

$$\begin{aligned}\sum_n |(xy)_n| &\le \sum_n \sum_m |x_m||y_{n-m}| \\ &= \sum_m |x_m| \sum_n |y_{n-m}| \\ &= \sum_m |x_m| \, \|y\| \\ &= \|x\| \, \|y\| \, .\end{aligned}$$

Thus $xy \in \ell^1(\mathbb{Z})$ and so A is a Banach algebra. Furthermore, A has a unit given by $(x_n) = (\delta_{0n}) = (\ldots, 0, 0, 1, 0, 0, \ldots)$ where the 1 appears in the 0^{th} position.

Definition 1.6. An element x in a unital Banach algebra A is said to be invertible (or non-singular) in A if there is some $z \in A$ such that $xz = zx = \mathbb{1}$. Note that if such a z exists, then it is unique; if $z'x = xz' = \mathbb{1}$, then $z = z\mathbb{1} = zxz' = \mathbb{1}z' = z'$. z is called the inverse of x, and is written x^{-1}, as usual. Evidently, the set of invertible elements forms a group. Non-invertible elements are also called singular.

Proposition 1.7. *The set $\mathcal{G}(A)$ of invertible elements in a unital Banach algebra A is open in A, and the inverse operation $x \mapsto x^{-1}$ is a continuous map from $\mathcal{G}(A)$ to $\mathcal{G}(A)$.*

Proof. First let $y \in A$ with $\|y\| < 1$, and put $s_n = \sum_{k=0}^{n} y^k$, $n \in \mathbb{N}$. Then (s_n) is a Cauchy sequence in A and so converges, since A is complete. Let w denote its limit; $w = \sum_{k=0}^{\infty} y^k$. We claim that w is the inverse of $\mathbb{1} - y$. Indeed, we have

$$\begin{aligned}(\mathbb{1} - y)w &= \lim_n (\mathbb{1} - y)s_n \\ &= \lim_n (\mathbb{1} - y^{n+1}) = \mathbb{1}\end{aligned}$$

and

$$\begin{aligned}w(\mathbb{1} - y) &= \lim_n s_n(\mathbb{1} - y) \\ &= \lim_n (\mathbb{1} - y^{n+1}) = \mathbb{1}\end{aligned}$$

which establishes the claim. Hence, if $x \in A$ with $\|\mathbb{1} - x\| < 1$, then writing $x = \mathbb{1} - (\mathbb{1} - x)$ and arguing as above (with $y = \mathbb{1} - x$), we see that x is invertible and its inverse x^{-1} is given by the convergent series $\sum_{k=0}^{\infty} (\mathbb{1} - x)^k$.

Let $x_0 \in \mathcal{G}(A)$. Then for any $x \in A$, we have $x = x_0 x_0^{-1} x$. Now,

$$\|\mathbb{1} - x_0^{-1}x\| = \|x_0^{-1}(x_0 - x)\| \le \|x_0^{-1}\| \, \|x_0 - x\|,$$

and so we conclude that if $\|x - x_0\| < \|x_0^{-1}\|^{-1}$ then $x_0^{-1}x$ is invertible with inverse given by

$$(x_0^{-1}x)^{-1} = \left(\sum_{k=0}^{\infty} y^k\right)$$

with $y = (\mathbb{1} - x_0^{-1}x)$. Hence x is invertible and

$$x^{-1} = \sum_{k=0}^{\infty}[x_0^{-1}(x_0 - x)]^k x_0^{-1}\,.$$

To see that $x \mapsto x^{-1}$ is continuous on $\mathcal{G}(A)$, suppose that $x_0 \in \mathcal{G}(A)$ and that (x_n) is a sequence in $\mathcal{G}(A)$ such that $x_n \to x_0$ as $n \to \infty$. Then for all sufficiently large n, $\|x_n - x_0\| < \|x_0^{-1}\|^{-1}$ and so

$$\begin{aligned} \|x_n^{-1} - x_0^{-1}\| &= \left\|\sum_{k=1}^{\infty}\left[x_0^{-1}(x_0 - x_n)\right]^k x_0^{-1}\right\| \\ &\le \sum_{k=1}^{\infty}\left[\|x_0^{-1}\|\|x_0 - x_n\|\right]^k \|x_0^{-1}\| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. ■

Definition 1.8. Let A be a unital algebra and let $x \in A$. The spectrum of x is the subset $\sigma_A(x)$ of $\mathbb{C}$ given by

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : x - \lambda\mathbb{1} \notin \mathcal{G}(A)\}.$$

The resolvent set $\rho_A(x)$ of x is the complement of the spectrum of x;

$$\rho_A(x) = \mathbb{C} \setminus \sigma_A(x).$$

The spectral radius $r_A(x)$ of an element $x \in A$ is defined as

$$r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$$

provided $\sigma_A(x)$ is not empty.

Example 1.9. Let A be the unital Banach algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. For $a \in A$ and $\lambda \in \mathbb{C}$, $a - \lambda\mathbb{1}$ is invertible in A if and only if λ is not an eigenvalue of a. In other words, $\sigma_A(a)$ is just the set of eigenvalues of the matrix a.

Example 1.10. Suppose that A is the commutative unital Banach algebra $\mathcal{C}[0,1]$, and $f \in A$. Then, for $\lambda \in \mathbb{C}$, $f - \lambda\mathbb{1}$ is invertible in A provided f does not take the value λ. Hence $\sigma_A(f)$ is equal to the set of values assumed by f, i.e., $\sigma_A(f) = \operatorname{ran} f$, the range of f.

Example 1.11. A normed algebra is defined in just the same way as a Banach algebra, except that the completeness of the space is no longer required, i.e., the space is merely a normed space rather than a Banach space. However, if A_0 is a normed algebra, then it is not difficult to see that its completion A, say, is in fact a Banach algebra. (To show this, suppose that A_0 is dense in A. Then one shows that the product in A_0 extends (by continuity) to a product on A, and that, when equipped with this product, A is a Banach algebra.)

Let $A_0 = \mathbb{C}[x]$, the algebra of complex polynomials in the indeterminate x, considered as a subalgebra of $A = \mathcal{C}([0,1])$. When equipped with the supremum norm, A_0 is a unital normed algebra whose completion, by Weierstrass' theorem, is just the unital Banach algebra A.

Let $f \in A_0$. For any $\lambda \in \mathbb{C}$, $f - \lambda\mathbb{1}$ fails to be invertible in A_0, unless f is a constant (i.e., a polynomial of degree zero) not equal to λ. That is, $\sigma_{A_0}(f) = \mathbb{C}$ whenever f is not a constant, but $\sigma_{A_0}(f) = \{\alpha\}$ when f is the constant α. In particular, an element of A_0 is invertible in A_0 if and only if it is a non-zero constant. Hence $\mathcal{G}(A_0) = \{f \in \mathbb{C}[x] : f = \alpha,\ \alpha \neq 0\}$. Thus we see that $\mathcal{G}(A_0)$ is not an open set in A_0. Indeed, $\mathbb{1}$, the constant polynomial 1, belongs to $\mathcal{G}(A_0)$, of course, but for any $\varepsilon > 0$, the polynomial $p(x) = 1 + \frac{1}{2}\varepsilon x$ satisfies $\|\mathbb{1} - p\| < \varepsilon$ and $p \notin A_0$. Thus, every open set containing $\mathbb{1}$ also contains singular elements of A_0.

Example 1.12. Let A denote the algebra of meromorphic functions and let B denote the algebra of entire functions. Clearly B is a subalgebra of A, and both A and B are unital; the unit being the constant function equal to 1 on $\mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $z \mapsto f(z) = z$, $z \in \mathbb{C}$. For any $\lambda \in \mathbb{C}$, the function $z \mapsto (z - \lambda)^{-1}$ is meromorphic, i.e., $f - \lambda\mathbb{1}$ is invertible in A. On the other hand, the function $z \mapsto (z - \lambda)^{-1}$ is not entire for any $\lambda \in \mathbb{C}$. Thus we see that $\sigma_A(f) = \mathbb{C}$ but $\sigma_B(f) = \varnothing$.

Theorem 1.13. *For any x in a unital Banach algebra A, the spectrum $\sigma_A(x)$ is a non-empty compact subset of $\mathbb{C}$ with $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$.*

Proof. For given $x \in A$, $x - \lambda\mathbb{1}$ is invertible whenever $|\lambda| > \|x\|$. Indeed, for any such λ, $(x - \lambda\mathbb{1}) = -\lambda\big(\mathbb{1} - x/\lambda\big)$ has inverse given by a convergent series expansion in powers of x/λ. Hence $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$, as claimed. In particular, this shows that $\sigma_A(x)$ is bounded.

Let f be the map $f : \lambda \mapsto x - \lambda\mathbb{1}$. Then $\lambda \notin \sigma_A(x)$ if and only if $x - \lambda\mathbb{1} \in \mathcal{G}(A)$, i.e., if and only if $\lambda \in f^{-1}(\mathcal{G}(A))$. It follows that $\mathbb{C} \setminus \sigma_A(x) = f^{-1}(\mathcal{G}(A))$. But it is clear that $f : \mathbb{C} \to A$ is continuous and so $\mathcal{G}(A)$ open implies that $f^{-1}(\mathcal{G}(A)) = \mathbb{C} \setminus \sigma_A(x)$ is open and so $\sigma_A(x)$ is closed. Hence $\sigma_A(x)$ is a closed, bounded subset of $\mathbb{C}$ and therefore compact.

We must show that $\sigma_A(x)$ is non-empty. Indeed, suppose the contrary, $\sigma_A(x) = \varnothing$. Then $(x - \lambda\mathbb{1})$ is invertible for all $\lambda \in \mathbb{C}$. We claim that the map $\lambda \mapsto (x - \lambda \mathbb{1})^{-1}$ is differentiable, with derivative $(x - \lambda \mathbb{1})^{-2}$, that is, we claim that

$$\frac{(x - (\lambda + \zeta)\mathbb{1})^{-1} - (x - \lambda \mathbb{1})^{-1}}{\zeta} \to (x - \lambda \mathbb{1})^{-2}$$

in A as $\zeta \to 0$ in $\mathbb{C}$ (with $\zeta \neq 0$).

To see this, note that

$$\begin{aligned}(x - \alpha \mathbb{1})^{-1} - (x - \beta \mathbb{1})^{-1} &= (x - \alpha \mathbb{1})^{-1}\big\{(x - \beta \mathbb{1}) - (x - \alpha \mathbb{1})\big\}(x - \beta \mathbb{1})^{-1} \\ &= (x - \alpha \mathbb{1})^{-1}(\alpha - \beta)(x - \beta \mathbb{1})^{-1}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{(x - (\lambda + \zeta)\mathbb{1})^{-1} - (x - \lambda \mathbb{1})^{-1}}{\zeta} &= \frac{(x - (\lambda + \zeta)\mathbb{1})^{-1}\,\zeta\,(x - \lambda \mathbb{1})^{-1}}{\zeta} \\ &\to (x - \lambda \mathbb{1})^{-2}\end{aligned}$$

in A as $\zeta \to 0$, since $x - (\lambda + \zeta) \to x - \lambda$ and the taking of the inverse is continuous. This proves the claim.

Now let $\varphi \in A^*$, the dual space of A (the space of continuous linear functionals on A). Then the map $\lambda \mapsto \varphi((x - \lambda \mathbb{1})^{-1})$ is everywhere differentiable in $\mathbb{C}$, that is, $g(\lambda) \equiv \varphi((x - \lambda \mathbb{1})^{-1})$ is an entire function. For all sufficiently large $|\lambda|$, we may write

$$\begin{aligned}(x - \lambda \mathbb{1})^{-1} &= (-\lambda)^{-1}(\mathbb{1} - \lambda^{-1}x)^{-1} \\ &= (-\lambda)^{-1}\sum_{n=0}^{\infty}(\lambda^{-1}x)^n \qquad (|\lambda| > \|x\|).\end{aligned}$$

Clearly, the right hand side converges to 0 as $|\lambda| \to \infty$, and so the same is true of the left hand side, thus $g(\lambda) \to 0$ as $|\lambda| \to \infty$. By Liouville's theorem, we deduce that g is identically zero on $\mathbb{C}$. But then we have $\varphi((x - \lambda \mathbb{1})^{-1}) = 0$ for all $\varphi \in A^*$, which implies that $(x - \lambda \mathbb{1})^{-1} = 0$. This is impossible because $(x - \lambda \mathbb{1})^{-1}(x - \lambda \mathbb{1}) = \mathbb{1} \neq 0$. We conclude that $\sigma_A(x) \neq \varnothing$. ■

Proposition 1.14. *Let A be a unital Banach algebra, and let $a \in A$.*

(i) *$\sigma_A(p(a)) = p(\sigma_A(a))$ for any complex polynomial p.*

(ii) *If a is invertible, $\sigma_A(a^{-1}) = \sigma_A(a)^{-1}$.*

Proof. (i) Suppose that p has degree $n \geq 1$. For any $\mu \in \mathbb{C}$, let $\lambda_1, \dots, \lambda_n$ be the n complex roots of the polynomial $p(\,\cdot\,) - \mu$. Then, for any $z \in \mathbb{C}$, $p(z) - \mu = \alpha(z - \lambda_1)\dots(z - \lambda_n)$, for some non-zero $\alpha \in \mathbb{C}$ and so $p(a) - \mu \mathbb{1} = \alpha(a - \lambda_1 \mathbb{1})\dots(a - \lambda_n \mathbb{1})$.

Now, if $a_1, \dots, a_n$ are mutually commuting elements of A (i.e., $a_i a_j = a_j a_i$, for any $1 \le i, j \le n$), then the product $a_1 \dots a_n$ is invertible if and only if each a_i is invertible. (If each a_i is invertible, then their product is invertible, regardless of the commutativity assumption. Conversely, if the product is invertible, then, for example, we have

$$\begin{aligned} a_2 a_1 a_3 \dots a_n (a_1 \dots a_n)^{-1} &= a_1 \dots a_n (a_1 \dots a_n)^{-1} = \mathbb{1} \\ &= (a_1 \dots a_n)^{-1} a_1 \dots a_n \\ &= (a_1 \dots a_n)^{-1} a_1 a_3 \dots a_n a_2, \end{aligned}$$

which shows that a_2 is invertible. Similarly, it is easy to see that any a_i is invertible.) Suppose that $\mu \in \sigma_A(p(a))$. Then $p(a) - \mu\mathbb{1}$ is singular and so therefore is $a - \lambda_i \mathbb{1}$, for some $1 \le i \le n$. That is, $\lambda_i \in \sigma_A(a)$. But $p(\lambda_i) = \mu$ which shows that $\mu \in p(\sigma_A(a))$. Conversely, suppose that $\lambda \in \sigma_A(a)$ and let $\mu = p(\lambda)$. Then, with the notation above, it follows that $\lambda = \lambda_i$ for some $1 \le i \le n$, and that $p(a) - \mu\mathbb{1}$ is singular. Thus $p(\lambda) \in \sigma_A(p(a))$, and the result follows.

(ii) Suppose that $a \in A$ is invertible, i.e., $0 \notin \sigma_A(a)$. For any $\lambda \in \mathbb{C}$, $\lambda \ne 0$, we have

$$a - \lambda\mathbb{1} = a(\mathbb{1} - \lambda a^{-1}) = a\lambda(\lambda^{-1}\mathbb{1} - a^{-1})$$

which implies that $a - \lambda\mathbb{1}$ is singular if and only if $a^{-1} - \lambda^{-1}\mathbb{1}$ is singular. ■

Suppose that x and y are elements of a unital Banach algebra A such that $\|x\| < 1$ and also $\|y\| < 1$. Then it follows that $\|xy\| \le \|x\|\,\|y\| < 1$, and similarly, $\|yx\| < 1$. We have seen that this implies that $\mathbb{1} - xy$ and $\mathbb{1} - yx$ are both invertible with inverses given, respectively, by $a = (\mathbb{1} - xy)^{-1} = \sum_{n=0}^{\infty} (xy)^n$ and $b = (\mathbb{1} - yx)^{-1} = \sum_{n=0}^{\infty} (yx)^n$. Evidently,

$$\begin{aligned} yax &= y(\mathbb{1} + xy + (xy)^2 + \dots)x \\ &= yx + (yx)^2 + (yx)^3 + \dots \end{aligned}$$

and so

$$\begin{aligned} \mathbb{1} + yax &= \mathbb{1} + yx + (yx)^2 + \dots \\ &= b\,. \end{aligned}$$

Now suppose that x and y are elements of A such that $\mathbb{1} - xy$ is invertible, but otherwise x and y are arbitrary. Let $a = (\mathbb{1} - xy)^{-1}$, and $b = \mathbb{1} + yax$. Is it still true that b is the inverse of $\mathbb{1} - yx$? That this is, indeed, still the case follows by straightforward calculation;

$$\begin{aligned} b(\mathbb{1} - yx) &= (\mathbb{1} + yax)(\mathbb{1} - yx) \\ &= \mathbb{1} - yx + yax - yaxyx \\ &= \mathbb{1} - yx + y\,\underbrace{a(\mathbb{1} - xy)}_{=\mathbb{1}}\,x \\ &= \mathbb{1} - yx + yx \\ &= \mathbb{1}\,. \end{aligned}$$

A similar calculation shows that $(\mathbb{1} - yx)b = \mathbb{1}$. (In fact, this demonstration that $\mathbb{1} + yax$ is the inverse of $\mathbb{1} - yx$ is valid in any ring with unit—only the motivation was carried out in a Banach algebra.) These observations lead directly to the following result.

Proposition 1.15. *For any x, y in a unital Banach algebra A,*

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

Proof. Suppose that $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then, using the observation above, we see that $\lambda\mathbb{1} - xy$ is invertible if and only if $\lambda(\mathbb{1} - xy/\lambda)$ is invertible if and only if $\lambda(\mathbb{1} - yx/\lambda)$ is invertible if and only if $\lambda - yx$ is invertible. Hence $\sigma_A(xy) \setminus \{0\} = \sigma_A(yx) \setminus \{0\}$, and the result follows. ∎

Remark 1.16. It is a consequence of this proposition, that an identity of the form $ab - ba = \mathbb{1}$ cannot possibly hold in any unital Banach algebra. Indeed, suppose the contrary. Then, by the proposition, $\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}$. On the other hand, we know from **??** that $\sigma_A(ab) = \sigma_A(\mathbb{1} + ba) = 1 + \sigma_A(ba)$. Therefore $\sigma_A(ba) \cup \{0\} = \{0\} \cup \{1 + \sigma_A(ba)\}$. This is impossible because $\sigma_A(ba)$ is bounded. (If $\alpha \in \sigma_A(ba)$ with $\operatorname{Re}\alpha \geq 0$, then $1 + \alpha \neq 0$ and belongs to the right hand side of the above equality. So it also belongs to $\sigma_A(ba)$. By induction, we see that $\alpha + n$ belongs to $\sigma_A(ba)$ for any $n \in \mathbb{N}$ which is not possible. On the other hand, if $\alpha \in \sigma_A(ba)$ with $\operatorname{Re}\alpha < 0$, then $\alpha \in 1 + \sigma_A(ba)$ and so $\alpha - 1 \in \sigma_A(ba)$. Again, by induction, it follows that $\alpha - n \in \sigma_A(ba)$ for any $n \in \mathbb{N}$ which, once again, is not possible.)

This result is of great importance in quantum mechanics where the position and momentum of a particle are represented by linear operators q and p acting on a Hilbert space and are supposed to satisfy the Heisenberg canonical commutation relation $pq - qp = \mathbb{1}$. It therefore follows that q and p cannot both be bounded linear operators on the Hilbert space. To salvage this situation, one must consider unbounded operators. (In fact, it turns out (by a result of von Neumann) that both q and p must be unbounded operators.)

Theorem 1.17. (Spectral radius formula) *Let A be a unital Banach algebra and let $x \in A$. Then the limit $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists and satisfies*

$$\lim_{n\to\infty} \|x^n\|^{1/n} = r_A(x) = \inf_n \|x^n\|^{1/n}.$$

Proof. Set
$$\gamma(x) = \inf\{\|x^n\|^{1/n} : n = 1, 2, \dots\}.$$
We shall show that $\|x^n\|^{1/n} \to \gamma(x)$ as $n \to \infty$. Given any $\varepsilon > 0$, let $k \in \mathbb{N}$ be such that
$$\|x^k\|^{1/k} < \gamma(x) + \varepsilon.$$
For any $n \in \mathbb{N}$, write n as $n = \alpha k + \beta$ where $0 \le \beta < k$ and $\alpha, \beta \in \mathbb{Z}^+$. (Note that k is fixed and α, β depend on n.) Then $\beta/n \to 0$ as $n \to \infty$ since β is always less than k. Also,
$$1 = \frac{n}{n} = \frac{\alpha k + \beta}{n} = \frac{\alpha k}{n} + \frac{\beta}{n}$$
and so $\dfrac{\alpha k}{n} \to 1$ as $n \to \infty$, that is, $\dfrac{\alpha}{n} \to \dfrac{1}{k}$, as $n \to \infty$. Now,
$$\begin{aligned}\|x^n\|^{1/n} &= \|(x^k)^\alpha x^\beta\|^{1/n}\\ &\le \|x^k\|^{\alpha/n} \|x\|^{\beta/n}.\end{aligned}$$
and the right hand side converges to $\|x^k\|^{1/k}$, as $n \to \infty$, which is less than $\gamma(x) + \varepsilon$. Hence, for all sufficiently large n,
$$\begin{aligned}\|x^n\|^{1/n} &\le \|x^k\|^{\alpha/n} \|x\|^{\beta/n}\\ &< \gamma(x) + \varepsilon.\end{aligned}$$
On the other hand, $\gamma(x) \le \|x^n\|^{1/n}$ for any $n = 1, 2, \dots$ and so
$$\gamma(x) \le \|x^n\|^{1/n} < \gamma(x) + \varepsilon$$
for all sufficiently large n. Thus $\|x^n\|^{1/n}$ converges to $\gamma(x)$ as $n \to \infty$, i.e., $\lim_n \|x^n\|^{1/n}$ exists and is equal to $\inf_m \|x^m\|^{1/m}$.

We must now show that the above limit is equal to $r_A(x)$. Recall, first, that for any $y \in A$, $\sigma_A(y) \subseteq \{\lambda : |\lambda| \le \|y\|\}$ and so $r_A(y) \le \|y\|$.

By **??**, it follows that $\{\lambda^n : \lambda \in \sigma_A(x)\} = \sigma_A(x^n)$ and so $r_A(x^n) = r_A(x)^n$, for any $n \in \mathbb{N}$. But $r_A(x^n) \le \|x^n\|$ and so we obtain
$$\begin{aligned} & r_A(x)^n = r_A(x^n) \le \|x^n\| \\ \implies \quad & r_A(x) \le \|x^n\|^{1/n}, \quad \text{for all } n, \\ \implies \quad & r_A(x) \le \gamma(x). \end{aligned}$$

We want to show now that $r_A(x) \ge \gamma(x)$. Let $\varphi \in A^*$. Then $g : \lambda \mapsto \varphi((x - \lambda \mathbb{1})^{-1})$ is analytic on $\mathbb{C} \setminus \sigma_A(x)$, and so has a Laurent series expansion on $\{\lambda : |\lambda| > r_A(x)\}$. But for $|\lambda| > \|x\|$, we know that g has the expansion
$$g(\lambda) = \sum_{n=0}^{\infty} \frac{\varphi(x^n)}{\lambda^{n+1}}.$$

This must therefore be absolutely convergent in the region $\{\lambda : |\lambda| > r_A(x)\}$. Fix λ with $|\lambda| > r_A(x)$. Then, in particular, $\varphi(x^n/\lambda^{n+1}) \to 0$ as $n \to \infty$. This holds for any $\varphi \in A^*$, and so, by the uniform boundedness principle, it follows that (x^n/λ^{n+1}) is a bounded sequence in A, that is, there is $\kappa > 0$ such that $\|x^n/\lambda^{n+1}\| \le \kappa$ for all n. Hence $\|x^n\| \le \kappa|\lambda|\,|\lambda^n|$ and so $\|x^n\|^{1/n} \le (\kappa|\lambda|)^{1/n}|\lambda|$. Letting $n \to \infty$ gives $\gamma(x) \le |\lambda|$. This holds for any λ with $|\lambda| > r_A(x)$ and so we deduce that $\gamma(x) \le r_A(x)$.

It follows that $r_A(x) = \gamma(x)$ and the proof is complete. ■

Remark 1.18. We have shown that

$$\inf_n \|x^n\|^{1/n} = \lim_n \|x^n\|^{1/n} = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}.$$

The right hand side is purely algebraic in that it involves (non-)existence of an inverse in A, whereas the left hand side involves the norm, that is, the metric aspects of the algebra. We see here an inter-relationship between purely metric and purely algebraic parts of the theory. By "enlarging" the algebra, the spectrum may change, but the spectral radius will not.

Example 1.19. Let A be the disc algebra. Then each f in A is uniquely determined by its values on the unit circle $\partial D = S^1$. Thus, A can be regarded as a subalgebra of $\mathcal{C}(S^1)$, the Banach algebra of continuous complex-valued functions on the circle S^1. This identification of A in $\mathcal{C}(S^1)$ is also norm preserving (by the maximum modulus principle).

Let $g(z) = z$, for $z \in D$. Then $g \in A$. Evidently $(z - \lambda\mathbb{1})$ fails to be an invertible analytic function on D if and only if $\lambda \in D$. Thus we see that

$$\sigma_A(g) = D = \{\lambda : |\lambda| \le 1\}.$$

On the other hand, considered as an element of $\mathcal{C}(S^1)$, g is the function $g(\theta) = e^{i\theta}$, with the obvious notation. Then $(g - \lambda\mathbb{1})$ fails to be invertible in $\mathcal{C}(S^1)$ if and only if $|\lambda| = 1$. Hence

$$\sigma_{\mathcal{C}(S^1)}(g) = \partial D = \{\lambda : |\lambda| = 1\}.$$

Note that $r_A(g) = r_{\mathcal{C}(S^1)}(g)$ $(= 1)$, as we know should be the case.

Theorem 1.20. (Gelfand-Mazur) *Let A be a unital Banach algebra such that each non-zero element of A is invertible. Then $A \simeq \mathbb{C}$.*

Proof. For any $x \in A$, $\sigma_A(x) \ne \varnothing$ and so there is $\lambda \in \mathbb{C}$ with $x - \lambda\mathbb{1} \notin \mathcal{G}(A)$. But then this must mean that $x - \lambda\mathbb{1} = 0$, that is $x = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$. ■

The next theorem concerns quotient spaces.

Proposition 1.21. *Let X be a normed space, and V a closed linear subspace of X. Then the quotient space X/V is a normed space with respect to the quotient norm defined by*

$$\|\operatorname{cl} x\| = \inf_{v \in V} \|x + v\| = \inf_{x' \sim x} \|x'\|.$$

where $\operatorname{cl} x$ denotes the equivalence class of x in X/V.

Proof. First recall that X/V is the set of equivalence classes in X determined by the relation $x \sim x'$ if and only if $x - x' \in V$. X/V is a linear space when equipped with the obvious operations; $\operatorname{cl} x + \operatorname{cl} y = \operatorname{cl}(x + y)$ and $\lambda \operatorname{cl} x = \operatorname{cl}(\lambda x)$, for $x, y \in A$ and $\lambda \in \mathbb{C}$ (—one can readily check that these operations are well-defined, i.e., independent of the representatives used). We must show that $\|\cdot\|$ is a norm.

If $\operatorname{cl} x = 0$ in X/V, then $x \in V$ and so, taking $v = -x$, we get $\|\operatorname{cl} x\| = \inf_{v \in V} \|x + v\| = 0$. On the other hand, if $\|\operatorname{cl} x\| = 0$, then there is a sequence (v_n) in V such that $\|x + v_n\| \to 0$ as $n \to \infty$. Hence $-v_n \to x$ in X. Since V is closed, we deduce that $x \in V$ and so $\operatorname{cl} x = 0$ in X/V.

For $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $x \in X$,

$$\begin{aligned} \|\operatorname{cl} \lambda x\| &= \inf_{v \in V} \|\lambda x + v\| = \inf_{v' \in V} \|\lambda x + \lambda v'\| \\ &= |\lambda| \inf_{v' \in V} \|x + v'\| = |\lambda| \, \|\operatorname{cl} x\|. \end{aligned}$$

For any x, $y \in X$,

$$\begin{aligned} \|\operatorname{cl} x + \operatorname{cl} y\| &= \inf_{v \in V} \|x + y + v\| \\ &= \inf_{v, w \in V} \|x + y + v + w\| \\ &\le \inf_{v, w \in V} \big(\|x + v\| + \|y + w\|\big) \\ &= \|\operatorname{cl} x\| + \|\operatorname{cl} y\|. \end{aligned}$$

Hence $\|\cdot\|$ is a norm on X/V. ■

If X is a Banach space, then so is X/V, as we shall now show. We shall use the following standard result from Banach space theory.

Proposition 1.22. *Let Y be a normed space. Then Y is complete if and only if it has the following property: if (y_m) is any sequence in Y such that $\sum_{m=1}^{\infty} \|y_m\| < \infty$, then there is $y \in Y$ such that $\sum_{m=1}^{n} y_m \to y$ as $n \to \infty$.*

Proof. If Y is complete and (y_n) satisfies $\sum_{k=1}^{\infty} \|y_k\| < \infty$, then it is clear that $\big(\sum_{k=1}^{n} y_k\big)$ is a Cauchy sequence and hence converges.

Conversely, suppose that Y has the stated property, and let (x_n) be a Cauchy sequence in Y. We construct a subsequence as follows. Let $n_1 \in \mathbb{N}$ be such that $\|x_{n_1} - x_m\| < \frac{1}{2}$ for all $m > n_1$. Now let $n_2 > n_1$ be such that $\|x_{n_2} - x_m\| < \frac{1}{4}$ for all $m > n_2$. Continuing in this way, we obtain a subsequence (x_{n_k}) of (x_n) such that

$$\|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^k}, \quad k = 1, 2, \dots$$

Set $y_k = x_{n_k} - x_{n_{k+1}}$ for $k = 1, 2, \dots$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} \|y_k\| &= \sum_{k=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| \\ &< \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty .\end{aligned}$$

By hypothesis, there is some $y \in Y$ such that

$$\begin{aligned}y &= \lim_{m \to \infty} \sum_{k=1}^{m} y_k \\ &= \lim_{m\to\infty} (x_{n_1} - x_{n_2}) + (x_{n_2} - x_{n_3}) + \dots + (x_{n_m} - x_{n_{m+1}}) \\ &= \lim_{m\to\infty} (x_{n_1} - x_{n_{m+1}}).\end{aligned}$$

That is, (x_{n_k}) converges in Y (to $x_{n_1} - y$). But if a subsequence of a Cauchy sequence converges, the whole sequence does; i.e., (x_n) converges in Y and we conclude that Y is complete. ■

Proposition 1.23. *Let X be a Banach space and suppose that V is a closed linear subspace of X. Then X/V is a Banach space with respect to the quotient norm*

$$\| \operatorname{cl} x\| = \inf_{v \in V} \|x + v\|.$$

Proof. We have already shown that X/V is a normed space. To show that it is indeed a Banach space we will use the last result. Let $(\operatorname{cl} x_n)$ be any sequence in X/V such that $\sum_{n=1}^{\infty} \|\operatorname{cl} x_n\| < \infty$. By definition of the infimum, for each n, there is $v_n \in V$ such that

$$\begin{aligned}\|x_n + v_n\| &< \inf_{v \in V} \|x_n + v\| + \frac{1}{2^n} \\ &= \|\operatorname{cl} x_n\| + \frac{1}{2^n}.\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \|x_n + v_n\| < \sum_{n=1}^{\infty} \Big(\|\operatorname{cl} x\| + \frac{1}{2^n} \Big) < \infty.$$

Now, X is a Banach space, so, by the previous proposition, it follows that there is $y \in X$ such that

$$y = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k + v_k).$$

We claim that $\sum_{k=1}^{n} \operatorname{cl} x_k \to \operatorname{cl} y$ in X/V. Indeed,

$$\begin{aligned} \|\operatorname{cl} y - \sum_{k=1}^{n} \operatorname{cl} x_k\| &= \|\operatorname{cl}(y - \sum_{k=1}^{n} x_k)\| \\ &= \inf_{v\in V} \|y - \sum_{k=1}^{n} x_k + v\| \\ &\le \|y - \sum_{k=1}^{n} x_k - \sum_{k=1}^{n} v_k\| \\ &= \|y - \sum_{k=1}^{n} (x_k + v_k)\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence $\sum_{k=1}^{n} \operatorname{cl} x_k$ converges in X/V (to $\operatorname{cl} y$) and so X/V is a Banach space, again by the previous proposition. ■

Definition 1.24. A linear subset V in an algebra A is a left (respectively, right) ideal if $av \in V$ (respectively, $va \in V$) for all $a \in A$ and $v \in V$. V is a two-sided ideal in A if V is both a left and a right ideal.

We can obtain new algebras by taking suitable quotients, as the next theorem shows.

Theorem 1.25. *Let A be a Banach algebra and suppose that V is a closed two-sided ideal in A. Then A/V is a Banach algebra with respect to the quotient norm*

$$\|\operatorname{cl} x\| = \inf_{v\in V} \|x + v\|.$$

If A is unital and V is proper, then A/V is unital. Moreover the identity of A/V has unit norm.

Proof. We have shown that A/V is a Banach space. Since V is a two-sided ideal it is easy to see that A/V is an algebra with respect to the multiplication $\operatorname{cl} x \operatorname{cl} y = \operatorname{cl} xy$. Furthermore,

$$\begin{aligned} \|\operatorname{cl} x \operatorname{cl} y\| &= \|\operatorname{cl} xy\| \\ &= \inf_{v\in V} \|xy + v\| \end{aligned}$$

$$\begin{aligned}
&\leq \inf_{v,w\in V} \|xy + \underbrace{xw + vy + vw}_{\in V}\| \\
&= \inf_{v,w\in V} \|(x+v)(y+w)\| \\
&\leq \inf_{v,w\in V} \|x+v\|\,\|y+w\| \\
&= \|\operatorname{cl} x\|\,\|\operatorname{cl} y\|.
\end{aligned}$$

Thus A/V is a Banach algebra.

If $V = A$, then A/V is trivial, that is, $\{0\}$. If V is proper, then A/V is not equal to $\{0\}$ and one sees that $\operatorname{cl}\mathbb{1}$ is a unit for A/V.

Furthermore, if $\|\mathbb{1}\| = 1$, then

$$\begin{aligned}
\|\operatorname{cl}\mathbb{1}\| &= \inf_{v\in V} \|\mathbb{1} + v\| \\
&\leq \|\mathbb{1}\|, \quad \text{(taking } v = 0\text{)}, \\
&= 1.
\end{aligned}$$

However, we know that the unit of a Banach algebra always has norm greater than or equal to 1, so we obtain $\|\operatorname{cl}\mathbb{1}\| = 1$. ■

Chapter 2

Gelfand Theory

In this section we shall investigate the interplay between the maximal ideals of a unital Banach algebra, the multiplicative linear functionals and associated function spaces.

Definition 2.1. An ideal in an algebra is said to be *maximal* if it is proper (i.e., not equal to the whole algebra) and is not contained as a proper subset of any other proper ideal. Thus, 'maximal' is synonymous with 'maximal proper'.

Proposition 2.2. *Every maximal ideal in a unital Banach algebra is closed.*

Proof. Let J be a maximal ideal in the unital Banach algebra A. Then J cannot contain any invertible elements, otherwise we would have $J = A$. Hence $J \subseteq A \setminus \mathcal{G}(A)$. Now, $\mathcal{G}(A)$ is open and so $A \setminus \mathcal{G}(A)$ is closed, hence

$$J \subseteq \overline{J} \subseteq A \setminus \mathcal{G}(A).$$

In particular, $\overline{J} \neq A$. But $\overline{J}$ is an ideal containing J, and so $\overline{J} = J$ since J is a maximal ideal. That is, J is closed. ∎

Proposition 2.3. *Every complex-valued homomorphism on a Banach algebra is continuous.*

Proof. Let A be a Banach algebra, and $\varphi : A \to \mathbb{C}$ a homomorphism. If $\varphi = 0$, then it is certainly continuous. So suppose that $\varphi \neq 0$ and suppose also that A is unital. For any $a \in A$, $\varphi(a) = \varphi(a\mathbb{1}) = \varphi(a)\varphi(\mathbb{1})$ and so $\varphi(\mathbb{1}) = 1$. If $a \in A$ and $\varphi(a) \neq 0$, then $b = a - \varphi(a)\mathbb{1}$ belongs to the kernel of φ and therefore is singular (—otherwise, $1 = \varphi(bb^{-1}) = \varphi(b)\varphi(b^{-1})$, which is impossible). This means that $\varphi(a)$ belongs to $\sigma_A(a)$ and it follows that $|\varphi(a)| \leq \|a\|$. This inequality remains valid when $\varphi(a) = 0$ and we conclude that φ is continuous on A.

If A is non-unital, we consider A_I instead. Define the map $\varphi' : A_I \to \mathbb{C}$ by $\varphi'((a, \lambda)) = \varphi(a) + \lambda$, $(a, \lambda) \in A_I$. Then it is straightforward to check that φ' is a homomorphism and therefore, by the above, is continuous on A_I. In particular, its restriction to A in A_I is continuous, i.e., φ is continuous. ∎

Definition 2.4. A non-zero complex-valued homomorphism on a Banach algebra is called a character (or multiplicative linear functional).

By the last proposition, characters are necessarily continuous.

Theorem 2.5. (Gelfand-Mazur) *There is a canonical bijection between the maximal ideals in a commutative unital Banach algebra A and its characters given by associating to each character its kernel; i.e., if ℓ is a character on A, $\ker \ell$ is a maximal ideal in A, and every maximal ideal has this form for some unique character.*

Proof. Suppose that $\ell : A \to \mathbb{C}$ is a character and let $J = \ker \ell$. We have $J \neq A$ since ℓ is non-zero. Let $a \notin J$. Then any $b \in A$ can be written as

$$b = a\frac{\ell(b)}{\ell(a)} + \left(b - a\frac{\ell(b)}{\ell(a)}\right).$$

Since $b - a\dfrac{\ell(b)}{\ell(a)} \in \ker \ell = J$, we see that $A = \mathbb{C}a + J$ and hence J is a maximal ideal.

Now suppose that J is a maximal ideal. Then J is closed and so A/J is a Banach algebra. We claim that the maximality of J implies that every non-zero element of A/J is invertible. To see this, suppose that $\mathrm{cl}\, a$ is a non-zero non-invertible element of A/J. Then $J + aA$ is a proper ideal of A which contains J as a proper subset. ($J + aA$ does not contain $\mathbb{1}$ since $\mathrm{cl}\, a$ is non-invertible in A/J.) This contradicts the supposed maximality of J, and the claim is established.

It then follows that every element in A/J has the form $\lambda\, \mathrm{cl}\, \mathbb{1}$, for $\lambda \in \mathbb{C}$. Let $\varphi : A/J \to \mathbb{C}$ denote this isomorphism, and let π denote the canonical projection $\pi : A \to A/J$. Then $\phi \circ \pi$ is a homomorphism from A to $\mathbb{C}$ with kernel equal to J, for

$$\begin{aligned}\phi \circ \pi(ab) &= \phi(\pi(ab)) = \phi(\mathrm{cl}\, ab)\\ &= \phi(\mathrm{cl}\, a\ \mathrm{cl}\, b) = \phi(\mathrm{cl}\, a)\, \phi(\mathrm{cl}\, b)\\ &= (\phi \circ \pi(a))(\phi \circ \pi(b)),\end{aligned}$$

and $\phi \circ \pi(a) = 0$ if and only if $\pi(a) = 0$ if and only if $a \in J$.

Thus we have a correspondence between maximal ideals J and characters ℓ with $\ker \ell = J$. This association is one-one, since ℓ is uniquely determined by its kernel. Indeed, suppose that ℓ and ℓ' have the same kernel. Then for any $a \in A$, $a - \ell(a)\mathbb{1}$ belongs to $\ker \ell = \ker \ell'$ and so $\ell'(a) = \ell(a)$ since $\ell'(\mathbb{1}) = 1$. ∎

Proposition 2.6. *Any commutative unital Banach algebra possesses at least one character.*

Proof. If all elements of the commutative unital Banach algebra A are invertible, then $A \simeq \mathbb{C}$ and the effecting isomorphism is a character. On the other hand, if there is some $x \in A$ such that x is not invertible, then xA is a proper ideal and so is contained in a maximal proper ideal J, say, by Zorn's lemma. (The set of proper ideals containing J is partially ordered by set-theoretic inclusion, and the union of any increasing family of such ideals is also a proper ideal containing J (—since none of these can contain the unit). Zorn's lemma states that there exists a maximal such set, i.e., proper and containing J.) But then we know that J is the kernel of a character on A. ■

Without the assumption of commutativity, there may be no characters at all on an algebra.

Example 2.7. Let $A = M_n(\mathbb{C})$, with $n > 1$, and let e_{ij} be the $n \times n$ matrix all of whose entries are 0 except for the ij-entry which is equal to 1. If ℓ were a character on A, then, for $i \neq j$, the equality $e_{ij}^2 = 0$ would imply that $\ell(e_{ij}) = 0$. Hence the equality $e_{ii} = e_{ij}e_{ji}$, applied with $i \neq j$, would imply that $\ell(e_{ii}) = 0$ for each $i = 1, 2, \dots n$. We would conclude that $\ell(\mathbb{1}) = \ell(e_{11}) + \cdots + \ell(e_{nn}) = 0$, which is impossible. Hence $M_n(\mathbb{C})$, for any $n > 1$, possesses no characters.

Definition 2.8. The set of characters of a commutative unital Banach algebra A is called the spectrum (or carrier space, or structure space, or maximal ideal space) of A, and is denoted $\operatorname{Sp} A$.

We recall the definition of the w^*-topology on A^*, the dual of the Banach space A.

Definition 2.9. The w^*-topology on A^* is that generated by the neighbourhoods

$$\mathfrak{N}(\varphi : \mathcal{S}, \varepsilon) = \{\omega \in A^* : |\omega(a) - \varphi(a)| < \varepsilon \quad \text{for all } a \in \mathcal{S}\}$$

where $\varphi \in A^*$, ε is any positive real number and $\mathcal{S}$ is any *finite* subset of A. Thus a set G in A^* is open in the w^*-topology if and only if for each $\psi \in G$ there is some $\mathfrak{N}(\psi : \mathcal{S}, \varepsilon)$ as above with $\mathfrak{N}(\psi : \mathcal{S}, \varepsilon) \subseteq G$.

This gives rise to a Hausdorff topology on A^*; indeed, for any $\varphi_1, \varphi_2 \in A^*$, with $\varphi_1 \neq \varphi_2$ there exists $a \in A$ such that $\varphi_1(a) \neq \varphi_2(a)$. Let $\varepsilon_0 = |\varphi_1(a) - \varphi_1(a)|/3$. Then evidently the neighbourhoods $\mathfrak{N}(\varphi_1 : \{a\}, \varepsilon_0)$ and $\mathfrak{N}(\varphi_2 : \{a\}, \varepsilon_0)$ have empty intersection.

In terms of nets, the w^*-topology is described as the weakest topology on A^* such that a net $(\omega_\alpha) \to \omega$ in A^* if and only if $\omega_\alpha(a) \to \omega(a)$ for each $a \in A$.

We shall use the following result—the Banach-Alaoglu theorem.

Theorem 2.10. *The closed unit ball of A^*, the dual of the Banach space A, is w^*-compact.*

Proposition 2.11. *The spectrum,* $\operatorname{Sp} A$, *of a commutative unital Banach algebra A is a w^*-closed subset of the unit ball of A^*, and hence is compact.*

Proof. We know from **??** that any character is bounded with norm equal to 1, and therefore $\operatorname{Sp} A$ is contained in the unit ball of A^*.

To show that $\operatorname{Sp} A$ is w^*-closed, we shall show that $A^* \setminus \operatorname{Sp} A$ is open. Let $\varphi \in A^* \setminus \operatorname{Sp} A$. If $\varphi = 0$, we have $0 = \varphi \in \mathfrak{N}(0; \{\mathbb{1}\}, \frac{1}{2}) \subset A^* \setminus \operatorname{Sp} A$, since $\ell(\mathbb{1}) = 1$ for any $\ell \in \operatorname{Sp} A$. Suppose that $\varphi \neq 0$. Then there is $a, b \in A$ such that $\varphi(ab) \neq \varphi(a)\varphi(b)$. Consider the neighbourhood $\mathfrak{N}(\varphi : \{a, b, ab\}, \varepsilon)$ of φ in A^*. This consists of all those $\omega \in A^*$ such that $|\omega(a) - \varphi(b)| < \varepsilon$, $|\omega(b) - \varphi(b)| < \varepsilon$ and $|\omega(ab) - \varphi(ab)| < \varepsilon$. Clearly, if $\varepsilon > 0$ is sufficiently small, then for any such ω, we have $\omega(ab) \neq \omega(a)\omega(b)$, iė; for sufficiently small $\varepsilon > 0$, $\mathfrak{N}(\varphi : \{a, b, ab\}, \varepsilon)$ is contained in $A^* \setminus \operatorname{Sp} A$. Hence $A^* \setminus \operatorname{Sp} A$ is open in A^*, and so $\operatorname{Sp} A$ is w^*-closed. $\operatorname{Sp} A$ is compact since it is a closed subset of a compact set. ∎

Remark 2.12. The fact that $\operatorname{Sp} A$ is w^*-closed can be demonstrated quite easily using nets. Suppose that (ℓ_α) is a net in $\operatorname{Sp} A$ converging to $\varphi \in A^*$. Then, by definition of the w^*-topology, $\ell_\alpha(x) \to \varphi(x)$ for each $x \in A$. But then for any $x, y \in A$

$$\begin{aligned} \varphi(xy) &= \lim \ell_\alpha(xy) \\ &= \lim \ell_\alpha(x)\, \ell_\alpha(y) \\ &= \varphi(x)\, \varphi(y) \end{aligned}$$

and it follows that $\varphi \in \operatorname{Sp} A$. Note that φ is non-zero since $\varphi(\mathbb{1}) = 1$.

Theorem 2.13. *Let A be a commutative unital Banach algebra. For $x \in A$ and $\ell \in \operatorname{Sp} A$, define $\widehat{x} : \operatorname{Sp} A \to \mathbb{C}$ by*

$$\widehat{x}(\ell) = \ell(x).$$

Then the range of the function $\widehat{x}$ on $\operatorname{Sp} A$ satisfies

$$\operatorname{ran} \widehat{x} = \sigma_A(x).$$

Furthermore, the map $\widehat{\ }$ is a homomorphism $\widehat{\ } : A \to C(\operatorname{Sp} A)$ and

$$\|\widehat{x}\|_\infty \leq \|x\|, \quad \text{for } x \in A.$$

$\widehat{\ }$ is called the Gelfand transform.

Proof. We have seen already that for any $x \in A$ and $\ell \in \operatorname{Sp} A$ we have $\ell(x) \in \sigma_A(x)$; iė; $\widehat{x}(\ell) \in \sigma_A(x)$ and so the range of $\widehat{x}$ satisfies the inclusion $\operatorname{ran} \widehat{x} \subseteq \sigma_A(x)$.

Let $\lambda \in \sigma_A(x)$. Then $x - \lambda \mathbb{1}$ is not invertible and so belongs to some maximal ideal, J, say. (In fact, $x - \lambda\mathbb{1}$ belongs to the proper ideal $A(x - \lambda\mathbb{1})$ which is contained in a maximal ideal, by Zorn's lemma.)

Let ℓ be that element of $\operatorname{Sp} A$ with $\ker \ell = J$. Then $x - \lambda\mathbb{1} \in J$ implies that $\ell(x) = \lambda$. Hence $\widehat{x}(\ell) = \ell(x) = \lambda$ and it follows that $\operatorname{ran} \widehat{x} = \sigma_A(x)$.

It is clear that $\widehat{\ }$ is a homomorphism; for example,

$$\begin{aligned} \widehat{xy}(\ell) &= \ell(xy) = \ell(x)\ell(y) \\ &= \widehat{x}(\ell)\,\widehat{y}(\ell) \quad \text{for any } x, y \in A,\ \ell \in \operatorname{Sp} A, \end{aligned}$$

and so $\widehat{xy} = \widehat{x}\,\widehat{y}$. Similarly, one sees that $\widehat{\ }$ is linear.

To show that $\widehat{x} \in \mathcal{C}(\operatorname{Sp} A)$, let U be any open set in $\mathbb{C}$. We must show that $\widehat{x}^{-1}(U)$ is open in $\operatorname{Sp} A$. If $\widehat{x}^{-1}(U) = \varnothing$, then we are done. So suppose that $\widehat{x}^{-1}(U) \neq \varnothing$. Let ℓ be any element of $\widehat{x}^{-1}(U)$. Then there is $\zeta \in U$ such that $\widehat{x}(\ell) = \zeta$. Since U is open in $\mathbb{C}$, there is $\varepsilon > 0$ such that $N_\varepsilon(\zeta) \equiv \{z \in \mathbb{C} : |z - \zeta| < \varepsilon\} \subseteq U$. Let $V = \mathfrak{N}(\ell : \{x\}, \varepsilon) \equiv \{\omega \in \operatorname{Sp} A : |\omega(x) - \ell(x)| < \varepsilon\}$. Then $\omega(x) = \widehat{x}(\omega) \in U$ for all $\omega \in V$, iė; $\ell \in V \subseteq \widehat{x}^{-1}(U)$. We deduce that $\widehat{x}^{-1}(U)$ is open in $\operatorname{Sp} A$ and hence $\widehat{x} : \operatorname{Sp} A \to \mathbb{C}$ is continuous; that is, $\widehat{x}(\,\cdot\,) \in \mathcal{C}(\operatorname{Sp} A)$. (Alternatively, the continuity of $\widehat{x}$ can easily be established using nets, as follows. Suppose that $\ell_\alpha \to \ell$ in $\operatorname{Sp} A$. Then $\widehat{x}(\ell_\alpha) = \ell_\alpha(x) \to \ell(x) = \widehat{x}(\ell)$, by definition of the w^*-topology. In other words, $\widehat{x}$ is continuous.)

Finally, we have that $\operatorname{ran} \widehat{x} = \sigma_A(x) \subseteq \{\lambda : |\lambda| \le \|x\|\}$ and so it follows that $|\widehat{x}(\ell)| \le \|x\|$, for all $\ell \in \operatorname{Sp} A$. Thus $\|\widehat{x}\|_\infty \le \|x\|$ for any $x \in A$. ∎

This theorem has a sharper form for C^*-algebras, as we will see in the next section.

Theorem 2.14. *Let A be a commutative unital Banach algebra generated by the single element a: that is, the set of polynomials in a is dense in A. Then the map $\widehat{a} : \operatorname{Sp} A \to \sigma_A(a) \subset \mathbb{C}$ is a homeomorphism.*

Proof. We know that $\widehat{a}$ is a continuous function on $\operatorname{Sp} A$ with $\operatorname{ran} \widehat{a} = \sigma_A(a)$, iė; $\widehat{a} : \operatorname{Sp} A \to \sigma_A(a)$ is continuous and onto. Now, both $\operatorname{Sp} A$ and $\sigma_A(a)$ are compact Hausdorff spaces, so we need only show that $\widehat{a}$ is injective. To see this, suppose that $\widehat{a}(\ell_1) = \widehat{a}(\ell_2)$, so that $\ell_1(a) = \ell_2(a)$. Using the multiplicativity of ℓ_1 and ℓ_2, we see that for given $N \in \mathbb{N}$ and $c_0, c_1, \dots, c_N$ in $\mathbb{C}$

$$\ell_1\Big(\sum_{n=0}^{N} c_n a^n\Big) = \ell_2\Big(\sum_{n=0}^{N} c_n a^n\Big).$$

Since ℓ_1 and ℓ_2 are continuous and a generates A, it follows that $\ell_1 = \ell_2$. ∎

Example 2.15. Let A be the subalgebra of $M_2(\mathbb{C})$ consisting of those elements of the form $\left(\begin{smallmatrix}\alpha & \beta \\ 0 & \alpha\end{smallmatrix}\right)$, with $\alpha,\ \beta \in \mathbb{C}$. Then

$$\begin{pmatrix}\alpha & \beta \\ 0 & \alpha\end{pmatrix} = \alpha\mathbb{1} + \beta q, \quad \text{where } q = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}.$$

We notice that $q^2 = 0$ (i.e., q is nilpotent). Evidently, A is a two-dimensional commutative Banach algebra with unit $\mathbb{1}$. We shall compute the spectrum, $\sigma_A(x)$, for $x = \left(\begin{smallmatrix}\alpha & \beta \\ 0 & \alpha\end{smallmatrix}\right)$. Indeed, for $\lambda \in \mathbb{C}$,

$$x - \lambda\mathbb{1} = \begin{pmatrix}\alpha - \lambda & \beta \\ 0 & \alpha - \lambda\end{pmatrix}$$

is invertible in $M_2(\mathbb{C})$ if and only if $\lambda \neq \alpha$. If $\lambda \neq \alpha$, then, in fact,

$$(x - \lambda\mathbb{1})^{-1} = \begin{pmatrix}(\alpha - \lambda)^{-1} & -\beta(\alpha - \lambda)^{-2} \\ 0 & (\alpha - \lambda)^{-1}\end{pmatrix},$$

which belongs to A. Hence, $\sigma_A(x) = \sigma_A(\alpha\mathbb{1} + \beta q) = \{\alpha\}$. In particular, $\sigma_A(q) = \sigma_A(\beta q) = \{0\}$, but $q \neq 0$.

Now we consider the characters of A. If ℓ is a character, then $\ell(xy) = \ell(x)\ell(y)$ implies that $\ell(q^2) = \ell(q)\ell(q)$. But $q^2 = 0$ and therefore $\ell(q) = 0$. Since $\ell(\mathbb{1}) = 1$, we find that $\ell(\alpha\mathbb{1} + \beta q) = \alpha$, for any $\alpha,\ \beta$ in $\mathbb{C}$. In other words, there is just one character on A; $\operatorname{Sp} A = \{\ell\}$, where ℓ is given uniquely by the action $\ell(\mathbb{1}) = 1$ and $\ell(q) = 0$.

The Gelfand transform is the map $x \mapsto \widehat{x}$, $\alpha\mathbb{1} + \beta q \mapsto \alpha\widehat{\mathbb{1}} + \beta\widehat{q}$. But $\widehat{\mathbb{1}} = 1$ and $\widehat{q}(\ell) = \ell(q) = 0$ so that $\widehat{q} = 0$ and we have $(\alpha\mathbb{1} + \beta q)^{\wedge} = \alpha$, for any $\alpha,\ \beta \in \mathbb{C}$. The transform $^{\wedge}$ has kernel $\{\beta q : \beta \in \mathbb{C}\}$, so we see that $^{\wedge}$ is not an isomorphism.

The algebra A has exactly one maximal ideal, namely, the kernel of ℓ.

A is the unital algebra generated by the element q, and so $\operatorname{Sp} A \simeq \sigma_A(q)$, via the homeomorphism $\widehat{q} : \operatorname{Sp} A \to \sigma_A(q)$, $\ell \mapsto \widehat{q}(\ell) = 0$. Indeed, both $\operatorname{Sp} A$ and $\sigma_A(q)$ are singleton sets!

Alternatively, we can determine the spectrum of $x \in A$ using the equality $\sigma_A(x) = \operatorname{ran} \widehat{x}$. For $x = \left(\begin{smallmatrix}\alpha & \beta \\ 0 & \alpha\end{smallmatrix}\right)$, we have

$$\begin{aligned}\sigma_A(x) &= \{\widehat{x}(\ell)\} = \{\ell(x)\} \\ &= \{\ell(\alpha\mathbb{1} + \beta q)\} \\ &= \{\alpha\ell(\mathbb{1}) + \beta\ell(q)\} \\ &= \{\alpha\}\end{aligned}$$

since $\ell(q) = 0$.

Chapter 3

C^*-algebras

Definition 3.1. A Banach *-algebra is a Banach algebra A together with an "involution" $a \mapsto a^*$ satisfying :

(i) * is conjugate linear, i.e., $(\alpha a)^* = \bar{\alpha} a^*$ for $\alpha \in \mathbb{C}$, $a \in A$;

(ii) $a^{**} = a$ for every $a \in A$;

(iii) $(ab)^* = b^* a^*$ for any $a, b \in A$;

(iv) $\|a^*\| = \|a\|$.

By (iv), we see that $^* : A \to A$ is continuous.
If A has a unit $\mathbb{1}$, then $\mathbb{1}^* = \mathbb{1}^* \mathbb{1} = (\mathbb{1}^* \mathbb{1})^* = \mathbb{1}^{**} = \mathbb{1}$.

A C^*-algebra is a Banach *-algebra for which

(v) $\|a^* a\| = \|a\|^2$, for all $a \in A$.

This property (v) is often referred to as the C^*-property of the norm.

Remark 3.2. We note that in a C^*-algebra, property (iv) follows from the others: indeed, for any a in a C^*-algebra A, property (v) implies that $\|a\|^2 = \|a^* a\| \leq \|a^*\| \, \|a\|$ and so $\|a\| \leq \|a^*\|$. (If $\|a\| = 0$, then $a = 0 = a^*$.) Replacing a by a^* gives $\|a^*\| \leq \|a\|$ and so their equality follows.

Now suppose that A is a C^*-algebra with unit $\mathbb{1}$. Then, by the C^*-property, $\|\mathbb{1}^* \mathbb{1}\| = \|\mathbb{1}\|^2$ and so $\|\mathbb{1}\| = \|\mathbb{1}\|^2$ which implies that $\|\mathbb{1}\| = 1$.

If A does not have a unit, then one can be adjoined but care must be taken not to spoil the C^*-property, property (v). We will return to this later.

Examples 3.3.

1. Let Ω be a compact space. Then $\mathcal{C}(\Omega)$ is a commutative C^*-algebra with unit, when equipped with the supremum norm and the involution $f \mapsto f^* = \overline{f}$. Since $f^*f = |f|^2$ it follows that $\|f^*f\|_\infty = \|f\|_\infty^2$.

2. Let Ω be a topological space. $\mathcal{C}_b(\Omega)$, the algebra of continuous bounded complex-valued functions on Ω, is a commutative C^*-algebra with unit, as above.

3. Let Ω be a non-compact, locally compact Hausdorff space and let $\mathcal{C}_0(\Omega)$ denote the set of continuous complex-valued functions on Ω vanishing at infinity. (That is, the continuous complex-valued function $f : \Omega \to \mathbb{C}$ belongs to $\mathcal{C}_0(\Omega)$ if and only if for any given $\varepsilon > 0$ there is a compact set in Ω outside of which $|f|$ is less than ε; iė; the set $\{\omega \in \Omega : |f(\omega)| \geq \varepsilon\}$ is compact.) Then $\mathcal{C}_0(\Omega)$ is a C^*-algebra without a unit. (A unit $f \in \mathcal{C}_0(\Omega)$ must satisfy $f = f^2$ and so can only take the values 0 and 1. The set $K = \{\omega \in \Omega : f(\omega) = 1\}$ is necessarily compact and so is a proper subset of Ω, since Ω is supposed to be non-compact. But then for any $g \in \mathcal{C}_0(\Omega)$, $g = gf$ implies that g vanishes on the non-empty open set $\Omega \setminus K$. Let $\omega' \in \Omega \setminus K$. Since Ω is locally compact, there is a compact set K_1 and an open set G such that $\omega' \in G \subseteq K_1$. By Urysohn's lemma, there is a continuous map $g : \Omega \to \mathbb{C}$ such that g takes the value 1 at ω' and vanishes on the closed set $\Omega \setminus G$. But this means that $\{\omega \in \Omega : f(\omega) \neq 0\} \subseteq K_1$ and so $g \in \mathcal{C}_0(\Omega)$. This contradicts the fact that g should vanish outside of K, and we conclude that $\mathcal{C}_0(\Omega)$ has no unit.)

4. It is quite possible for $\mathcal{C}_0(\Omega)$ to possess a unit when Ω is locally compact, non-compact and non-Hausdorff. Indeed, let $\Omega = \{0\} \cup \mathbb{N}$, and define a topology on Ω by declaring a non-empty set to be open if it is equal to $\{0\}$ or if it contains the element 1. Then the points 1 and 2, for example, cannot be separated by disjoint open sets (since any open set containing 2 also contains 1) so Ω is non-Hausdorff, and $\{0\}$, $\{1\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}, \ldots$ is an open cover of Ω with no finite subcover, so Ω is non-compact. However,any $\omega \in \Omega$ is contained in the set $\{\omega\} \cup \{1\}$, which is open and compact, and therefore Ω is locally compact. Let $f : \Omega \to \mathbb{C}$ be continuous. Then $f(n) = f(1)$ for all $n \in \mathbb{N}$; to see this, note that $f^{-1}(\mathbb{C} \setminus \{f(1)\})$ is an open set in Ω not containing 1. Since the only open sets in Ω which do not contain 1 are $\varnothing$ and $\{0\}$, we conclude that $n \notin f^{-1}(\mathbb{C} \setminus \{f(1)\})$ for any $n \in \mathbb{N}$. In other words, $f(n) = f(1)$ for $n \in \mathbb{N}$.

   Suppose now that $f \in \mathcal{C}_0(\Omega)$. Then, by definition, for any given $\varepsilon > 0$ there is a compact set K, say, in Ω such that $|f(\omega)| < \varepsilon$ for all $\omega \in \Omega \setminus K$. The only compact sets in Ω are those with a finite number of elements, so for any given $\varepsilon > 0$, we must have $|f(n)| < \varepsilon$ for all sufficiently large n. However, since f is continuous, we have $f(n) = f(1)$, for all $n \in \mathbb{N}$. It follows that $f(n) = f(1) = 0$, for all $n \in \mathbb{N}$. Thus, $\mathcal{C}_0(\Omega)$ consists of all functions $f : \Omega \to \mathbb{C}$ such that $f(1) = f(2) = f(3) = \cdots = 0$. Evidently, the function $e : \mathcal{C}_0(\Omega) \to \mathbb{C}$ given by $e(0) = 1$, $e(1) = e(2) = \cdots = 0$, is a unit for the C^*-algebra $\mathcal{C}_0(\Omega)$.

Of course, $\mathcal{C}_0(\Omega)$ is not a particularly interesting C^*-algebra—it is isomorphic to $\mathbb{C}$, via the map $f \mapsto f(0)$.

5. $\mathbb{C}$, with the obvious structure, is a C^*-algebra. (This is just example 1, above, when Ω consists of a single point.)

6. We denote the linear space of bounded linear operators on the complex Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. For any $x \in \mathcal{B}(\mathcal{H})$ let $\|x\|$ denote the operator norm of x (given by $\sup\{\|x\xi\| : \|\xi\| \le 1, \xi \in \mathcal{H}\}$) and let * be the operator adjoint. Then $\mathcal{B}(\mathcal{H})$, equipped with this structure, is a unital C^*-algebra. (If $\dim \mathcal{H} = 1$, then we have the previous example.)

7. Any norm closed subalgebra of $\mathcal{B}(\mathcal{H})$ which contains x^* if it contains x is a C^*-algebra. This is *the* typical C^*-algebra, as we will see.

8. $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices, with * being the adjoint is a unital C^*-algebra. The norm is the operator norm obtained by considering $M_n(\mathbb{C})$ as an algebra of linear operators on the finite-dimensional complex Hilbert space $\mathbb{C}^n$. (This is the same as defining $\|x\|$ to be the square root of the largest eigenvalue of the positive self-adjoint matrix x^*x, $x \in M_n(\mathbb{C})$.) This is just example 6 when $\mathcal{H}$ is n-dimensional.

9. $\ell^\infty(\mathbb{N})$ and $\ell^\infty(\mathbb{Z})$ equipped with component-wise operations and the sup-norm are commutative unital C^*-algebras. (These are examples of 2 above.)

10. The set $\mathcal{K}$ of compact operators on a Hilbert space $\mathcal{H}$ is a C^*-algebra. Note, however, that $\mathcal{K}$ has a unit if and only if $\mathcal{H}$ is finite-dimensional.

11. Given any family $\{A_\alpha\}$ of C^*-algebras, let A denote the subset of the Cartesian product $\prod_\alpha A_\alpha$ consisting of those elements (a_α) for which $\sup_\alpha \|a_\alpha\|$ is finite. Then A is a Banach space with respect to the norm $\|(a_\alpha)\| = \sup_\alpha \|a_\alpha\|$. Furthermore, A is a *-algebra when equipped with component-wise operations; for example, the adjoint of (a_α) is just (a_α^*), and the product of (a_α) with (b_α) is $(a_\alpha b_\alpha)$. It is straightforward to check that, with this structure, A is a C^*-algebra. Moreover, A is unital if and only if each A_α is unital. (If (e_α) is a unit for A, then it is readily see that for any α_0, e_{α_0} is a unit for A_{α_0}. The converse is clear.)

The C^*-algebra A is called the direct sum of the C^*-algebras $\{A_\alpha\}$. If α runs over Λ and $A_\alpha = \mathbb{C}$, for each $\alpha \in \Lambda$, then $A = \ell^\infty(\Lambda)$.

Definition 3.4. An element x in a C^*-algebra A is called self-adjoint (or symmetric or hermitian) if $x^* = x$. A projection in A is a self-adjoint element p, say, such that $p^2 = p$. An element $x \in A$ is called normal if $xx^* = x^*x$. An element u in a unital C^*-algebra A is called unitary if $uu^* = u^*u = \mathbb{1}$.

Remark 3.5. We can write any $x \in A$ as the linear combination

$$x = \frac{1}{2}(x + x^*) + i\frac{1}{2i}(x - x^*).$$

We see that $(x + x^*)/2$ and $(x - x^*)/2i$ are both self-adjoint elements of A. Conversely, if h and k are self-adjoint of A and $x = h + ik$, then $x^* = h - ik$ so that $h = \frac{1}{2}(x + x^*)$ and $k = \frac{1}{2i}(x - x^*)$. In other words, the decomposition $x = h + ik$ with $h = h^*$, $k = k^*$ in A is unique. (These are often referred to as the real and imaginary parts of x, respectively.)

If A is a unital C^*-algebra and $x \in A$, we shall denote by $\mathcal{A}(x)$ the unital C^*-algebra of A generated by x; that is, $\mathcal{A}(x)$ is the closure in A of the *-algebra of complex polynomials in x, x^* and $\mathbb{1}$. Clearly, $\mathcal{A}(x)$ is commutative if and only if x is normal.

Proposition 3.6. *Let A be a unital C^*-algebra, and let $h \in A$ be self-adjoint. Then $\sigma_A(h) \subset \mathbb{R}$.*

Proof. Suppose that $h \in A$ is self-adjoint, and consider the commutative unital C^*-algebra $\mathcal{A}(h)$. For $t \in \mathbb{R}$, put

$$u_t = e^{ith} \equiv \sum_{n=0}^{\infty} \frac{(it)^n}{n!} h^n.$$

(—the series converges in $\mathcal{A}(h)$ since $\mathcal{A}(h)$ is complete). Then by the continuity of the involution *, we see that

$$\begin{aligned} u_t^* &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{(it)^k}{k!} h^k \right)^* = \lim_{n\to\infty} \sum_{k=0}^{n}, \frac{(-it)^k}{k!} h^k \\ &= u_{-t}\,. \end{aligned}$$

By multiplying the series (as in the complex case), we see that

$$u_t^* u_t = u_{-t} u_t = u_0 = \mathbb{1}.$$

Hence $1 = \|u_t^* u_t\| = \|u_t\|^2$, and so $\|u_t\| = 1$ for all $t \in \mathbb{R}$.

Let $\ell \in \operatorname{Sp} \mathcal{A}(h)$. Then, since ℓ is continuous,

$$\begin{aligned}
\ell(u_t) &= \ell\Big(\sum_{n=0}^{\infty} \frac{(it)^n}{n!}\, h^n \Big) \\
&= \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\, \ell(h)^n \\
&= e^{it\ell(h)}.
\end{aligned}$$

Since $\|\ell\| = 1$, we have $|\ell(u_t)| \le \|u_t\| = 1$, and hence $|e^{it\ell(h)}| \le 1$ for all $t \in \mathbb{R}$. It follows that $\ell(h) \in \mathbb{R}$. This holds for all $\ell \in \operatorname{Sp} \mathcal{A}(h)$ and so $\widehat{h}$ is real-valued on $\operatorname{Sp} \mathcal{A}(h)$. But $\sigma_{\mathcal{A}(h)}(h) = \operatorname{ran} \widehat{h}$ and therefore $\sigma_{\mathcal{A}(h)}(h) \subset \mathbb{R}$. Now $\mathcal{A}(h) \subseteq A$ and so $\sigma_A(h) \subseteq \sigma_{\mathcal{A}(h)}(h)$ and we conclude that $\sigma_A(h) \subset \mathbb{R}$. ∎

Theorem 3.7. (Gelfand-Naimark) *Suppose that A is a commutative unital Banach *-algebra. The Gelfand transform $\widehat{\ }: A \to \mathcal{C}(\operatorname{Sp} A)$ is an isometric *-isomorphism if and only if A is a C^*-algebra.*

Proof. If $\widehat{\ }$ is an isometric *-isomorphism, then for any $x \in A$,

$$\begin{aligned}
\|x^*x\| &= \|\widehat{x^*x}\|_\infty = \|\widehat{(x^*)}\,\widehat{x}\|_\infty \\
&= \|\,\overline{\widehat{x}}\,\widehat{x}\,\|_\infty = \|\,|\widehat{x}|^2\,\|_\infty \\
&= \|\,\widehat{x}\,\|_\infty^2 = \|x\|^2
\end{aligned}$$

which is precisely the C^*-property (v). So A is a C^*-algebra. (Indeed, $\mathcal{C}(\operatorname{Sp} A)$ *is* a C^*-algebra, so if A is isometrically *-isomorphic to $\mathcal{C}(\operatorname{Sp} A)$, then A must also be a C^*-algebra.)

Conversely, suppose that A is a C^*-algebra. Then for any $h = h^* \in A$, we know that $\sigma_A(h) \subset \mathbb{R}$; that is, $\widehat{h}$ is real-valued. For any $x \in A$, write $x = (x + x^*)/2 + i(x - x^*)/2i$. Then

$$\begin{aligned}
\widehat{(x^*)}(\ell) &= \ell(x^*) = \ell\big(\frac{x + x^*}{2} - i\frac{(x - x^*)}{2i}\big) \\
&= \ell\big(\frac{x + x^*}{2}\big) - i\,\ell\big(\frac{x - x^*}{2i}\big) \\
&= \Big(\ell\big(\frac{x + x^*}{2}\big) + i\,\ell\big(\frac{x - x^*}{2i}\big) \Big)^{-} \\
&= \overline{\ell(x)} \\
&= \overline{\widehat{x}(\ell)},
\end{aligned}$$

using the fact that $\ell(h)$ is real if h is symmetric. It follows that $\widehat{\ }$ is a *-homomorphism.

To show that $\widehat{\ }$ is isometric (and hence also an injection) consider again $h = h^* \in A$. Then, by the C^*-property of the norm $\|h\|^2 = \|h^2\|$ and so

$\|h\|^{2^n} = \|h^{2^n}\|$. Therefore

$$\|\widehat{h}\|_\infty = r(h) = \lim_{n\to\infty} \|h^{2^n}\|^{1/2^n} = \|h\|$$

and so $\widehat{\ }$ is isometric on self-adjoint elements. For any $x \in A$ we have

$$\begin{aligned} \|\,\widehat{x}\,\|_\infty^2 &= \|\,\overline{\widehat{x}}\,\widehat{x}\,\|_\infty \\ &= \|\,\widehat{x^*x}\,\|_\infty\,, \text{ since } \widehat{\ } \text{ is a } {}^*\text{-homomorphism}, \\ &= \|x^*x\|\,, \text{ since } x^*x \text{ is self-adjoint}, \\ &= \|x\|^2\,, \text{ by the } C^*\text{-property of the norm.} \end{aligned}$$

Hence $\widehat{\ }$ is isometric.

We must now show that $\widehat{\ }$ is surjective. To see this, we note that A is complete and since $\widehat{\ }$ is isometric it follows that $\operatorname{ran}\widehat{\ }$ is closed in $\mathcal{C}(\operatorname{Sp} A)$. Now, $\widehat{\ }$ is a *-homomorphism and so $\operatorname{ran}\widehat{\ }$ is a closed *-subalgebra of $\mathcal{C}(\operatorname{Sp} A)$ which contains the constant function $1 = \widehat{\mathbb{1}}$. Furthermore, if $\ell_1 \neq \ell_2$, then, by definition, there is $x \in A$ such that $\ell_1(x) \neq \ell_2(x)$, that is, $\widehat{x}(\ell_1) \neq \widehat{x}(\ell_2)$. Thus $\operatorname{ran}\widehat{\ }$ separates points of $\operatorname{Sp} A$. It follows from the Stone-Weierstrass theorem that $\operatorname{ran}\widehat{\ } = \mathcal{C}(\operatorname{Sp} A)$. ■

Remark 3.8. Let A be a unital C^*-algebra, and let $x \in A$ be normal. Then $\mathcal{A}(x)$ is commutative and so $\mathcal{A}(x) \simeq \mathcal{C}(\operatorname{Sp}\mathcal{A}(x))$. Let $f : \sigma_A(x) \to \mathbb{C}$ be continuous. Since $\operatorname{ran}\widehat{x} = \sigma_A(x)$, it follows that $f \circ \widehat{x} : \operatorname{Sp}\mathcal{A}(x) \to \mathbb{C}$ is well-defined and is continuous, iė; $f \circ \widehat{x} \in \mathcal{C}(\operatorname{Sp}\mathcal{A}(x))$. Hence there exists $y \in \mathcal{A}(x) \subseteq A$ such that $\widehat{y} = f \circ \widehat{x}$, and y is unique since $\widehat{\ }$ is an isomorphism. We write $y = f(x)$. In this way, we can define $f(x)$ as an element of A for any normal (and, in particular, for any self-adjoint) element $x \in A$ and any function f continuous on the spectrum of x. If f is a polynomial, then $f(x)$ is the obvious polynomial in x.

We have seen that the spectrum of an element can depend on the Banach algebra it is considered to belong to. The following is the key result telling us that this is not so for C^*-algebras.

Theorem 3.9. *Let A be a unital C^*-algebra and suppose that $x \in A$ is invertible. Then x^{-1} belongs to the C^*-subalgebra of A generated by $\mathbb{1}$, x and x^* (iė; the closure in A of the set of complex polynomials in $\mathbb{1}$, x, x^*).*

Proof. Suppose first that $x = x^*$, and let $\mathcal{A}$ denote the unital C^*-algebra generated by x, and $\mathcal{B}$ that generated by x and x^{-1}. Evidently $\mathcal{B}$ is commutative and $\mathcal{A} \subseteq \mathcal{B} \subseteq A$. We know that the Gelfand transform $\widehat{\ } : \mathcal{B} \to \widehat{\mathcal{B}} = \mathcal{C}(\operatorname{Sp}\mathcal{B})$ is an (isometric *-) isomorphism and, since $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}$, it follows that $\widehat{\mathcal{A}}$ (the range of $\mathcal{A}$ under $\widehat{\ }$) is a C^*-subalgebra of $\widehat{\mathcal{B}}$.

Let $\ell_1, \ell_2 \in \operatorname{Sp}\mathcal{B}$ and suppose that $\ell_1(x) = \ell_2(x)$. For any $\ell \in \operatorname{Sp}\mathcal{B}$, $\ell(xx^{-1}) = \ell(x)\ell(x^{-1}) = \ell(\mathbb{1}) = 1$. Hence $\ell_1(x^{-1}) = \ell_1(x)^{-1} = \ell_2(x)^{-1} = \ell_2(x^{-1})$. Since $\mathcal{B}$ is generated by x and x^{-1}, we deduce that $\ell_1 = \ell_2$.

Thus, if $\ell_1 \neq \ell_2$ then $\ell_1(x) \neq \ell_2(x)$ or $\widehat{x}(\ell_1) \neq \widehat{x}(\ell_2)$. That is, $\widehat{\mathcal{A}}$ separates points of $\operatorname{Sp}\mathcal{B}$. By the Stone-Weierstrass theorem, it follows that $\widehat{\mathcal{A}} = \widehat{\mathcal{B}}$ and so $\mathcal{A} = \mathcal{B}$ and therefore $x^{-1} \in \mathcal{A}$.

Now let $x \in \mathcal{A}$ be arbitrary with inverse x^{-1}. Then x^*x is invertible with inverse $x^{-1}(x^{-1})^*$. But x^*x is self-adjoint and so $x^{-1}(x^{-1})^*$ belongs to the C^*-algebra generated by $\mathbb{1}$ and x^*x which is contained in that generated by $\mathbb{1}$, x, and x^*. But then $x^{-1} = x^{-1}(x^{-1})^*x^* = (x^*x)^{-1}x^*$ is also contained in this last algebra. ∎

Corollary 3.10. *Let $A \subseteq B$ be unital C^*-algebras with the same unit, and let $x \in A$. Then $\sigma_A(x) = \sigma_B(x)$.*

Proof. Suppose that $x - \lambda\mathbb{1}$ is invertible in B. Then this inverse belongs to the C^*-algebra generated by $\mathbb{1}$, $x - \lambda\mathbb{1}$ and $(x - \lambda\mathbb{1})^*$, which is contained in A. Hence $x - \lambda\mathbb{1}$ is invertible in A. It follows that $\mathbb{C} \setminus \sigma_B(x) \subseteq \mathbb{C} \setminus \sigma_A(x)$ and so $\sigma_B(x) \supseteq \sigma_A(x)$. But $A \subseteq B$ implies that $\sigma_A(x) \supseteq \sigma_B(x)$ and therefore we have $\sigma_A(x) = \sigma_B(x)$. ∎

Example 3.11. Let A be the C^*-subalgebra of $M_2(\mathbb{C})$ consisting of those 2×2 complex matrices of the form $\left(\begin{smallmatrix}\alpha & 0\\ 0 & 0\end{smallmatrix}\right)$, $\alpha \in \mathbb{C}$, and let B be the C^*-subalgebra consisting of those matrices of the form $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$, $\alpha, \beta \in \mathbb{C}$. Then $A \subset B$, and A and B are both unital with units given by $\mathbb{1}_A = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $\mathbb{1}_B = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Note that $\mathbb{1}_A \in B$ but $\mathbb{1}_A \neq \mathbb{1}_B$ in B. Now if $a = \left(\begin{smallmatrix}\alpha & 0\\ 0 & 0\end{smallmatrix}\right) \in A$, then $\sigma_A(a) = \{\alpha\}$, but $\sigma_B(a) = \{0, \alpha\}$. Evidently $\sigma_A(a) \neq \sigma_B(a)$. In fact, we have $\sigma_A(a) \subset \sigma_B(a)$ despite the fact that $A \subset B$.

Theorem 3.12. *Let A be a unital C^*-algebra generated by a single normal element h. Then there is an isometric *-isomorphism between A and the algebra of continuous functions on $\sigma_A(h)$ which maps polynomials in h to the same polynomial on $\sigma_A(h)$.*

Proof. A is a commutative C^*-algebra and is isomorphic as a C^*-algebra to $\mathcal{C}(\operatorname{Sp} A)$ via the Gelfand transform $\widehat{\ } : A \to \mathcal{C}(\operatorname{Sp} A)$. On the other hand, $\widehat{h} : \operatorname{Sp} A \to \sigma_A(h)$ is a homeomorphism. Define $\alpha : \mathcal{C}(\operatorname{Sp} A) \to \mathcal{C}(\sigma_A(h))$ by $\alpha(f) = f \circ \widehat{h}^{-1}$ so that $\alpha(f)(\lambda) = f \circ \widehat{h}^{-1}(\lambda))$ and, in particular, $\alpha(\widehat{h})(\lambda) = \lambda$ for $\lambda \in \sigma_A(h)$. Then α is an isometric *-isomorphism from $\mathcal{C}(\operatorname{Sp} A)$ onto $\mathcal{C}(\sigma_A(h))$. Hence $\alpha \circ \widehat{\ } : A \to \mathcal{C}(\sigma_A(h))$ is an isometric *-isomorphism.

Let p be a polynomial. Then

$$\begin{aligned}\big(\alpha \circ \widehat{\ }(p(h))\big)(\lambda) &= \alpha(p(\widehat{h}))(\lambda)\\ &= p(\alpha(\widehat{h}))(\lambda)\\ &= p(\alpha(\widehat{h})(\lambda))\\ &= p(\lambda)\end{aligned}$$

for any $\lambda \in \sigma_A(h)$. ∎

Theorem 3.13. *Let Ω be a compact Hausdorff space, and let A be the commutative unital C^*-algebra $\mathcal{C}(\Omega)$. Then* $\operatorname{Sp} A \simeq \Omega$.

Proof. For each ω in Ω, define $\varphi_\omega : A \to \mathbb{C}$ by $\varphi_\omega(a) = a(\omega)$, $a \in A$. Then, clearly, $\varphi_\omega \in \operatorname{Sp} A$. Moreover, if $\varphi_{\omega_1} = \varphi_{\omega_2}$, then $a(\omega_1) = a(\omega_2)$ for all $a \in A$. Since $A = \mathcal{C}(\Omega)$ separates points of Ω, we have $\omega_1 = \omega_2$. Thus $\omega \mapsto \varphi_\omega$ is one-one, i.e., it is an identification of Ω as a subset of $\operatorname{Sp} A$. We shall show that this mapping is onto $\operatorname{Sp} A$.

To see this, let $\ell \in \operatorname{Sp} A$, and suppose that ℓ is not of the form φ_ω for any $\omega \in \Omega$. Then $\ell - \varphi_\omega \neq 0$ for all $\omega \in \Omega$. This means that for each $\omega \in \Omega$ there is some element $a_\omega \in A$ such that $\ell(a_\omega) - \varphi_\omega(a_\omega) \neq 0$; that is, $\ell(a_\omega) \neq a_\omega(\omega)$. Put $b_\omega = a_\omega - \ell(a_\omega)\mathbb{1}$. Then $b_\omega \neq 0$, indeed, $b_\omega(\omega) \neq 0$, but $\ell(b_\omega) = 0$.

Now, b_ω is continuous on Ω, and so there is a neighbourhood N_ω of ω such that b_ω does not vanish on N_ω. As ω varies over Ω, we obtain an open cover $\{N_\omega : \omega \in \Omega\}$ of Ω. Since Ω is compact, there is a finite subcover $\{N_{\omega_1}, \dots, N_{\omega_k}\}$, say. Put

$$x = |b_{\omega_1}|^2 + \cdots + |b_{\omega_k}|^2.$$

Then $x \in A = \mathcal{C}(\Omega)$, and $x(\omega) > 0$ for all $\omega \in \Omega$. Furthermore,

$$\begin{aligned}\ell(x) &= \ell(b_{\omega_1}^* b_{\omega_1}) + \cdots + \ell(b_{\omega_k}^* b_{\omega_k}) \\ &= \ell(b_{\omega_1}^*)\ell(b_{\omega_1}) + \cdots + \ell(b_{\omega_k}^*)\ell(b_{\omega_k}) \\ &= 0\end{aligned}$$

since $\ell(b_{\omega_j}) = 0$ for each $j = 1, \dots, k$. Thus, $x \in \ker \ell$ and $x > 0$ on Ω. But $x > 0$ implies that x^{-1} exists in $A = \mathcal{C}(\Omega)$, and therefore $\ell(\mathbb{1}) = \ell(x)\ell(x^{-1}) = 0$, which is impossible. We deduce that there is some $\omega_0 \in \Omega$ such that $\ell = \varphi_{\omega_0}$. Hence the map $\varphi : \omega \mapsto \varphi_\omega$ is one-one and onto $\operatorname{Sp} A$.

It remains to show that this is a homeomorphism. Since both Ω and $\operatorname{Sp} A$ are compact, it is enough to show that φ is continuous (for then φ^{-1} is automatically also continuous). Let $U \subseteq \operatorname{Sp} A$ be a non-empty open set and suppose that $\omega_0 \in \varphi^{-1}(U)$, so that $\varphi_{\omega_0} \in U$. Since U is open, there is $\varepsilon > 0$ and a finite set $S \subset A$ such that the w^*-neighbourhood $\mathfrak{N}(\varphi_{\omega_0} : S, \varepsilon) \subseteq U$. Each $x \in S$ is continuous at ω_0 and so there are open neighbourhoods V_x of ω_0 in Ω such that $|x(\omega) - x(\omega_0)| < \varepsilon$ for all $\omega \in V_x$. Put $V = \bigcap_{x \in S} V_x$. Then V is open in Ω and $\omega_0 \in V$. For any $\omega \in V$, we have $|x(\omega) - x(\omega_0)| < \varepsilon$, for all $x \in S$, that is, $|\varphi_\omega(x) - \varphi_{\omega_0}(x)| < \varepsilon$, for all $x \in S$. Hence $\varphi(V) \subseteq \mathfrak{N}(\varphi_{\omega_0} : S, \varepsilon)$ and thus $\omega_0 \in V \subseteq \varphi^{-1}(\mathfrak{N}(\varphi_{\omega_0} : S, \varepsilon))$ and it follows that φ is continuous.

The continuity of φ can also be easily seen using nets. Indeed, suppose that $\omega_\alpha \to \omega$ in Ω. Then, for each $x \in A = \mathcal{C}(\Omega)$, we have $\varphi_{\omega_\alpha}(x) = x(\omega_\alpha) \to x(\omega)$ since x is a continuous function on Ω. But $x(\omega) = \varphi_\omega(x)$, and so we conclude that $\varphi_{\omega_\alpha} \to \varphi_\omega$ with respect to the w^*-topology on $\operatorname{Sp} A$. ∎

We shall now consider the problem of giving a unit to a C^*-algebra when one does not already exist. Let A be a C^*-algebra, and let $\mathcal{B}(A)$ denote the Banach algebra of bounded operators on A. Let $\widetilde{A}$ denote the normed subalgebra of $\mathcal{B}(A)$ consisting of those elements of the form $L_a + \alpha\mathbb{1}$, with $a \in A$ and $\alpha \in \mathbb{C}$, where L_a is the operator $x \mapsto ax$, $x \in A$, and $\mathbb{1}$ is the unit operator in $\mathcal{B}(A)$. Then $\|L_a x\| = \|ax\| \le \|a\|\|x\|$, so we have $\|L_a\| \le \|a\|$.

The map $a \mapsto L_a$ from A into $\widetilde{A}$ is linear, and since $ab \mapsto L_{ab} = L_a L_b$, we see that it is an algebra homomorphism. Furthermore,

$$\begin{aligned}\|L_a a^*\| &= \|aa^*\| = \|a^*\|^2 \\ &= \|a\|^2 = \|a\|\|a^*\|\end{aligned}$$

which, together with the inequality $\|L_a\| \le \|a\|$, implies that $\|L_a\| = \|a\|$. Thus the map $a \mapsto L_a$ is isometric and, in particular, injective.

Define an involution on $\widetilde{A}$ by $(L_a + \alpha\mathbb{1})^* = L_{a^*} + \overline{\alpha}\mathbb{1}$. Then the map $a \mapsto L_a$ of A into $\widetilde{A}$ is an isometric *-homomorphism.

If A has a unit, then $L_{\alpha\mathbb{1}} = \alpha\mathbb{1}$, so that every element of $\widetilde{A}$ has the form L_a for some $a \in A$. In this case, $a \mapsto L_a$ is an isometric *-isomorphism of A onto $\widetilde{A}$ (and so $\widetilde{A}$ is also a C^*-algebra).

If A does not have a unit, then clearly, identifying A with $\{L_a : a \in A\}$, we see that A is a *-subalgebra of $\widetilde{A}$ with codimension 1.

We claim that $\widetilde{A}$ is a unital C^*-algebra. We must show that

$$\|(L_a + \alpha\mathbb{1})^*(L_a + \alpha\mathbb{1})\| = \|L_a + \alpha\mathbb{1}\|^2$$

and that $\widetilde{A}$ is complete.

For $\alpha \in \mathbb{C}$, and $a, x \in A$, we have

$$\begin{aligned}\|(L_a + \alpha\mathbb{1})x\|^2 &= \|ax + \alpha x\|^2 \\ &= \|(ax + \alpha x)^*(ax + \alpha x)\| \\ &= \|(x^*a^* + \overline{\alpha}x^*)(ax + \alpha x)\| \\ &= \|x^*(L_a + \alpha\mathbb{1})^*(L_a + \alpha\mathbb{1})x\| \\ &\le \|x^*\|\, \|(L_a + \alpha\mathbb{1})^*(L_a + \alpha\mathbb{1})\|\, \|x\| \\ &= \|(L_a + \alpha\mathbb{1})^*(L_a + \alpha\mathbb{1})\|\, \|x\|^2.\end{aligned}$$

Therefore,

$$\|L_a + \alpha\mathbb{1}\|^2 \le \|(L_a + \alpha\mathbb{1})^*(L_a + \alpha\mathbb{1})\| \qquad (*)$$

It follows that

$$\begin{aligned}\|L_a + \alpha\mathbb{1}\|^2 &\le \|(L_a + \alpha\mathbb{1})^*(L_a + \alpha\mathbb{1})\| \\ &\le \|(L_a + \alpha\mathbb{1})^*\|\, \|L_a + \alpha\mathbb{1}\|.\end{aligned}$$

We observe now that if $\|L_a + \alpha\mathbb{1}\| = 0$, then $L_a + \alpha\mathbb{1} = 0$ and therefore $ax = -\alpha x$ for all $x \in A$. If α were non-zero, this would imply that $-a/\alpha$ is a left unit for A, and that $-a^*/\overline{\alpha}$ is a right unit for A, which would mean that A has a unit. Since A does not have a unit, we must have $\alpha = 0$. But then $L_a = 0$ and so it follows that $a = 0$ (since $a \mapsto L_a$ is injective). The last inequality above therefore implies that

$$\begin{aligned} \|L_a + \alpha \mathbb{1}\| &\le \|(L_a + \alpha \mathbb{1})^*\| \\ &= \|L_{a^*} + \overline{\alpha}\, \mathbb{1}\|. \end{aligned}$$

Replacing a by a^* and α by $\overline{\alpha}$, we deduce that

$$\|L_a + \alpha \mathbb{1}\| = \|(L_a + \alpha \mathbb{1})^*\|.$$

This, together with the inequality $(*)$, gives

$$\begin{aligned} \|L_a + \alpha \mathbb{1}\|^2 \le \|L_a + \alpha \mathbb{1}^* L_a + \alpha \mathbb{1}\| &\le \|L_a + \alpha \mathbb{1}^*\| \, \|L_a + \alpha \mathbb{1}\| \\ &= \|L_a + \alpha \mathbb{1}\|^2 \end{aligned}$$

and so $\|L_a + \alpha \mathbb{1}^* L_a + \alpha \mathbb{1}\| = \|L_a + \alpha \mathbb{1}\|^2$, which is the C^*-property.

It remains to show that $\widetilde{A}$ is complete. To see this, let us define $\phi : \widetilde{A} \to \mathbb{C}$ to be the map $\phi(L_a + \alpha \mathbb{1}) = \alpha$. Note that ϕ is a well-defined linear functional on $\widetilde{A}$ because $L_a + \alpha \mathbb{1} = L_b + \beta \mathbb{1}$ implies that $L_{a-b} + (\alpha - \beta)\mathbb{1} = 0$ and hence, as above, $a = b$ and $\alpha = \beta$. The kernel of ϕ is precisely the set $\{L_a : a \in A\}$. Now, A is complete, and $a \mapsto L_a$ is isometric, and hence the range $\{L_a : a \in A\}$ of this mapping is a closed subset of $\mathcal{B}(A)$, i.e., the kernel of ϕ is a closed subset of $\mathcal{B}(A)$. It follows that ϕ is a continuous linear functional on $\widetilde{A}$.

Suppose that $(L_{a_n} + \alpha_n \mathbb{1})$ is a Cauchy sequence in $\widetilde{A}$. Then, evidently, $(\phi(L_{a_n} + \alpha_n \mathbb{1}))$ is a Cauchy sequence in $\mathbb{C}$, that is, (α_n) is a Cauchy sequence in $\mathbb{C}$, and so converges to α, say. Now

$$\|L_{a_n} - L_{a_m}\| \le \|(L_{a_n} - \alpha_n \mathbb{1}) - (L_{a_m} - \alpha_m \mathbb{1})\| + \underbrace{\|\alpha_n \mathbb{1} - \alpha_m \mathbb{1}\|}_{=|\alpha_n - \alpha_m|}$$

and so (L_{a_n}) is a Cauchy sequence in $\mathcal{B}(A)$. But $\|a_n - a_m\| = \|L_{a_n} - L_{a_m}\|$ implies that (a_n) is a Cauchy sequence in A. Hence there is $a \in A$ such that $a_n \to a$. It follows that $L_{a_n} \to L_a$ and $\alpha_n \mathbb{1} \to \alpha \mathbb{1}$ in $\widetilde{A}$, which implies that $L_{a_n} + \alpha_n \mathbb{1} \to L_a + \alpha \mathbb{1}$ in $\widetilde{A}$, and we deduce that $\widetilde{A}$ is complete.

Identifying A with $\{L_a : a \in A\}$, via the isometric *-isomorphism $a \mapsto L_a$ allows us to regard A as a C^*-subalgebra (and also a closed two-sided *-ideal) in $\widetilde{A}$, with codimension 1. We have thus proved the following theorem.

Theorem 3.14. *Let A be a C^*-algebra without a unit. Then A is isometrically *-isomorphic to a C^*-subalgebra of codimension 1 in a unital C^*-algebra.*

From the construction, we see that $\widetilde{A}$ is just $A \oplus \mathbb{C}$ with the involution $a \oplus \alpha \mapsto a^* \oplus \overline{\alpha}$, multiplication $(a \oplus \alpha)(b \oplus \beta) = (ab + \alpha b + \beta a) \oplus \alpha\beta$ and norm $\|(a \oplus \alpha)\| = \sup\{\|ax + \alpha x\| : x \in A, \quad \|x\| \leq 1\}$. The unit is $0 \oplus 1$. (Note that if A has a unit, then we saw that $\widetilde{A} = A$ and so, in this case, $\widetilde{A}$ is *not* the direct sum $A \oplus \mathbb{C}$.)

By virtue of this theorem, it is often possible to assume that there is a unit in a C^*-algebra under consideration. This may be undesirable, however; for example, the C^*-algebra of compact operators on an infinite-dimensional Hilbert space does not have a unit, and one may not be willing to step outside the compact operators for the privilege.

Theorem 3.15. *The norm on a C^*-algebra is unique.*

Proof. Suppose that $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are two norms on A with respect to each of which A is a C^*-algebra. Adjoin a unit to A if it does not have one (—note that $\widetilde{A}$, as a set, is just the set of pairs (a, λ) with $a \in A$ and $\lambda \in \mathbb{C}$, and so is defined independently of the norm on A. Let $h = h^* \in A$. Then

$$\begin{aligned} \|h\| &= \|\widehat{h}\|_\infty = \sup\{|\lambda| : \lambda \in \sigma(h)\} \\ &= |\!|\!|h|\!|\!| \end{aligned}$$

since inverses are defined algebraically (independently of the norm).

In general, for any $x \in A$,

$$\begin{aligned} \|x\|^2 = \|x^*x\| &= |\!|\!|x^*x|\!|\!| = |\!|\!|x|\!|\!|^2 \\ \implies \quad \|\cdot\| &= |\!|\!|\cdot|\!|\!|. \end{aligned}$$

■

Remark 3.16. This result also follows from the observation that for $h = h^* \in A$, $\|h\| = \sup\{|\ell(h)| : \ell \in \mathrm{Sp}(\mathfrak{A})\}$, where $\mathfrak{A}$ is the commutative unital C^*-algebra generated by h if A is unital, otherwise it is the commutative unital C^*-algebra obtained by adjoining a unit to the commutative C^*-algebra generated (in A) by h. The character space of $\mathfrak{A}$ is defined without reference to the norm.

Note that because of the uniqueness of the norm, we can assert that for any element a in a non-unital C^*-algebra A, the unital C^*-algebra generated by a in $\widetilde{A}$ is the same as the C^*-algebra obtained by adjoining a unit to the C^*-algebra generated by a in A.

Example 3.17. Consider $\mathcal{K}(\mathcal{H})$, the C^*-algebra of compact operators on the Hilbert space $\mathcal{H}$, where we suppose that $\mathcal{H}$ is infinite-dimensional. Let A be the unital *-subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of those operators of the form $a + \alpha\mathbb{1}$, with $a \in \mathcal{K}(\mathcal{H})$, $\alpha \in \mathbb{C}$ and where $\mathbb{1}$ denotes the identity in $\mathcal{B}(\mathcal{H})$. Notice that $a + \alpha\mathbb{1} = 0$ if and only if $a = 0$ and $\alpha = 0$, so that A and $\widetilde{\mathcal{K}(\mathcal{H})}$

are *algebraically* *-isomorphic. According to the preceeding discussion, A is a C^*-algebra when equipped with the norm

$$|||a+\alpha\mathbb{1}||| = \sup\{\|ax+\alpha x\| : x \in \mathcal{K}(\mathcal{H}),\ \|x\| \le 1\}\,.$$

With this norm, A is isomorphic to $\widetilde{\mathcal{K}(\mathcal{H})}$. The question is whether or not A and $\widetilde{\mathcal{K}(\mathcal{H})}$ are isomorphic as C^*-algebras when A is equipped with the operator norm inherited from $\mathcal{B}(\mathcal{H})$. If we knew that A were a C^*-algebra with respect to the operator norm then this would follow from the uniqueness of the norm in a C^*-algebra. To show this, we must show that A is complete with respect to the norm in $\mathcal{B}(\mathcal{H})$. This can be established by the same argument as that used above to show the completeness of $\widetilde{A}$. This done, we conclude that A is precisely $\widetilde{\mathcal{K}(\mathcal{H})}$, the C^*-algebra obtained by adjoining a unit to the C^*-algebra $\mathcal{K}(\mathcal{H})$.

Alternatively, we can see this by showing directly that the operator norm on A coincides with the above C^*-norm. To see this, let ξ be a unit vector in $\mathcal{H}$, and let $p : \mathcal{H} \to \mathcal{H}$ be the one-dimensional (orthogonal) projection in $\mathcal{B}(\mathcal{H})$ with $p\xi = \xi$. Then $p \in \mathcal{K}(\mathcal{H})$ and $\|p\| = 1$ so that

$$\begin{aligned}\|(a+\alpha\mathbb{1})\xi\| = \|a\xi+\alpha\xi\| &= \|(ap+\alpha p)\xi\| \\ &\le \|ap+\alpha p\| \\ &\le |||a+\alpha\mathbb{1}|||\,.\end{aligned}$$

Taking the supremum over all such $\xi \in \mathcal{H}$, we obtain

$$\|a+\alpha\mathbb{1}\| \le |||a+\alpha\mathbb{1}|||\,.$$

On the other hand, for any $x \in \mathcal{K}(\mathcal{H})$ with $\|x\| \le 1$,

$$\begin{aligned}\|ax+\alpha x\| &\le \|a+\alpha\mathbb{1}\|\|x\| \\ &\le \|a+\alpha\mathbb{1}\|\end{aligned}$$

and so, taking the supremum over all such x,

$$|||a+\alpha||| \le \|a+\alpha\mathbb{1}\|\,.$$

We conclude that the norm on A defined above coincides with the operator norm induced from $\mathcal{B}(\mathcal{H})$, and therefore $A = \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathbb{1}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, and is the C^*-algebra, $\widetilde{\mathcal{K}(\mathcal{H})}$, obtained by adjoining a unit to $\mathcal{K}(\mathcal{H})$.

We have seen that a commutative, unital C^*-algebra A is isometrically *-isomorphic to the C^*-algebra of continuous functions on a compact Hausdorff space, namely, $\mathcal{C}(\operatorname{Sp} A)$, where $\operatorname{Sp} A$ is the space of characters on A. Suppose now that A is a commutative C^*-algebra without a unit. Evidently, $\widetilde{A}$ is a commutative unital C^*-algebra and so is isomorphic to the C^*-algebra of continuous functions on *its* character space, $\operatorname{Sp}\widetilde{A}$. It is clear that any character on A extends uniquely to a character on $\widetilde{A}$ (—by simply assigning it the value 1 on $\mathbb{1}$, the unit in $\widetilde{A}$). Conversely, every character on $\widetilde{A}$ defines, by restriction, a complex-valued homomorphism on A. Such a restriction will also be a character provided it is non-zero on A. In other words, we can say that $\widetilde{A}$ has one more character than A, that being the character κ_0, say, given by $\kappa_0(a) = 0$ for all $a \in A \subset \widetilde{A}$ (—and, of course, $\kappa_0(\mathbb{1}) = 1$). This observation leads to the following result which complements **??**.

Theorem 3.18. *Let A be a commutative C^*-algebra without a unit. Then there is a locally compact, non-compact, Hausdorff space X such that A is isometrically *-isomorphic to $\mathcal{C}_0(X)$, the C^*-algebra of continuous complex-valued functions on X vanishing at infinity.*

Proof. Let $K = \operatorname{Sp}\widetilde{A}$. Then K is a compact Hausdorff space and $\widetilde{A} \simeq \mathcal{C}(K)$, by **??**. Thus A is isometrically *-isomorphic to a C^*-subalgebra of $\mathcal{C}(K)$ via the Gelfand transform on $\widetilde{A}$. Let $\kappa_0 \in K$ be the character on $\widetilde{A}$ as above;

$$\kappa_0(a) = \begin{cases} 0, & \text{for } a \in A \subset \widetilde{A} \\ 1, & a = \mathbb{1}\,. \end{cases}$$

Then, for any $a \in A$, $\widehat{a}(\kappa_0) = 0$, so that the image of A under $\widehat{\ }$ consists of functions in $\mathcal{C}(K)$ which vanish at $\kappa_0 \in K$. Conversely, suppose that $f \in \mathcal{C}(K)$ is such that $f(\kappa_0) = 0$. Let $x \in \widetilde{A}$ be such that $\widehat{x} = f$. Now x can be written as $x = a + \mu\mathbb{1}$, for some $a \in A$ and $\mu \in \mathbb{C}$, and we have

$$\begin{aligned} f(\kappa_0) = 0 &\implies \widehat{x}(\kappa_0) = 0 \\ &\implies \kappa_0(x) = 0 \\ &\implies \kappa_0(a) + \mu\kappa_0(\mathbb{1}) = 0 \\ &\implies \mu = 0, \text{ since } \kappa_0(a) = 0, \text{ for } a \in A. \end{aligned}$$

Thus $x \in A$, and we see that the Gelfand transform on $\widetilde{A}$ maps A onto the subalgebra of $\mathcal{C}(K)$ consisting of those functions which vanish on κ_0. (We could also see this by noting that the kernel of κ_0 is a maximal proper ideal in $\widetilde{A}$ which contains A, and hence must equal A, since A itself is a maximal proper ideal. But the kernel of κ_0 acting on $\mathcal{C}(K)$, via the Gelfand transform, is precisely the set of functions above.) Let $X = K \setminus \{\kappa_0\}$. Then X is locally compact and the map $g \mapsto g \restriction X$ defines an isometric *-isomorphism between $\{g \in \mathcal{C}(K) : g(\kappa_0) = 0\}$ and $\mathcal{C}_0(X)$. Hence $A \simeq \mathcal{C}_0(X)$.

It remains to verify that X is not compact. If X were compact, then κ_0 would be an isolated point of K and the element $e \in A \subset \widetilde{A}$ corresponding to the continuous function $\widehat{e}(\kappa) = \begin{cases} 0, & \kappa = \kappa_0 \\ 1, & \text{otherwise} \end{cases}$ would be a unit for A, contrary to hypothesis. Thus X is not compact. ∎

Chapter 4

The Spectral Theorem

We shall present several formulations of the spectral theorem. Each realises operators as "multiplication" operators in some sense. This corresponds to the concept of diagonalisation of matrices.

Theorem 4.1. *For any bounded normal operator a on a Hilbert space $\mathcal{H}$ there exists a family $\{\mu_\alpha\}$ of real regular Borel measures on $\sigma(a)$, the spectrum of a, such that $\mathcal{H}$ is unitarily equivalent to $\bigoplus_\alpha L^2(\sigma(a), d\mu_\alpha)$ and a is unitarily equivalent to multiplication by λ, $\lambda \in \sigma(a)$; i.e. if $\zeta \in \mathcal{H}$ corresponds to $f \in \bigoplus_\alpha L^2(\sigma(a), d\mu_\alpha)$, then $a\zeta$ corresponds to the map $\lambda \mapsto \lambda f(\lambda)$, for $\lambda \in \sigma(a)$.*

Proof. Let $\mathcal{A}$ be the commutative unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by a. Then we have seen (**??**) that the map $\psi : x \mapsto \widehat{x}(\widehat{a}\,(\cdot)^{-1})$, defines an isometric *-isomorphism from $\mathcal{A}$ onto $\mathcal{C}(\sigma(a))$ such that $\psi(ax)(\lambda) = \lambda\psi(x)(\lambda)$ for $x \in \mathcal{A}$ and $\lambda \in \sigma(a)$. Let $\phi : \mathcal{C}(\sigma(a)) \to \mathcal{A}$ denote the inverse of the map ψ.

Suppose first that there is a vector $\xi \in \mathcal{H}$ such that $\{\mathcal{A}\xi\}$ is dense in $\mathcal{H}$ (—such a vector ξ is called a cyclic vector for $\mathcal{A}$), and define μ_ξ to be the map

$$f \mapsto \mu_\xi(f) = (\phi(f)\xi, \xi)$$

for $f \in \mathcal{C}(\sigma(a))$. Then μ_ξ is a continuous positive linear functional on $\mathcal{C}(\sigma(a))$. By the Riesz-Markov theorem, there is a (regular Borel) measure μ_ξ on $\sigma(a)$ such that

$$\mu_\xi(f) = \int_{\sigma(a)} f \, d\mu_\xi \, .$$

Define $u : \mathcal{C}(\sigma(a)) \to \mathcal{H}$ by $uf = \phi(f)\xi$. Then

$$\begin{aligned} \|uf\|^2 &= (\phi(f)^*\phi(f)\xi, \xi) = (\phi(f^*f)\xi, \xi) \\ &= \int_{\sigma(a)} |f|^2 d\mu_\xi \\ &= \|f\|^2_{L^2(\sigma(a), d\mu_\xi)}. \end{aligned}$$

Hence u is isometric with a dense range (since ξ is assumed cyclic) and with a dense domain of definition in $L^2(\sigma(a), d\mu_\xi)$, namely $\mathcal{C}(\sigma(a))$. Therefore u extends uniquely to define a unitary operator : $L^2(\sigma(a), d\mu_\xi) \to \mathcal{H}$.

For any $f \in \mathcal{C}(\sigma(a))$, we have

$$\begin{aligned}(u^{-1} a\, u\ f)(\lambda) &= (u^{-1} a\, \phi(f)\xi)(\lambda)\\ &= \big(u^{-1}\phi(\phi^{-1}(a)f)\xi\big)(\lambda), \quad \text{since } \phi(\phi^{-1}(a)f) = a\phi(f),\\ &= (\phi^{-1}(a)f)(\lambda)\\ &= \phi^{-1}(a)(\lambda)\, f(\lambda)\\ &= \lambda\, f(\lambda), \quad \text{by the definition of } \phi^{-1}.\end{aligned}$$

By continuity, this holds for any $f \in L^2(\sigma(a), d\mu_\xi)$ and gives the required unitary equivalence.

If $\xi \in \mathcal{H}$ does not exist as above (iė; if there is no cyclic vector ξ), then, by Zorn's lemma, there is a family of orthogonal subspaces $\{\mathcal{H}_\alpha\}$ in $\mathcal{H}$, with $\bigoplus_\alpha \mathcal{H}_\alpha = \mathcal{H}$, and vectors $\xi_\alpha \in \mathcal{H}_\alpha$ such that $\mathcal{A}\xi_\alpha$ is dense in $\mathcal{H}_\alpha$.

As above, we construct $u_\alpha : L^2(\sigma(a), d\mu_{\xi_\alpha}) \to \mathcal{H}_\alpha$, for each α. Then $u = \bigoplus_\alpha u_\alpha$ gives the required unitary equivalence. ∎

The more conventional form of the spectral theorem is the following.

Theorem 4.2. *Let a be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a family $\{e_\lambda : \lambda \in \mathbb{R}\}$ of projections in $\mathcal{H}$ satisfying:*

(i) *e_λ is a strong limit of polynomials in a,*

(ii) *$e_\lambda e_\mu = e_\mu$ if $\mu \le \lambda$,*

(iii) $\text{s-lim}_{\varepsilon\downarrow 0}\, e_{\lambda+\varepsilon} = e_\lambda$, $\text{s-lim}_{\lambda\to-\infty}\, e_\lambda = 0$, $\text{s-lim}_{\lambda\to\infty}\, e_\lambda = \mathbb{1}$,

(iv) $a = \int_{\mathbb{R}} \lambda\, de_\lambda = \lim_{\varepsilon\downarrow 0} \int_{-\|a\|-\varepsilon}^{\|a\|} \lambda\, de_\lambda$, *where the integral is a norm convergent Stieltjes integral.*

Moreover, the family $\{e_\lambda\}$ is uniquely determined by (ii), (iii) and (iv).

Proof. To construct such a family $\{e_\lambda\}$, let $\mathcal{A}$ be the (commutative) unital C^*-algebra generated by a. Then $\mathcal{A} \simeq \mathcal{C}(K)$, via the Gelfand transform $\widehat{\ }$, where $K = \operatorname{Sp}\mathcal{A}$, and $\widehat{a}(\cdot)$ is real-valued, since a is self-adjoint. For given $\lambda \in \mathbb{R}$, define a function $p_\lambda(\cdot)$ on K by

$$p_\lambda(\cdot) = \chi_{\{\kappa\in K : \widehat{a}(\kappa)\le\lambda\}}(\cdot)\,,$$

iė; $p_\lambda(\kappa) = 1$ if $\widehat{a}(\kappa) \le \lambda$, otherwise $p_\lambda(\kappa) = 0$. Write $K_\lambda = \{\kappa \in K : \widehat{a}(\kappa) \le \lambda\}$. Since $\widehat{a}$ is continuous, it follows that K_λ is closed in K.

Let $\zeta(\cdot)$ be such that $\zeta \in \mathcal{C}(K)$, $0 \le \zeta \le 1$, $\zeta(\kappa) = 1$ for all $\kappa \in K_\lambda$, and $0 \le \zeta(\kappa) < 1$ for all $\kappa \notin K_\lambda$. (Such a ζ exists by Urysohn's lemma.) Clearly $\zeta^n(\kappa) \to p_\lambda(\kappa)$ for each $\kappa \in K$, as $n \to \infty$.

Let $\xi \in \mathcal{H}$. The map $\widehat{x} \mapsto (x\xi, \xi)$, $x \in \mathcal{A}$, defines a positive linear functional on $\mathcal{C}(K)$, and so, by the Riesz-Markov representation theorem, there is a regular Borel measure μ_ξ on K such that

$$(x\xi, \xi) = \int_K \widehat{x}(\kappa)\, d\mu_\xi(\kappa)\,.$$

Now, by the dominated convergence theorem, we see that $\zeta^n \to p_\lambda$ in $L^2(K, d\mu_\xi)$. In particular, (ζ^n) is L^2-Cauchy. Since $\zeta \in \mathcal{C}(K)$, there is $y \in \mathcal{A}$ such that $\zeta = \widehat{y}$, and so $\zeta^n = (y^n)\widehat{\ }$. Note that $\|y\| \le 1$. We have

$$\begin{aligned}
\|\zeta^n - \zeta^m\|^2_{L^2(K, d\mu_\xi)} &= \int_K |\widehat{y^n} - \widehat{y^m}|^2\, d\mu_\xi(\kappa) \\
&= ((y^n - y^m)^*(y^n - y^m)\xi, \xi) \\
&= \|(y^n - y^m)\xi\|^2
\end{aligned}$$

and so we see that $(y^n\xi)$ is a Cauchy sequence in $\mathcal{H}$, for each $\xi \in \mathcal{H}$. It follows that $\text{s-lim}_{n\to\infty} y^n$ exists, and defines a bounded operator which we denote by e_λ. Since ζ is real-valued, each y^n is self-adjoint, and so $e_\lambda = e_\lambda^*$. Furthermore, since $(\|y^n\|)$ is bounded (by 1) we have

$$e_\lambda^2 = \text{s-lim}_{n\to\infty} y^n y^n = \text{s-lim}_{n\to\infty} y^{2n} = e_\lambda .$$

Thus e_λ is a projection on $\mathcal{H}$ for each $\lambda \in \mathbb{R}$. Moreover, e_λ is a strong limit of polynomials in a because it is a strong limit of a sequence in $\mathcal{A}$, and each element of A is a norm limit of polynomials in a. This proves (i).

Note that if ζ' is another element of $\mathcal{C}(K)$ such that $0 \le \zeta'(\kappa) < 1$, for $\kappa \notin K_\lambda$, and $\zeta'(\kappa) = 1$, for $\kappa \in K$, then we repeat the proof to obtain $y' \in \mathcal{A}$ with $y'^n \to e'_\lambda$ strongly as $n \to \infty$. But both $(\widehat{y}^n)$ and $(\widehat{y'}^n)$ converge to p_λ in L^2 and so we can consider the combined sequence

$$y''_n = \begin{cases} y^n, & n \text{ even} \\ y'^n, & n \text{ odd.} \end{cases}$$

This is L^2-Cauchy, and, as before, there exists $e''_\lambda = \text{s-lim}\, y''_n$. Hence $e''_\lambda = e'_\lambda = e_\lambda$; i.e., e_λ is independent of the choice of ζ subject to its defining requirements.

To prove (ii), we note that if $\lambda \le \mu$, then $p_\lambda p_\mu = p_\mu$. Suppose that $\zeta_\lambda \in \mathcal{C}(K)$ determines e_λ and that $\zeta_\mu \in \mathcal{C}(K)$ determines e_μ, as above. Then $\zeta_\mu\zeta_\lambda$ will also give e_μ since $(\zeta_\mu\zeta_\lambda)^n(\kappa) \to p_\mu(\kappa)$ as $n \to \infty$, for each $\kappa \in K$. Hence, with the obvious notation, $(\widehat{y}_\lambda = \zeta_\lambda$ etc.)

$$e_\lambda e_\mu = \text{s-lim}_{n\to\infty}\, y^n_\lambda y^n_\mu$$

$$= \text{s-lim}_{n\to\infty} y^n_\mu \,, \text{ by the previous observation}$$
$$= e_\mu .$$

This proves (ii).

To prove (iii): for $\xi \in \mathcal{H}$, we have

$$\begin{aligned} \|(e_{\lambda+\varepsilon} - e_\lambda)\xi\|^2 &= \lim_{n\to\infty} \|(y^n_{\lambda+\varepsilon} - y_\lambda)\xi\|^2 \\ &= \lim_{n\to\infty} \int_K |(\zeta^n_{\lambda+\varepsilon} - \zeta^n_\lambda)(\kappa)|^2 d\mu_\xi \\ &= \int_K |p_{\lambda+\varepsilon(\kappa)} - p_\lambda(\kappa)|^2 d\mu_\xi \,. \end{aligned}$$

But $p_{\lambda+\varepsilon}(\kappa) \to p_\lambda(\kappa)$ as $\varepsilon \downarrow 0$ for each $\kappa \in K$, so by the dominated convergence theorem, the integral converges to 0, iė; $e_{\lambda+\varepsilon} \to e_\lambda$ strongly as $\varepsilon \downarrow 0$. Similarly, one verifies that $\|(\mathbb{1} - e_\lambda)\xi\| \to 0$ as $\lambda \to \infty$ and that $\|e_\lambda \xi\| \to 0$ as $\lambda \to -\infty$, which completes the proof of (iii). In fact, $e_\lambda \xi = 0$ for any $\lambda < -\|a\|$, and $e_\lambda \xi = \xi$ for all $\lambda > \|a\|$.

In order to establish (iv), we first observe that $|\widehat{a}(\,\cdot\,)| \leq \|a\|$. Divide the interval $(-\|a\| - \varepsilon, \|a\|]$ into n open-closed intervals, $(\lambda_j, \lambda_{j+1}]$, $1 \leq j \leq n$, of equal length. Then

$$\chi_{\{\kappa\in K\,:\,\widehat{a}(\kappa)\in I_j\}}(\,\cdot\,) = (p_{\lambda_{j+1}} - p_{\lambda_j})(\,\cdot\,).$$

Put $s_n(\kappa) = \sum_{j=1}^n \lambda_{j+1}(p_{\lambda_{j+1}} - p_{\lambda_j})(\kappa)$. Then it is easy to see that $s_n \to \widehat{a}$ uniformly on K as $n \to \infty$. Thus, for given $\delta > 0$, there is N such that for all $n > N$, and any $\xi \in \mathcal{H}$,

$$\begin{aligned} \int_K |s_n - \widehat{a}|^2 d\mu_\xi &< \delta \int_K d\mu_\xi \\ &= \delta(\xi, \xi) \\ &= \delta \|\xi\|^2. \end{aligned}$$

That is, for $n > N$, and $\xi \in \mathcal{H}$,

$$\left\| \left(\sum_{j=1}^n \lambda_{j+1}(e_{\lambda_{j+1}} - e_{\lambda_j}) - a \right) \xi \right\|^2 < \delta \, \|\xi\|^2$$

and so we see that the sum converges in norm to a;

$$a = \int_{\|a\|-\varepsilon}^{\|a\|} \lambda \, de_\lambda \,, \quad \varepsilon > 0 \text{ arbitrary.}$$

If $\lambda < \|a\|$, then $p_\lambda = 0$ implies that $e_\lambda = 0$, and if $\lambda > \|a\|$ then $p_\lambda = 1$ implies that $e_\lambda = \mathbb{1}$, so we can write

$$a = \int_{-\infty}^{\infty} \lambda \, de_\lambda$$

which completes the proof of part (iv).

To see that $\{e_\lambda\}$ is unique, we first note that since $e_\lambda - e_\mu$ is orthogonal to $e_\beta - e_\alpha$ whenever $(\mu, \lambda] \cap (\alpha, \beta] = \varnothing$ (since $\mu \le \lambda$ implies that $e_\lambda e_\mu = e_\mu$) it follows that

$$a^2 = \int \lambda \, de_\lambda \,, \; a^3 = \int \lambda^3 \, de_\lambda \,, \ldots, a^n = \int \lambda^n \, de_\lambda$$

(—just square the approximating sum etc.). Now let $g(\cdot)$ be the characteristic function of the interval $(-\|a\| - 1, \mu]$. Then, for any $\xi \in \mathcal{H}$,

$$\begin{aligned} \int g(\lambda) \, d(e_\lambda \xi, \xi) &= (e_\mu \xi, \xi) \,, \text{ since } e_\lambda = 0 \text{ for } \lambda < -\|a\| \,, \\ &= \|e_\mu \xi\|^2. \end{aligned}$$

Let $(\mathcal{P}_n(\lambda))$ be a sequence of polynomials converging pointwise to $g(\lambda)$ on $(-\|a\| - 1, \|a\|]$, and uniformly bounded on this interval. Then

$$\begin{aligned} \|e_\mu \xi\|^2 &= \int g(\lambda) \, d(e_\lambda \xi, \xi) \\ &= \lim_n \int \mathcal{P}_n(\lambda) \, d(e_\lambda \xi, \xi) \\ &= \lim_n (\mathcal{P}_n(a)\xi, \xi). \end{aligned}$$

Since $(\mathcal{P}_n(a)\xi, \xi)$ is defined independently of e_μ, we deduce that e_μ is, indeed, uniquely determined by a. ∎

Definition 4.3. The projections $\{e_\lambda\}$ are called the spectral projections of the self-adjoint operator a.

From (i) and (iv), we see that an operator $b \in \mathcal{B}(\mathcal{H})$ commutes with a if and only if b commutes with all spectral projections of a.

Note also that (i) gives strong convergence; e_λ need not belong to A. Indeed, A may contain no non-trivial projections at all; for example, if A is the unital C^*-algebra generated by the operator of multiplication by x acting on the Hilbert space $L^2([0, 1])$, then, by Weierstrass' theorem, $A \simeq \mathbb{C}[0, 1]$. (Note that the operator norm in A is the function sup-norm in this example.)

Theorem 4.4. *Let $\mathfrak{A}$ be a commutative unital C^*-algebra of operators on a Hilbert space $\mathcal{H}$. Then there is a compact Hausdorff space K and Borel measures $\{\mu_\alpha\}$ on K such that $\mathcal{H}$ is unitarily equivalent to $\bigoplus_\alpha L^2(K, d\mu_\alpha)$ and each $a \in \mathfrak{A}$ is unitarily equivalent to multiplication by a continuous function on K.*

Proof. The proof is just as before, with $K = \operatorname{Sp} \mathfrak{A}$, $\mu = \bigoplus_\alpha \mu_\alpha$ is the measure on K given by the vectors $\zeta_\alpha \in \mathcal{H}_\alpha$ such that $\mathfrak{A}\zeta_\alpha$ is dense in $\mathcal{H}_\alpha$, and $\bigoplus_\alpha \mathcal{H}_\alpha = \mathcal{H}$. The unitaries $u_\alpha : \mathcal{C}(K) \to \mathfrak{A}\zeta_\alpha$ are defined by $u_\alpha \widehat{a} = a\zeta_\alpha$, for $a \in \mathfrak{A}$. ∎

Chapter 5

Positive elements of a C^*-algebra

The notion of positivity plays a fundamental rôle in the development of the theory of C^*-algebras.

Definition 5.1. An element a in a C^*-algebra A is said to be positive if and only if $a = h^2$ for some self-adjoint element $h \in A$. We write $a \geq 0$. We write $a \geq b$ if and only if $a - b \geq 0$.

Proposition 5.2. *Let A be a C^*-algebra, and let $a \in A$ with $a = a^*$. Then $a \geq 0$ if and only if $\sigma(a) \subset [0, \infty)$, where $\sigma(a)$ is the spectrum of a considered as an element of $\widetilde{A}$ if A has no unit.*

Proof. Suppose that $\sigma(a) \subset [0, \infty)$. Let $\mathcal{A}(a)$ be the unital C^*-subalgebra of A (or $\widetilde{A}$ if A does not have a unit) generated by a. Then $\mathcal{A}(a) \simeq \mathcal{C}(\operatorname{Sp} \mathcal{A}(a))$ and $\operatorname{ran} \widehat{a} = \sigma(a) \subset [0, \infty)$, and so $\widehat{a}(\,\cdot\,) \geq 0$ as a function on $\operatorname{Sp} \mathcal{A}(a)$. Let $f \in \mathcal{C}(\operatorname{Sp} \mathcal{A}(a))$ be the positive square root of $\widehat{a}(\,\cdot\,)$. Then there is $h = h^* \in \mathcal{A}$ with $\widehat{h} = f$ and hence $\widehat{h}^2 = \widehat{a}$ which implies that $h^2 = a$. That is, $a \geq 0$ as an element of $\mathcal{A}(a)$. To see that, in fact, $a \geq 0$ as an element of A, we note that as a consequence of Weierstrass' theorem, there is a sequence of polynomials (p_n), say, with $p_n(0) = 0$ such that $p_n(t) \to \sqrt{t}$, uniformly on $[0, \|a\|]$. Since $\sigma(a) \subseteq [0, \|a\|]$, we deduce that

$$\begin{aligned} \widehat{h} &= \lim p_n(\widehat{h}^2), \text{ in } \mathcal{C}(\operatorname{Sp} \mathcal{A}(a)), \\ &= \lim p_n(\widehat{a}). \end{aligned}$$

It follows that $h = \lim p_n(a) \in A$, and we conclude that $a \geq 0$, as required.

Conversely, suppose that $a \geq 0$. Then $a = h^2$ for some $h = h^* \in A$. Hence $a = a^*$ and the unital C^*-algebra $\mathcal{A}(a)$ generated by a is commutative. It follows that $\sigma(a) = \operatorname{ran} \widehat{a} = \operatorname{ran} \widehat{h}^2 \subset [0, \infty)$, since $\widehat{h}$ is real-valued. ∎

Corollary 5.3. *Let $a \in A$ where A is a C^*-algebra without a unit. Then $a \geq 0$ in A if and only if $a \geq 0$ in $\widetilde{A}$.*

Proof. This follows immediately from the proposition. If $a \geq 0$ in A, then certainly $a \geq 0$ in $\widetilde{A}$. On the other hand, if $a \in A$ and $a \geq 0$ in $\widetilde{A}$, then $\sigma_{\widetilde{A}}(a) \subset [0, \infty)$. By the proposition, $a \geq 0$ in A. ∎

Corollary 5.4. *Suppose that the element a in a C^*-algebra A satisfies $a \geq 0$ and $a \leq 0$. Then $a = 0$.*

Proof. We have $a \geq 0$ and $-a \geq 0$, so that $\sigma(a) \subset [0, \infty)$ and $\sigma(-a) \subset [0, \infty)$. This last inclusion is equivalent to $\sigma(a) \subset [0, \infty)$. It follows that $\sigma(a) = \{0\}$. But then $\operatorname{ran} \widehat{a} = \{0\}$ (where $\widehat{a}$ is the Gelfand transform of a as an element of the commutative unital C^*-algebra it generates), and so $\widehat{a} = 0$ and we conclude that $a = 0$. ■

Corollary 5.5. *Every positive element of a C^*-algebra has a unique positive square root; i.e., if A is a C^*-algebra and $a \in A$ with $a \geq 0$, then there is a unique $s \in A$ with $s \geq 0$ and $s^2 = a$. Furthermore, if a is invertible, then so is the square root s.*

Proof. The existence of $s \in A$ with $s \geq 0$ and $s^2 = a$ follows from the construction in the first part of the proposition. To prove the uniqueness, suppose that also $t \in A$ with $t \geq 0$ and $t^2 = a$. Let $\mathcal{A}(t)$ be the commutative unital C^*-algebra generated by t. Then $\mathcal{A}(a)$, the C^*-algebra generated by a, is contained in $\mathcal{A}(t)$, so that $s, a \in \mathcal{A}(a) \subseteq \mathcal{A}(t)$. Now $\mathcal{A}(t) \simeq \mathcal{C}(\operatorname{Sp} \mathcal{A}(t))$, via the Gelfand map, and we have $\widehat{a} = \widehat{s}^2 = \widehat{t}^2$ where $\widehat{s} \geq 0$ and $\widehat{t} \geq 0$ since $s \geq 0$ and $t \geq 0$. It follows that $\widehat{s} = \widehat{t}$, and therefore $s = t$.

If a is invertible, then $0 \notin \sigma(a)$, i.e., $0 \notin \operatorname{ran} \widehat{a}$. Hence $\widehat{a} > 0$ on $\operatorname{Sp} \mathcal{A}(a)$. It follows that $\widehat{s} > 0$ on $\operatorname{Sp} \mathcal{A}(a)$ and so $0 \notin \sigma(s)$, i.e., s is invertible. ■

Theorem 5.6. *Let A be a C^*-algebra and let $h, k \in A$ with $h \geq 0$ and $k \geq 0$. Then $h + k \geq 0$.*

Proof. Suppose $a = a^* \in A$ and $\|a\| \leq 1$. Then $a \geq 0$ if and only if $\widehat{a} \geq 0$ as a function on $\operatorname{Sp} \mathcal{A}(a)$, where $\mathcal{A}(a)$ is the commutative unital C^*-algebra generated by a, i.e., if and only if $|1 - \widehat{a}(\,\cdot\,)| \leq 1$, i.e., if and only if $\|\mathbb{1} - a\| \leq 1$. (Note that $\|a\| \leq 1$ if and only if $|\widehat{a}(\,\cdot\,)| \leq 1$.) We have

$$\begin{aligned}
\left\| \mathbb{1} - \frac{h+k}{\|h\| + \|k\|} \right\| &= \frac{\| \, \|h\| + \|k\| - h - k \|}{\|h\| + \|k\|} \\
&\leq \frac{\| \|h\| - h \| + \| \|k\| - k \|}{\|h\| + \|k\|} \\
&= \frac{\|h\| \|(\mathbb{1} - h/\|h\|)\| + \|k\| \|(\mathbb{1} - k/\|k\|)\|}{\|h\| + \|k\|} \\
&\leq 1 \quad \text{since } h/\|h\| \geq 0 \text{ and } k/\|k\| \geq 0.
\end{aligned}$$

Since $\left\| \dfrac{h+k}{\|h\| + \|k\|} \right\| \leq 1$, we deduce that $\dfrac{h+k}{\|h\| + \|k\|} \geq 0$, and hence $h + k \geq 0$. ■

Corollary 5.7. *If $a \leq b$ and $b \leq c$ in a C^*-algebra A, then $a \leq c$ in A.*

Proof. We have $c - a = c - b + b - a \geq 0$ since $c - b \geq 0$ and $b - a \geq 0$. ■

Theorem 5.8. *Let A be a C^*-algebra, and let $a \in A$. Then $a \geq 0$ if and only if $a = x^*x$ for some $x \in A$.*

Proof. If $a \geq 0$, then $a = h^2$ for some $h = h^* \in A$.

Conversely, suppose that $a = x^*x$, for some $x \in A$. Then clearly $a = a^*$. To show that a is positive we may suppose that A is unital (—if not, we simply work with $\widetilde{A}$). Suppose that $a = x^*x$ is not positive. Then $\widehat{a} \in \mathcal{C}(\operatorname{Sp}\mathcal{A}(a))$ is a non-positive continuous function and so there is $\ell_0 \in \operatorname{Sp}(\mathcal{A}(a))$ such that $\widehat{a}(\ell_0) < 0$. Thus there is a neighbourhood $\mathfrak{N}$ of ℓ_0 in $\operatorname{Sp}\mathcal{A}(a)$ such that $\widehat{a}(\ell) < 0$ for all $\ell \in \mathfrak{N}$. Let $f \in \mathcal{C}(\operatorname{Sp}\mathcal{A}(a))$ be such that f vanishes on the closed set $\operatorname{Sp}\mathcal{A}(a) \setminus \mathfrak{N}$, $0 \leq f \leq 1$ on $\operatorname{Sp}\mathcal{A}(a)$ and $f(\ell_0) = 1$ (—such an f exists, by Urysohn's lemma). Then $f\widehat{a}f \leq 0$ on $\operatorname{Sp}\mathcal{A}(a)$ and $(f\widehat{a}f)(\ell_0) < 0$.

Let $b \in \mathcal{A}(a)$ be such that $\widehat{b} = f$. Then $b = b^*$, and $\widehat{b}\,\widehat{a}\,\widehat{b} = f\widehat{a}f \leq 0$ implies that $bab \leq 0$ (i.e., $-bab \geq 0$). Furthermore, $bab \neq 0$ since $\ell_0(bab) = (f\widehat{a}f)(\ell_0) < 0$.

Write $xb = h + ik$, with $h = h^* = \frac{1}{2}(xb + (xb)^*) \in \mathcal{A}(a)$ and $k = k^* = \frac{1}{2i}(xb - (xb)^*) \in \mathcal{A}(a)$. Then

$$\begin{aligned}(xb)^*(xb) &= (h - ik)(h + ik)\\ &= h^2 + k^2 + ihk - ikh\end{aligned}$$

and

$$(xb)(xb)^* = h^2 + k^2 + ikh - ihk.$$

Adding, we obtain that $(xb)^*(xb) + (xb)(xb)^* = 2(h^2 + k^2) \geq 0$ since $h^2 \geq 0$ and $k^2 \geq 0$. But $-(xb)^*(xb) = -bx^*xb = -bab \geq 0$, and so $(xb)(xb)^* = 2(h^2 + k^2) + (-(xb)^*(xb)) \geq 0$, being the sum of two positive elements. So we have that $\sigma((xb)^*(xb)) \subset (-\infty, 0]$ and $\sigma((xb)(xb)^*) \subset [0, \infty)$. But for any $y \in A$, the sets $\sigma(y^*y)$ and $\sigma(yy^*)$ differ by at most $\{0\}$, and so we conclude that $\sigma((xb)^*(xb)) = \{0\}$, i.e., $\sigma(bx^*xb) = \sigma(bab) = \{0\}$. But then $\operatorname{ran}\widehat{bab} = \sigma(bab) = \{0\}$ and so $\widehat{bab} = 0$ which implies that $bab = 0$. This is a contradiction, and we conclude that $x^*x \geq 0$. ■

Corollary 5.9. *Let A be a C^*-algebra and let $a \in A$ with $a \geq 0$, then $b^*ab \geq 0$ for any $b \in A$.*

Proof. This follows immediately from the theorem; if $a = x^*x$ for some $x \in A$, then $b^*ab = b^*x^*xb = (xb)^*(xb) \geq 0$. ■

Proposition 5.10. *Let a, b be self-adjoint elements of a C^*-algebra A.*

(i) *Suppose that $-b \leq a \leq b$. Then $\|a\| \leq \|b\|$.*

(ii) *Suppose that* $0 \le a \le b$ *and* a *and* b *are invertible. Then* $b^{-1} \le a^{-1}$.

Proof. By adjoining a unit, if necessary, and looking at Gelfand transforms, we see that $-\|b\| \le \widehat{b}(\,\cdot\,) \le \|b\|$ on $\mathrm{Sp}(\mathcal{A}(b))$ and so $b \le \|b\|\mathbb{1}$. Hence we have

$$-\|b\|\mathbb{1} \le -b \le a \le b \le \|b\|\mathbb{1}.$$

It follows that $-\|b\|\mathbb{1} \le a \le \|b\|\mathbb{1}$ and so $-\|b\| \le \widehat{a}(\,\cdot\,) \le \|b\|$ on $\mathrm{Sp}(\mathcal{A}(a))$. Hence $\|a\| \le \|b\|$. This proves (i).

To prove (ii), we note that $0 \le a \le b$ implies that $b^{-1/2}ab^{-1/2} \le \mathbb{1}$. Note that the invertibility of a and b requires that A be unital. Let c denote $b^{-1/2}ab^{-1/2}$. Then $c = c^*$, c is invertible and $0 < \widehat{c}(\,\cdot\,) \le 1$. Hence $\widehat{c}^{-1} \ge 1$ on $\mathrm{Sp}(\mathcal{A}(c))$ and so $c^{-1} \ge \mathbb{1}$. It follows that $b^{1/2}a^{-1}b^{1/2} \ge \mathbb{1}$ and therefore $a^{-1} \ge b^{-1/2}\mathbb{1}b^{-1/2} = b^{-1}$. ∎

Note that if, in fact, $0 \le a \le b$ and if a is invertible, then b is also invertible. To see this, we note that since a is invertible, there is $\gamma \in \mathbb{R}$ with $\gamma > 0$ such that $0 < \gamma \le \widehat{a}$ on $\mathrm{Sp}\,\mathcal{A}(a)$, where, as usual, $\mathcal{A}(a)$ is the commutative unital C^*-subalgebra of A generated by a. Hence $a \ge \gamma\mathbb{1}$. But then, $b - \gamma\mathbb{1} = (b-a) + (a - \gamma\mathbb{1})$ is the sum of two positive elements of A and therefore is itself positive. This means that $\widehat{b} \ge \gamma > 0$ on $\mathrm{Sp}\,\mathcal{A}(b)$, and so $\widehat{b}$ has an inverse in $C(\mathrm{Sp}(\mathcal{A}(b)))$, which implies that b is invertible.

Proposition 5.11. *Let* a *and* b *be elements of a unital* C^**-algebra* A*. If* $0 \le a \le b$*, then* $a(\mathbb{1}+a)^{-1} \le b(\mathbb{1}+b)^{-1}$.

Proof. Using Gelfand transforms, it is evident that both $(\mathbb{1}+a)$ and $(\mathbb{1}+b)$ are invertible. We have

$$\begin{aligned} & 0 \le a \le b \\ \implies & \mathbb{1} + a \le \mathbb{1} + b \\ \implies & (\mathbb{1}+b)^{-1} \le (\mathbb{1}+a)^{-1} \\ \implies & -(\mathbb{1}+a)^{-1} \le -(\mathbb{1}+b)^{-1} \\ \implies & \mathbb{1} - (\mathbb{1}+a)^{-1} \le \mathbb{1} - (\mathbb{1}+b)^{-1} \end{aligned}$$

i.e., $a(\mathbb{1}+a)^{-1} \le b(\mathbb{1}+b)^{-1}$. ∎

Proposition 5.12. *Let* $a \in A$*, where* A *is a unital* C^**-algebra. Then* a *can be written as a linear combination of*

(i) *two self-adjoint elements of* A*,*

(ii) *four positive elements of* A*,*

(iii) *four unitary elements of* A*.*

Proof. (i) $a = \frac{1}{2}(a + a^*) + i\frac{1}{2i}(a - a^*)$.

(ii) Let $h = h^* \in A$. Let $|h|$ denote the positive square root of h^2. Then, by looking at $\widehat{h}$, we see that $|h| \pm h$ are both ≥ 0. Write $h = \frac{1}{2}(h + |h|) - \frac{1}{2}(|h| - h)$ and use (i).

(iii) Let $h = h^* \in A$. Then $h^2 \geq 0$. Suppose that $\|h\| < 1$. Then $\mathbb{1} - h^2 \geq 0$ and so has a positive square root, $(\mathbb{1} - h^2)^{1/2}$. Put $u = h + i(\mathbb{1} - h^2)^{1/2}$. Then one checks that $u^*u = uu^* = \mathbb{1}$, i.e., u is unitary. Moreover, $h = \frac{1}{2}(u + u^*)$.

If $\|h\| \geq 1$ then consider αh with $\alpha = \dfrac{1}{2\|h\|}$. Then, as above, we can write $\alpha h = \frac{1}{2}(v + v^*)$ with v unitary. Thus $h = \frac{1}{2\alpha}(v + v^*)$. For general $h \in A$, the result now follows from part (i). ■

Remark 5.13. Notice that parts (i) and (ii) remain valid even if A has no unit. This is because the elements a, a^*, h and $|h|$ all belong to A, if a does.

The positive elements $h_\pm = |h| \pm h$ in part (ii) satisfy $h_+ h_- = 0$ as is readily computed. The decomposition of a self-adjoint element h into two such positive elements with "disjoint support" is unique. This is seen as follows. Suppose that $h = h^*$ can be written as $h = k_+ - k_-$ where $k_\pm \geq 0$ and $k_+ k_- = 0$. Then we have

$$\begin{aligned} |h|^2 = h^2 &= (k_+ - k_-)^2 \\ &= k_+^2 + k_-^2 \\ &= (k_+ + k_-)^2 . \end{aligned}$$

Now, $(k_+ + k_-)$ is positive and so, by the uniqueness of the positive square root, we must have $(k_+ + k_-) = |h|$. This, together with the equality $h = k_+ - k_-$, implies that $2k_+ = |h| + h = 2h_+$ and $2k_- = |h| - h = 2h_-$ which establishes the uniqueness. The positive elements $h_+ = \frac{1}{2}(|h| + h)$ and $h- = \frac{1}{2}(|h| - h)$ are called the positive and negative parts of h.

Example 5.14. Suppose that p is a non-trivial projection in the unital C^*-algebra$A$. Set $h = \cos\alpha\mathbb{1} = \cos\alpha p + \cos\alpha p^\perp$, where, say, $0 < \alpha < \pi/2$ and $p^\perp = \mathbb{1} - p$. Clearly $h = h^*$ and $\|h\| < 1$. Then $u = e^{i\alpha}\mathbb{1}$ and $v = e^{i\alpha}p + e^{-i\alpha}p^\perp$ are distinct unitary elements of A, but we have $h = \frac{1}{2}(u + u^*) = \frac{1}{2}(v + v^*)$.

As another example, let A be the direct sum of C^*-algebras (A_α), and let $u = (u_\alpha) \in A$, with each u_α unitary. Let $v = (v_\alpha) \in A$ be obtained from u by changing some of the u_α's to u_α^*. Then u and v are unitary in $A$and, in general, $u \neq v$. However, $u + u^*$ can also be written as $v + v^*$.

Proposition 5.15. *Let A be a C^*-algebra. Then A_+, the set of positive elements of A, is closed in A.*

Proof. Let (h_n) be a sequence in A_+ such that $h_n \to h$ in A. We must show that $h \in A_+$. First we observe that if $h = h^*$, $h \in A$, then $h \geq 0$ if and only if $\|h - \|h\|\mathbb{1}\| \le \|h\|$ (in $\widetilde{A}$, if A does not have a unit). Indeed,

$$\begin{aligned}\|h - \|h\|\mathbb{1}\| &= \sup_{\kappa \in \operatorname{Sp}\mathcal{A}(h)} |\widehat{h}(\kappa) - \|h\|| \\ &= \sup_{\kappa}(\|h\| - \widehat{h}(\kappa)), \text{ since } \operatorname{ran}\widehat{h} \in \mathbb{R} \text{ and } \|h\| = \|\widehat{h}\|_\infty \ge \widehat{h}(\kappa), \\ &\le \|h\|, \text{ if and only if } \widehat{h}(\kappa) \ge 0 \text{ for all } \kappa,\end{aligned}$$

i.e., if and only if $\sigma(h) \subset [0, \infty)$ if and only if $h \ge 0$.

Now, $h_n = h_n^*$ and therefore $h^* = \lim h_n^* = \lim h_n = h$. We also have $\|h_n\| \to \|h\|$. By the above remark, $\|h_n - \|h_n\|\mathbb{1}\| \le \|h_n\|$. Letting $n \to \infty$, we see that $\|h - \|h\|\mathbb{1}\| \le \|h\|$ and so, again by the above remark, it follows that $h \ge 0$. ∎

Proposition 5.16. *Let $a, b \in A$ and suppose that ab is self-adjoint. Then $\|ab\| \le \|ba\|$.*

Proof. By adjoining a unit, if necessary, we may suppose that A is unital. Since $ab = (ab)^*$, it follows that

$$\begin{aligned}\|ab\| &= r(ab), \text{ the spectral radius of } ab \ (= \|\widehat{ab}\|_\infty), \\ &= \sup\{|\lambda| : \lambda \in \sigma(ab)\} \\ &= \sup\{|\lambda| : \lambda \in \sigma(ba)\}, \text{ since } \sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}, \\ &\le \|ba\|,\end{aligned}$$

as required. ∎

Proposition 5.17. *Let $0 \le a \le b$ be elements of a C^*-algebra A. Then $a^{1/2} \le b^{1/2}$.*

Proof. We may suppose that A is unital (if not, we simply consider $\widetilde{A}$). Suppose first that a and b are both invertible. Then $a \le b$ implies that $0 \le b^{-1/2}ab^{-1/2} \le \mathbb{1}$. Hence,

$$\begin{aligned}\|b^{-1/4}a^{1/2}b^{-1/4}\|^2 &\le \|b^{-1/2}a^{1/2}\|^2, \text{ as above}, \\ &= \|b^{-1/2}a^{1/2}a^{1/2}b^{-1/2}\|, \text{ by the } C^*\text{-property}, \\ &= \|b^{-1/2}ab^{-1/2}\| \\ &\le 1.\end{aligned}$$

Therefore $\|b^{-1/4}a^{1/2}b^{-1/4}\| \le 1$ and so $0 \le b^{-1/4}a^{1/2}b^{-1/4} \le \mathbb{1}$. That is, $a^{1/2} \le b^{1/2}$.

In general, for any $\varepsilon > 0$, $0 \le a \le b$ implies that $0 \le \varepsilon\mathbb{1} \le a + \varepsilon\mathbb{1} \le b + \varepsilon\mathbb{1}$, and $a + \varepsilon\mathbb{1}$ and $b + \varepsilon\mathbb{1}$ are invertible. So, by the above argument, we see that $(a + \varepsilon\mathbb{1})^{1/2} \le (b + \varepsilon\mathbb{1})^{1/2}$, that is, $(b + \varepsilon\mathbb{1})^{1/2} - (a + \varepsilon\mathbb{1})^{1/2} \ge 0$.

Now, as $\varepsilon \downarrow 0$, $a + \varepsilon\mathbb{1} \to a$ in A, and since the map $t \mapsto \sqrt{t}$ is uniformly continuous on $[0, M]$, with $M = \|a\| + 1$, say, it follows that $\sqrt{\widehat{a} + \varepsilon} - \sqrt{\widehat{a}} \to 0$ uniformly on $\operatorname{Sp}\mathcal{A}(a)$, as $\varepsilon \downarrow 0$. In other words, $\|\sqrt{\widehat{a} + \varepsilon} - \sqrt{\widehat{a}}\|_\infty \to 0$ as $\varepsilon \downarrow 0$, i.e., $\|(a + \varepsilon\mathbb{1})^{1/2} - a^{1/2}\| \to 0$ as $\varepsilon \downarrow 0$.

Similarly, $\|(b + \varepsilon\mathbb{1})^{1/2} - b^{1/2}\| \to 0$ in A as $\varepsilon \downarrow 0$. Hence $b^{1/2} - a^{1/2} = \lim_{\varepsilon\downarrow 0}(b + \varepsilon\mathbb{1})^{1/2} - (a + \varepsilon\mathbb{1})^{1/2}$ belongs to A_+, since A_+ is closed. ∎

Remark 5.18. In general, the inequality $a \le b$ does not imply that $a^2 \le b^2$.

We shall now turn to the discussion of approximate units. Let J be a left ideal in a unital C^*-algebra A, and let Λ be the collection of finite subsets of J ordered by set inclusion $\subseteq$. Then Λ is a directed set. For each $\lambda \in \Lambda$, define $y_\lambda = \sum_{x\in\lambda} x^*x$, and set $u_\lambda = n_\lambda y_\lambda(\mathbb{1} + n_\lambda y_\lambda)^{-1}$, where n_λ denotes the number of members of λ. Note that $n_\lambda y_\lambda \ge 0$ and so $(\mathbb{1} + n_\lambda y_\lambda)$ is invertible. We note also that $u_\lambda \in J$, $u_\lambda = u_\lambda^*$ and that $0 \le u_\lambda \le \mathbb{1}$.

Theorem 5.19. *Let J and (u_λ) be as above.*

(i) *If $\lambda \le \mu$, then $u_\lambda \le u_\mu$.*

(ii) *For any $x \in J$, $x - xu_\lambda \to 0$ along the directed set Λ.*

Proof. (i) If $\lambda \le \mu$, then clearly $n_\lambda y_\lambda \le n_\mu y_\mu$. Hence

$$\begin{aligned} u_\lambda &= \mathbb{1} - (\mathbb{1} + n_\lambda y_\lambda)^{-1} \\ &\le \mathbb{1} - (\mathbb{1} + n_\mu y_\mu)^{-1} \\ &= u_\mu. \end{aligned}$$

(ii) Let $x \in J$. Then

$$\begin{aligned} \|x - xu_\lambda\|^2 &= \|(x - xu_\lambda)^*\|^2 \\ &= \|x(\mathbb{1} - u_\lambda)(\mathbb{1} - u_\lambda)x^*\| \\ &\le \|x(\mathbb{1} - u_\lambda)x^*\|, \text{ since } (\mathbb{1} - u_\lambda)^2 \le (\mathbb{1} - u_\lambda), \\ &= \|x(\mathbb{1} + n_\lambda y_\lambda)^{-1}x^*\| \\ &\le \|x(\mathbb{1} + mx^*x)^{-1}x^*\| \end{aligned}$$

whenever λ has at least m elements and contains x; since in this case, $n_\lambda y_\lambda \ge mx^*x$,

$$\begin{aligned} &\le \|x(\mathbb{1} + mx^*x)^{-1}\|\,\|x^*\| \\ &= \|(\mathbb{1} + mx^*x)^{-1}x^*x(\mathbb{1} + mx^*x)^{-1}\|^{1/2}\|x^*\| \\ &\le \frac{1}{m^{1/2}}\|mx^*x(\mathbb{1} + mx^*x)^{-1}\|^{1/2}\|x^*\| \text{ since } \|(\mathbb{1} + mx^*x)^{-1}\| \le 1, \\ &\le \frac{\|x^*\|}{m^{1/2}} \end{aligned}$$

since $\|mx^*x(\mathbb{1} + mx^*x)^{-1}\| \le 1$. ∎

The properties of the net (u_λ) are the basis for the following definition.

Definition 5.20. An approximate unit for a left ideal J in a C^*-algebra A is a net (u_λ) of elements of J, indexed by Λ, say, satisfying

(i) $0 \le u_\lambda$, $\|u_\lambda\| \le 1$ and $u_\lambda \le u_\mu$ whenever $\lambda \le \mu$, $\lambda,\, \mu \in \Lambda$.

(ii) For any $x \in J$, $\|x - xu_\lambda\| \to 0$ along Λ.

An approximate unit for a right ideal J is a net (u_λ) in J satisfying (i) together with the following obvious modification of (ii).

(ii)′ For any $x \in J$, $\|x - u_\lambda x\| \to 0$ along Λ.

Evidently, if J is a left ideal, then $J^* = \{j^* : j \in J\}$ is a right ideal of A. Suppose that (u_λ) is an approximate unit for J, then, for any $x \in J^*$,

$$\begin{aligned}\|x - u_\lambda x\| &= \|(x - u_\lambda x)^*\| \\ &= \|x^* - x^* u_\lambda\| \\ &\to 0, \qquad \text{since } x^* \in J,\end{aligned}$$

so that (u_λ) is an approximate unit for the right ideal J^*.

An approximate unit for a C^*-algebra A is a net (u_λ) in A satisfying condition (i) together with the following.

(ii)″ For any $a \in A$, $\|a - au_\lambda\| \to 0$ along Λ, and hence also (as above) $\|a - u_\lambda a\| \to 0$ along Λ.

Theorem 5.21. *Suppose that J is a left (resp., right) ideal in a C^*-algebra A. Then J has an approximate unit.*

Proof. Let J be a left ideal of A. We may assume, without loss of generality, that A has a unit. (If not, then we can embed A as a two-sided ideal into $\widetilde{A}$, and then J is a left ideal of $\widetilde{A}$.) An approximate unit is then constructed as above.

Now if J is a right ideal, then J^* is a left ideal and so has an approximate unit, as above. But then we have seen that this is also an approximate unit for the right ideal J. ∎

Corollary 5.22. *Let A be a C^*-algebra. Then A possesses an approximate unit.*

Proof. This follows immediately from the theorem by taking $A = J$ and noting that $A = A^*$. ∎

Of course, if A is unital, then we may take u_λ to be $\mathbb{1}$, for each λ. The point is that even if A is not unital, it still has an approximate unit. The following results show the usefulness of approximate units.

Theorem 5.23. *Let J be a closed two-sided ideal in a C^*-algebra A. Then $J = J^*$, i.e., $j \in J$ if and only if $j^* \in J$.*

Proof. Let $j \in J$, and let u_λ be an approximate unit for J as a left ideal. Then $j = \lim j u_\lambda$ and it follows that $j^* = \lim u_\lambda j^*$. But $u_\lambda j^* \in J$ since $u_\lambda \in J$ and J is a two-sided ideal, and, since J is closed, we deduce that $j^* \in J$. ∎

Theorem 5.24. *Let J be a closed two-sided ideal in a C^*-algebra A and let (u_λ) be an approximate unit for J. Then for any $x \in A$*

$$\lim \|x - xu_\lambda\| = \|\operatorname{cl} x\|$$

where $\|\operatorname{cl} x\|$ is the norm of $\operatorname{cl} x$ in A/J.

Proof. Let $x \in A$. By definition, $\|\operatorname{cl} x\| = \inf_{j \in J} \|x + j\|$. Put $\alpha = \|\operatorname{cl} x\|$. Then

$$\begin{aligned}
\alpha^2 &= \inf_{j \in J} \|x + j\|^2 \le \|x - xu_\lambda\|^2, \quad \text{since } xu_\lambda \in J, \\
&= \|x(\mathbb{1} - u_\lambda)^2 x^*\| \\
&\le \|x(\mathbb{1} - u_\lambda)x^*\|.
\end{aligned}$$

Now, $x(\mathbb{1} - u_\lambda)x^*$ is decreasing in λ and so $\|x(\mathbb{1} - u_\lambda)x^*\|$ is decreasing and hence converges to its infimum, β, say. It follows that $\alpha^2 \le \beta$.

On the other hand, for given $\varepsilon > 0$, there is $j \in J$ such that $\|x + j\| < \alpha + \varepsilon$. Then

$$\begin{aligned}
(\alpha + \varepsilon)^2 > \|x + j\|^2 &= \|(x + j)(x^* + j^*)\| \\
&\ge \|(x + j)(\mathbb{1} - u_\lambda)(x^* + j^*)\| \\
&= \|x(\mathbb{1} - u_\lambda)x^* + j(\mathbb{1} - u_\lambda)x^* + x(\mathbb{1} - u_\lambda)j^* + j(\mathbb{1} - u_\lambda)j^*\|
\end{aligned}$$

for any λ. Now, $j(\mathbb{1} - u_\lambda), (\mathbb{1} - u_\lambda)j^*, j(\mathbb{1} - u_\lambda)j^* \to 0$. Furthermore, $\|x(\mathbb{1} - u_\lambda)x^*\| \to \beta$. It follows that $(\alpha + \varepsilon)^2 \ge \beta$ for all $\varepsilon > 0$, and therefore $\alpha^2 \ge \beta$. Thus $\alpha^2 = \beta$ and

$$\begin{aligned}
\alpha^2 &\le \|x - xu_\lambda\|^2 \\
&\le \|x(\mathbb{1} - u_\lambda)x^*\| \\
&\to \beta = \alpha^2.
\end{aligned}$$

Hence $\alpha = \lim \|x - xu_\lambda\|$, as required. ∎

Theorem 5.25. *Suppose that J is a closed two-sided ideal in a C^*-algebra A. Then A/J is a C^*-algebra.*

Proof. We know that A/J is a Banach algebra. Define an involution on A/J by $(\operatorname{cl} x)^* = \operatorname{cl} x^*$; since J is closed, $J = J^*$, and this is a well-defined involution on A/J. Furthermore,

$$\begin{aligned}\|(\operatorname{cl} x)^*\| = \|\operatorname{cl} x^*\| &= \inf_{j\in J} \|x^* + j\| \\ &= \inf_{j\in J} \|x^* + j^*\| = \inf_{j\in J} \|x + j\| \\ &= \|\operatorname{cl} x\|.\end{aligned}$$

Hence

$$\begin{aligned}\|\operatorname{cl} x^* \operatorname{cl} x\| &\leq \|\operatorname{cl} x^*\| \|\operatorname{cl} x\| \\ &= \|\operatorname{cl} x\|^2.\end{aligned}$$

We must show that equality holds. However, if (u_λ) is an approximate unit for J (as a left ideal), we have

$$\begin{aligned}\|\operatorname{cl} x^* \operatorname{cl} x\|^2 &= \lim \|x^* x(\mathbb{1} - u_\lambda)\|^2 \\ &= \lim \|(\mathbb{1} - u_\lambda)x^* x x^* x(\mathbb{1} - u_\lambda)\| \\ &\geq \lim \|(\mathbb{1} - u_\lambda)x^* x(\mathbb{1} - u_\lambda)(\mathbb{1} - u_\lambda)x^* x(\mathbb{1} - u_\lambda)\| \\ &= \lim \|(\mathbb{1} - u_\lambda)x^* x(\mathbb{1} - u_\lambda)\|^2 \\ &= \lim \|x(\mathbb{1} - u_\lambda)\|^4 \\ &= \|\operatorname{cl} x\|^4\end{aligned}$$

by the previous theorem, and the result follows. ∎

Chapter 6

Homomorphisms

We consider now further interplay between the algebraic structure and the metric structure of a C^*-algebra. In particular, we shall see that homomorphisms are necessarily continuous, isomorphisms are isometric and the range of a homomorphism is a C^*-algebra.

Definition 6.1. Let A and B be unital C^*-algebras. A homomorphism $\varphi : A \to B$ is a linear map satisfying

(i) $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$;

(ii) $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$;

(iii) $\varphi(\mathbb{1}_A) = \mathbb{1}_B$.

A linear map $\varphi : A \to B$ is said to be order preserving if $a \geq 0$ in A implies that $\varphi(a) \geq 0$ in B.

We will only consider unital C^*-algebras in this chapter.

Proposition 6.2. *Let $\varphi : A \to B$ be a homomorphism. Then φ is order preserving and norm decreasing. In particular, φ is continuous.*

Proof. Let $a \in A$, $a \geq 0$. Then there is $x \in A$ with $a = x^*x$. Hence $\varphi(a) = \varphi(x^*x) = \varphi(x^*)\varphi(x) = \varphi(x)^*\varphi(x) \geq 0$, which shows that φ is order preserving.

Now let $a \in A$ with $a = a^*$. Then $|\widehat{a}(\kappa)| \leq \|\widehat{a}\|_\infty$ for all $\kappa \in \operatorname{Sp}\mathcal{A}(a)$ implies that

$$-\|\widehat{a}\|_\infty \leq \widehat{a}(\,\cdot\,) \leq \|\widehat{a}\|_\infty$$

and so $\|\widehat{a}\|_\infty \mathbb{1} - \widehat{a}(\,\cdot\,) \geq 0$ and $\widehat{a}(\,\cdot\,) + \|\widehat{a}\|_\infty \geq 0$. Hence (using $\|\widehat{a}\|_\infty = \|a\|$), $\|a\|\mathbb{1} - a \geq 0$ and $a + \|a\|\mathbb{1} \geq 0$, that is, $-\|a\|\mathbb{1} \leq a \leq \|a\|\mathbb{1}$. It follows that $-\|a\|\mathbb{1} \leq \varphi(a) \leq \|a\|\mathbb{1}$ since $\varphi(\mathbb{1}) = \mathbb{1}$ and φ is order preserving.

By considering the Gelfand transform of $\varphi(a)$ $(= \varphi(a)^*)$, we see that $-\|a\|\mathbb{1} \leq \widehat{\varphi(a)} \leq \|a\|\mathbb{1}$ and so $\|\widehat{\varphi(a)}\|_\infty = \|\varphi(a)\| \leq \|a\|$ for any $a \in A$ with $a = a^*$.

For any $x \in A$, we have

$$\begin{aligned}\|\varphi(x)\|^2 &= \|\varphi(x)^*\varphi(x)\| = \|\varphi(x^*x)\| \\ &\le \|x^*x\| \\ &= \|x\|^2.\end{aligned}$$

■

Proposition 6.3. *Let $\varphi : A \to B$ be a homomorphism, and suppose that φ is injective. Then $\varphi^{-1} : \varphi(A) \to A$ is order preserving, and φ is norm preserving (i.e., φ is isometric).*

Proof. Let $y \in \varphi(A)$ be such that $y \ge 0$. Then there is a unique $a \in A$ such that $y = \varphi(a)$. We must show that $\varphi^{-1}(y) = a \ge 0$.

First we shall show that $a = a^*$. To see this, observe that $\varphi(a^*) = \varphi(a)^* = y^* = y$. Hence $a = a^*$, since φ is 1–1. Write $a = x_+ - x_-$, where $x_\pm \in A$ are the positive and negative parts of a. Then $y = \varphi(x_+ - x_-) = \varphi(x_+) - \varphi(x_-)$. We will show that $x_- = 0$.

For any $b \in B$, $b^*\varphi(x_+)b - b^*\varphi(x_-)b = b^*yb$. Let $b = \varphi(x_-^{1/2})$. Then the above equality becomes

$$\underbrace{\varphi(x_-^{1/2}x_+x_-^{1/2})}_{=0} - \varphi(x_-^2) = b^*yb\,.$$

Hence $-\varphi(x_-^2) = b^*yb$ and we deduce that $\varphi(x_-^2) \le 0$. But $\varphi(x_-^2) = \varphi(x_-)^2 \ge 0$, since $\varphi(x_-)$ is self-adjoint. Thus

$$\begin{aligned} & \varphi(x_-^2) = 0 \\ \implies\ & \varphi(x_-)\varphi(x_-) = 0 \\ \implies\ & \varphi(x_-) = 0, \qquad \text{e.g., using the Gelfand map,} \\ \implies\ & x_- = 0, \end{aligned}$$

since φ is 1–1. Thus $a = x_+ \ge 0$, as claimed, which shows that φ^{-1} is order preserving.

Now, for any $x \in A$, $\|\varphi(x)\| \le \|x\|$, by the previous proposition. On the other hand, putting $y = \varphi(x)$, we have $\varphi^{-1}(y) = x$ and $\|\varphi^{-1}(y)\| \le \|y\|$ again by the previous proposition. Hence $\|x\| \le \|\varphi(x)\| \le \|x\|$ which gives the required equality $\|\varphi(x)\| = \|x\|$, for any $x \in A$. ■

Corollary 6.4. *Let $\varphi : A \to B$ be an injective homomorphism. Then $\varphi(A)$ is a C^*-subalgebra of B.*

Proof. Evidently, $\varphi(A)$ is a *-subalgebra of B. Since φ is isometric, it follows that $\varphi(A)$ is closed in B, and hence is complete. ■

Theorem 6.5. *Let $\varphi : A \to B$ be a homomorphism. Then $\varphi(A)$ is a C^*-subalgebra of B.*

Proof. Let $J = \ker \varphi$. Then it is clear that J is a closed two-sided *-ideal of A. Define $\psi : A/J \to B$ by $\psi(\operatorname{cl} a) = \varphi(a)$. It is readily seen that ψ is a homomorphism from A/J into B, and that ψ is 1–1. Hence $\psi(A/J)$ is a C^*-subalgebra of B, i.e., $\varphi(A)$ is a C^*-subalgebra of B. ∎

Definition 6.6. A one-one homomorphism of A onto itself is called an automorphism of A. The collection of automorphisms of A is denoted $\operatorname{Aut} A$. Evidently, $\operatorname{Aut} A$ is a group under composition.

For any unitary $u \in A$, the map $a \mapsto uau^*$, $a \in A$, is an automorphism of A. Automorphisms of this form are said to be inner. In general, not all automorphisms are inner. Indeed, in a commutative C^*-algebra, only the identity automorphism is inner.

Definition 6.7. A representation of a C^*-algebra is a pair $(\mathcal{H}, \pi)$ consisting of a Hilbert space $\mathcal{H}$ and a homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. The representation $(\mathcal{H}, \pi)$ is said to be faithful if $\ker \pi = \{0\}$.

Thus, a faithful representation $(\mathcal{H}, \pi)$ is 1–1 and so is isometric. Conversely, if π is isometric, then, clearly, $(\mathcal{H}, \pi)$ is a faithful representation.

Definition 6.8. A C^*-algebra is said to be simple if it has no non-trivial closed two-sided ideals.

Remark 6.9. Since the kernel of any homomorphism $\varphi : A \to B$ is a closed two-sided ideal of A, it follows that for a simple C^*-algebra, all homomorphisms are 1–1, and so are isometric. In particular, all representations are faithful.

Chapter 7

States on a C^*-algebra

To begin with, we shall be mainly concerned with unital C^*-algebras. We will see later that this is no essential loss of generality.

Definition 7.1. A state on a unital C^*-algebra A is a positive linear functional ω, say, with $\omega(\mathbb{1}) = 1$.
That is, $\omega : A \to \mathbb{C}$, such that

(i) ω is linear;

(ii) $a \in A,\ a \geq 0 \implies \omega(a) \geq 0$;

(iii) $\omega(\mathbb{1}) = 1$.

Condition (iii) is really just a normalization condition.

The term "state" is borrowed from mathematical physics. The observables of a physical system are represented by self-adjoint elements of a C^*-algebra and the value $\omega(a)$ is supposed to be the expected value of the observable a in the "state" ω. To know the expected values of the observables of the system is to know the "state" of the system.

Examples 7.2.

1. Let $A = \mathcal{B}(\mathcal{H})$, and let ω be any map of the form $x \mapsto (x\xi, \xi)$, where $\xi \in \mathcal{H}$ has unit norm. Such a state is called a vector state.
2. Let $A = \mathcal{B}(\mathcal{H})$, and ω the map $x \mapsto \alpha(x\xi, \xi) + (1 - \alpha)(x\eta, \eta)$, where $\xi, \eta \in \mathcal{H}$ have unit norm, and $0 \leq \alpha \leq 1$.
3. Let $A = \mathcal{C}(X)$, where X is a compact Hausdorff space, and let ω be the map $f \mapsto f(x)$, where x is any point in X and $f \in \mathcal{C}(X)$.

Proposition 7.3. *Let $\varphi : A \to \mathbb{C}$ be a positive linear functional on the C^*-algebra A. Then $\varphi(a^*) = \overline{\varphi(a)}$, for any $a \in A$. In particular, $\varphi(h) \in \mathbb{R}$ whenever $h = h^* \in A$. Furthermore, φ satisfies Schwarz' inequality*

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b)$$

for any $a, b \in A$.

Proof. Let $a = h + ik$, $h = h^*$, $k = k^*$, and write $h = h_+ - h_-$, $k = k_+ - k_-$, with $h_\pm \geq 0$ and $k_\pm \geq 0$. Then

$$\begin{aligned}\varphi(a^*) &= \varphi(h_+ - h_- - i(k_+ - k_-)) \\ &= \varphi(h_+) - \varphi(h_-) - i\varphi(k_+) + i\varphi(k_-) \\ &= (\varphi(h_+) - \varphi(h_-) + i\varphi(k_+) - i\varphi(k_-))^- \\ &= \overline{\varphi(a)},\end{aligned}$$

where we have used $\varphi(h_\pm) \geq 0$ and $\varphi(k_\pm) \geq 0$.

Schwarz' inequality follows immediately from the fact that $\langle a, b\rangle = \varphi(b^* a)$ defines a sesquilinear form on A, with $\langle a, a\rangle \geq 0$. ■

Note that the existence of a unit is not required in the above argument.

Theorem 7.4. *Let ω be a positive linear functional on a unital C^*-algebra A. Then ω is bounded and $\|\omega\| = \omega(\mathbb{1})$.*

Proof. For any $h = h^* \in A$, we have $-\|h\|\mathbb{1} \leq h \leq \|h\|\mathbb{1}$, and so the linearity and positivity of ω give $-\|h\|\omega(\mathbb{1}) \leq \omega(h) \leq \|h\|\omega(\mathbb{1})$ which yields $|\omega(h)| \leq \omega(\mathbb{1})\|h\|$.

For general $a \in A$, we use Schwarz' inequality to obtain

$$\begin{aligned}|\omega(a)|^2 = |\omega(\mathbb{1}^* a)|^2 &\leq \omega(\mathbb{1}^*\mathbb{1})\omega(a^* a) \\ &\leq \omega(\mathbb{1})\omega(\mathbb{1})\|a^* a\| \quad \text{as above, with } h = a^* a, \\ &= \omega(\mathbb{1})^2\|a\|^2.\end{aligned}$$

Thus $|\omega(a)| \leq \omega(\mathbb{1})\|a\|$, so that ω is bounded and $\|\omega\| \leq \omega(\mathbb{1})$. But then we conclude that $\|\omega\| = \omega(\mathbb{1})$. ■

Theorem 7.5. *Suppose that ω is a bounded linear functional on a unital C^*-algebra A, satisfying $\|\omega\| = \omega(\mathbb{1})$. Then ω is positive.*

Proof. We shall first show that if $h = h^* \in A$ then $\omega(h) \in \mathbb{R}$. To see this, write $\omega(h) = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$. Then $\omega(h + i\lambda\mathbb{1}) = \alpha + i(\beta + \lambda\omega(\mathbb{1}))$ for all $\lambda \in \mathbb{R}$. Since $\omega(\mathbb{1}) = \|\omega\|$, it follows, in particular, that $\omega(\mathbb{1}) \in \mathbb{R}$, and therefore $|\omega(h + i\lambda\mathbb{1})| \geq |\beta + \lambda\omega(\mathbb{1})|$.

On the other hand,

$$\begin{aligned}|\omega(h + i\lambda\mathbb{1})| &\leq \|\omega\|\,\|h + i\lambda\mathbb{1}\| \\ &= \omega(\mathbb{1})\left(\|h\|^2 + \lambda^2\right)^{1/2}\end{aligned}$$

as is readily seen by using the Gelfand transform. Therefore

$$|\beta + \lambda\omega(\mathbb{1})|^2 \leq \omega(\mathbb{1})^2\left(\|h\|^2 + \lambda^2\right)$$

for all $\lambda \in \mathbb{R}$, which is impossible unless $\beta = 0$. Hence $\omega(h) \in \mathbb{R}$, as claimed.

Now suppose that $h \geq 0$, and, without loss of generality, suppose that $\|h\| \leq 1$. Then

$$\begin{aligned} |\omega(\mathbb{1}) - \omega(h)| &= |\omega(\mathbb{1} - h)| \leq \|\omega\| \, \|\mathbb{1} - h\| \\ &= \omega(\mathbb{1}) \|\mathbb{1} - h\| \\ &\leq \omega(\mathbb{1}) \end{aligned}$$

which implies that $\omega(h) \geq 0$. ∎

Remark 7.6. Thus, a state ω on a unital C^*-algebra is a positive linear functional with $\|\omega\| = 1$, or equivalently, a linear functional with $\|\omega\| = \omega(\mathbb{1}) = 1$.

Theorem 7.7. *Let $A \subseteq B$ be unital C^*-algebras (with the same unit), and let ω be a state on A. Then ω has an extension to a state on B, i.e., there is a state ρ on B such that $\rho \restriction A = \omega$.*

Proof. Since ω is a state on A, we have $\|\omega\| = \omega(\mathbb{1}) = 1$. By the Hahn-Banach theorem, there is a continuous linear functional ρ, say, on B, such that $\|\rho\| = \|\omega\|$ and $\rho \restriction A = \omega$. Hence $\rho(\mathbb{1}) = \omega(\mathbb{1}) = 1$, since $\mathbb{1} \in A \subseteq B$, and so $\|\rho\| = \|\omega\| = \omega(\mathbb{1}) = \rho(\mathbb{1}) = 1$. Thus ρ (is positive and) is a state on B. ∎

Theorem 7.8. *The set of states on a unital C^*-algebra A separates the points of A, i.e., for any $a, b \in A$ with $a \neq b$, there is a state ω on A with $\omega(a) \neq \omega(b)$.*

Proof. Let $a, b \in A$ with $a \neq b$ be given. Set $x = a - b$ and write $x = h + ik$ with $h = h^*$ and $k = k^*$. Then either $h \neq 0$ or $k \neq 0$ (otherwise $x = 0$). If ω is a state, then $\omega(h)$ and $\omega(k)$ are both real, and so $\omega(x) \neq 0$ if and only if either $\omega(h) \neq 0$ or $\omega(k) \neq 0$. So the theorem is proved if we can show that for any $h = h^* \in A$ with $h \neq 0$, there is a state ω with $\omega(h) \neq 0$.

To see this we use the identification $\mathcal{A}(h) \simeq \mathcal{C}(\operatorname{Sp} \mathcal{A}(h))$. Since $h \neq 0$ there is $\kappa' \in \operatorname{Sp} \mathcal{A}(h)$ such that $\widehat{h}(\kappa') \neq 0$. Define ρ on $\mathcal{A}(h)$ by $\rho(a) = \widehat{a}(\kappa')$ for $a \in \mathcal{A}(h)$. Clearly, ρ is a state on $\mathcal{A}(h)$, and so, by the previous theorem, has an extension to A. Then $\omega(h) = \rho(h) = \widehat{h}(\kappa') \neq 0$. ∎

Theorem 7.9. *The involution of a C^*-algebra is unique.*

Proof. Let A be a C^*-algebra. By considering $\widetilde{A}$ instead of A, we may assume that A has a unit. Let * and † be involutions on A with respect to which A is a C^*-algebra. Let ω be a state on A. Note that a state ω is any bounded linear functional on A with $\|\omega\| = \omega(\mathbb{1}) = 1$, so that the notion of state is independent of the involution. But then we know that ω is positive with respect to both * and †. Hence, for any $x \in A$, $\omega(x^*) = \overline{\omega(x)} = \omega(x^\dagger)$, and so $\omega(x^* - x^\dagger) = 0$ for all states ω on A. Since the set of states separates the points of A, we deduce that $x^* = x^\dagger$ for any $x \in A$. ∎

We now turn to a discussion of the non-unital case. We have seen (**??**) that a positive linear functional on a unital C^*-algebra is automatically continuous. This result is true even if there is no unit, as we shall now show.

Theorem 7.10. *Suppose that ω is a positive linear functional on a C^*-algebra A. Then ω is bounded.*

Proof. We proceed by contradiction; suppose that ω is not bounded. Then there is a sequence (x_n) in A such that $\|x_n\| \to 0$ but $\omega(x_n) = 1$. By positivity, **??**, $\omega(x_n^*) = \overline{\omega(x_n)} = 1$. Setting $y_n = \frac{1}{2}(x_n + x_n^*)$, we have $\|y_n\| \le \|x_n\| \to 0$ and $\omega(y_n) = 1$. Now, by **??**, each y_n can be written as $y_n = z_n' - z_n''$, where $z_n' \ge 0$ and $z_n'' \ge 0$ are given by $z_n' = \frac{1}{2}(|y_n| + y_n)$, $z_n'' = \frac{1}{2}(|y_n| - y_n)$. Thus $\|z_n'\| \le \|y_n\| \to 0$, and $\omega(z_n') = \omega(y_n) + \omega(z_n'') \ge 1$, by positivity of ω. Putting $a_n = \omega(z_n')^{-1} z_n'$, we have $a_n \in A$, $a_n \ge 0$, $\|a_n\| \to 0$ and $\omega(a_n) = 1$. By passing to a subsequence, if necessary, we may assume that $\|a_n\| \le 2^{-n}$, for each $n \in \mathbb{N}$. Set $b_n = \sum_{k=1}^{n} a_n$. Then $b_n \ge 0$ for each $n \in \mathbb{N}$, and there is $b \in A$ such that $b_n \to b$ as $n \to \infty$. Since A_+ is closed, it follows that $b \ge 0$.

Similarly, we see that for any $n \in \mathbb{N}$, $b - b_n = \lim_{m\to\infty} b_m - b_n \ge 0$. But ω is positive and so $\omega(b - b_n) \ge 0$, that is,

$$\omega(b) \ge \omega(b_n) = \sum_{k=1}^{n} \omega(a_n) = n$$

for any $n \in \mathbb{N}$, which is impossible. The result follows. ∎

We shall now consider the problem of extending a positive functional on a C^*-algebra A, without unit, to one on $\widetilde{A}$, the C^*-algebra obtained from A by adjoining a unit. First we need the following lemma.

Lemma 7.11. *For any positive functional ω on a C^*-algebra A,*

$$|\omega(k)| \le \|\omega\|^{1/2} \omega(k^2)^{1/2}$$

for all self-adjoint $k \in A$.

Proof. Note that, by **??**, ω is bounded so the claim of the lemma is meaningful.

Let (u_λ) be an approximate unit for A. Then $\|k - u_\lambda k\| \to 0$ so that $\omega(k) = \lim_\lambda \omega(u_\lambda k)$, for any $k = k^* \in A$. However, Schwarz' inequality gives

$$\begin{aligned} |\omega(u_\lambda k)| &\le \sqrt{\omega(u_\lambda^2)\omega(k^2)} \\ &\le \|\omega\|^{1/2} \omega(k^2)^{1/2} \end{aligned}$$

since $|\omega(u_\lambda^2)| \le \|\omega\| \|u_\lambda^2\| \le \|\omega\|$ for all λ. The result follows. ∎

Theorem 7.12. *Suppose that ω is a positive linear functional on a C^*-algebra A without unit. For any fixed $\mu \geq \|\omega\|$ define $\widetilde{\omega}(a + \lambda \mathbb{1}) = \omega(a) + \lambda\mu$, for $a \in A$ and $\lambda \in \mathbb{C}$. Then $\widetilde{\omega}$ is a positive linear functional on $\widetilde{A}$ such that $\widetilde{\omega} \upharpoonright A = \omega$. Moreover, all positive extensions of ω to $\widetilde{A}$ are of this form for suitable $\mu \geq \|\omega\|$.*

Proof. It is clear that $\widetilde{\omega}$ is a well-defined linear functional on $\widetilde{A}$. Suppose that $z \in \widetilde{A}$ with $z \geq 0$. Then $z = h^2$ for some $h = h^* \in \widetilde{A}$. Writing $h = k + \alpha\mathbb{1}$, it follows that $k = k^*$ and $\alpha = \overline{\alpha}$. By replacing h by $-h$, if necessary, we may suppose that $\alpha \geq 0$. We have

$$\begin{aligned}
\widetilde{\omega}(z) &= \widetilde{\omega}(k^2 + 2\alpha k + \alpha^2 \mathbb{1}) \\
&= \omega(k^2) + 2\alpha\omega(k) + \alpha^2\mu \\
&= \bigl(\omega(k^2)^{1/2} - \alpha\mu^{1/2}\bigr)^2 + 2\alpha\omega(k^2)^{1/2}\mu^{1/2} + 2\alpha\omega(k) \\
&\geq 0
\end{aligned}$$

by the lemma. Thus $\widetilde{\omega}$ is positive.

If ρ is a positive linear functional on $\widetilde{A}$, then $\|\rho\| = \rho(\mathbb{1})$, so that

$$\begin{aligned}
\rho(a + \lambda\mathbb{1}) &= \rho(a) + \lambda\rho(\mathbb{1}) \\
&= \rho(a) + \lambda\|\rho\|
\end{aligned}$$

for any $a \in A$ and $\lambda \in \mathbb{C}$. If $\rho \upharpoonright A = \omega$, then clearly $\|\rho\| \geq \|\omega\|$ and therefore $\rho = \widetilde{\omega}$, as above, with $\mu = \|\rho\|$. ■

Corollary 7.13. *Let A be a C^*-algebra without a unit, and suppose that ω is a positive linear functional on A with $\|\omega\| \leq 1$. Then ω has a unique extension to a state on $\widetilde{A}$.*

Proof. Define $\widetilde{\omega}$ on $\widetilde{A}$ by

$$\widetilde{\omega}(a + \lambda\mathbb{1}) = \omega(a) + \lambda\,,$$

for $a \in A$ and $\lambda \in \mathbb{C}$. Then, since, $\|\omega\| \leq 1$, it follows that $\widetilde{\omega}$ is positive. But $\widetilde{\omega}(\mathbb{1}) = 1$ and so $\widetilde{\omega}$ is a state on $\widetilde{A}$.

If ρ is state on $\widetilde{A}$ such that $\rho \upharpoonright A = \omega$, then $\rho(a + \lambda\mathbb{1}) = \rho(a) + \lambda\rho(\mathbb{1}) = \omega(a) + \lambda = \widetilde{\omega}(a + \lambda\mathbb{1})$, which gives the uniqueness of the extension. ■

Corollary 7.14. *A positive linear functional ω on a C^*-algebra A without unit has a unique extension to a positive linear functional with the same norm on $\widetilde{A}$.*

Proof. The formula $\widetilde{\omega}(a + \lambda\mathbb{1}) = \omega(a) + \lambda\|\omega\|$, for $a \in A$, $\lambda \in \mathbb{C}$, defines a positive linear extension of ω to $\widetilde{A}$. Furthermore, $\|\widetilde{\omega}\| = \widetilde{\omega}(\mathbb{1}) = \|\omega\|$.

Any positive extension ρ of ω to $\widetilde{A}$ satisfies $\|\rho\| = \rho(\mathbb{1})$, and so the equality $\|\rho\| = \|\omega\|$ implies that $\rho(\mathbb{1}) = \|\omega\|$ and the uniqueness follows. ■

Definition 7.15. A state ω on a C^*-algebra A without a unit is a positive linear functional on A such that $\|\omega\| = 1$.

Remark 7.16. According to the preceding discussion, such a functional then has a unique extension to a positive linear functional, $\widetilde{\omega}$, say, on $\widetilde{A}$, with $\|\widetilde{\omega}\| = 1$ (or, equivalently, with $\widetilde{\omega}(\mathbb{1}) = 1$). We see that ω is the restriction to A of a unique state, as earlier defined, on the unital C^*-algebra $\widetilde{A}$. In other words, there are no new features involved in considering positive functionals (and states) on non-unital C^*-algebras.

Chapter 8

The Gelfand, Naimark, Segal construction

We consider now the construction of a representation of a C^*-algebra A from a state on A. This will lead to the conclusion that any C^*-algebra can be realised as a C^*-algebra of operators on a Hilbert space.

Definition 8.1. Let A be a unital C^*-algebra and $(\mathcal{H}, \pi)$ a representation of A. A vector $\xi \in \mathcal{H}$ is said to be a cyclic vector (for the representation) if $\pi(A)\xi$ is dense in $\mathcal{H}$. If $(\mathcal{H}, \pi)$ has a cyclic vector, then it is called a cyclic representation.

Now, given any representation $(\mathcal{H}, \pi)$ of A and any unit vector $\xi \in \mathcal{H}$, the map $x \mapsto (\pi(x)\xi, \xi)$ is clearly a state on A. So from any given representation we can easily construct states. The following fundamental construction establishes the converse.

Theorem 8.2. (Gelfand, Naimark, Segal) *Let A be a unital C^*-algebra and ω a state on A. Then there is a cyclic representation $(\mathcal{H}, \pi)$ of A with unit cyclic vector $\Omega \in \mathcal{H}$ such that $\omega(a) = (\pi(a)\Omega, \Omega)$ for all $a \in A$.*

The triple $(\mathcal{H}, \pi, \Omega)$ is unique up to unitary equivalence, i.e., if $(\mathcal{H}', \pi', \Omega')$ is another such triple, then there is a unitary operator $u : \mathcal{H}' \to \mathcal{H}$ such that $u\Omega' = \Omega$ and $u\pi'(a)u^{-1} = \pi(a)$ for all $a \in A$.

Proof. Let $N = \{x \in A : \omega(x^*x)) = 0\}$. Then for any $a \in A$ and $x \in N$, Schwarz' inequality gives

$$\begin{aligned}\omega((ax)^*ax) &= \omega(x^*a^*ax)\\ &\le \omega(x^*x)^{1/2}\omega(y^*y)^{1/2}, \text{ with } y = a^*ax,\\ &= 0\end{aligned}$$

which shows that N is a left ideal in A.

Let $K = A/N$ as a vector space, and for $\xi, \eta \in K$ define $\langle \xi, \eta \rangle = \omega(y^*x)$, where $x \in \xi$ and $y \in \eta$. It is straightforward to verify that $\langle \cdot, \cdot \rangle$ is a well-defined sesquilinear form on K, i.e., defines an inner product. Moreover, if $\|\xi\|_\omega^2 \equiv \langle \xi, \xi \rangle = 0$, then $\omega(x^*x) = 0$, for any $x \in \xi$. Hence, $x \in N$, and so $\xi = 0$ in $K = A/N$. In other words, $\| \cdot \|_\omega$ is a norm on K.

We define an action of A on K by $L_a\xi = \operatorname{cl} ax$ for $a \in A$, where $x \in \xi$. This action is well-defined since N is a left ideal. Furthermore,

$$\begin{aligned} \|L_a\xi\|_\omega^2 &= \langle L_a\xi, L_a\xi\rangle \\ &= \omega((ax)^*ax)\,, \quad x \in \xi\,, \\ &= \omega(x^*a^*ax). \end{aligned}$$

Set $\rho(b) = \omega(x^*bx)$ for any $b \in A$. Then ρ is a positive linear functional on A and so $\|\rho\| = \rho(\mathbb{1})$, i.e., $|\rho(b)| \le \rho(\mathbb{1})\|b\|$, for $b \in A$. Hence $|\omega(x^*bx)| \le \omega(x^*x)\|b\|$. So, with $= a^*a$, we get

$$\begin{aligned} |\omega(x^*a^*ax)| &\le \omega(x^*x)\|a^*a\| \\ &= \omega(x^*x)\|a\|^2 \\ &= \langle \xi, \xi\rangle \|a\|^2. \end{aligned}$$

That is, $\|L_a\xi\|_\omega \le \|a\|\,\|\xi\|_\omega$. Hence L_a defines a bounded linear operator on $K = A/N$. The following relations are readily checked;

$$\begin{aligned} L_{a+b} &= L_a + L_b\,, \\ L_{ab} &= L_a L_b\,, \\ L_{\mathbb{1}} &= \mathbb{1}_K\,, \end{aligned}$$

and $\langle L_{a^*}\xi, \eta\rangle = \omega(y^*a^*x) = \omega((ay)^*x) = \langle \xi, L_a\eta\rangle$, where $x \in \xi$ and $y \in \eta$.

Let $\mathcal{H}$ be the completion of K with respect to the norm $\|\cdot\|_\omega$. Then $\mathcal{H}$ is a Hilbert space and contains (an isomorphic copy of) K as a dense subset. Let $\Omega \in K$ be given by $\Omega = \operatorname{cl} \mathbb{1}$. Then any $\xi \in K$ has the form $\xi = L_x\Omega$, with $x \in \xi$.

For each $a \in A$, L_a is a bounded linear map from K into K and so has a unique bounded linear extension, $\pi(a)$, say, from $\mathcal{H}$ into $\mathcal{H}$. The above relations remain true, and so we see that π is a representation of A on $\mathcal{H}$. Since $K = \{L_x\Omega : x \in A\} = \{\pi(x)\Omega : x \in A\}$ is dense in $\mathcal{H}$, it follows that Ω is a cyclic vector for the representation $(\mathcal{H}, \pi)$.

We note that, for any $a \in A$, $\langle \pi(a)\Omega, \Omega\rangle = \langle L_a \operatorname{cl}\mathbb{1}, \operatorname{cl}\mathbb{1}\rangle = \omega(a)$.

To establish the uniqueness, up to unitary equivalence, suppose that $(\mathcal{H}', \pi', \Omega')$ is another such triple. Define $u : \mathcal{H}' \to \mathcal{H}$ by $u\pi'(a)\Omega' = \pi(a)\Omega$. Then

$$\begin{aligned} \|u\pi'(a)\Omega'\|_{\mathcal{H}}^2 &= \|\pi(a)\Omega\|_{\mathcal{H}}^2 \\ &= \omega(a^*a) \\ &= \|\pi'(a)\Omega'\|_{\mathcal{H}'}^2. \end{aligned}$$

So u is an isometric linear operator from a dense set in $\mathcal{H}'$ to a dense set in $\mathcal{H}$ and thus extends to define a unitary operator from $\mathcal{H}'$ onto $\mathcal{H}$. This unitary provides the required equivalence; to see this, we consider

$$u\pi'(a)u^{-1}\pi(b)\Omega = u\pi'(a)\pi'(b)\Omega' = u\pi'(ab)\Omega$$

$$= \pi(ab)\Omega = \pi(a)\pi(b)\Omega$$

for all $a, b \in A$. Since $\pi(b)\Omega$ is dense in $\mathcal{H}$, we deduce that $u\pi'(a)u^{-1} = \pi(a)$ for $a \in A$. ∎

Remark 8.3. $(\mathcal{H}, \pi, \Omega)$ is called the Gelfand, Naimark, Segal (GNS) representation (or triple) associated with ω on A.

Examples 8.4.

1. Let ω be the state on $\mathcal{C}([0,1])$ given by $x \mapsto \int_0^1 x(s)\, ds$. Then the GNS triple $(\mathcal{H}, \pi, \Omega)$ is given by $\mathcal{H} = L^2([0,1])$, with Lebesgue measure, Ω is the vector 1 in $\mathcal{H}$, and $\pi(x)$ is given on $\mathcal{H}$ by multiplication by the function $x \in \mathcal{C}([0,1])$.

2. Suppose ω is the state on $\mathcal{C}([0,2])$ given, as above, by $x \mapsto \int_0^1 x(s)\, ds$. Then the GNS triple $(\mathcal{H}, \pi, \Omega)$ here is exactly the same as for example 1.

3. Let $\mathcal{H}_0$ be a Hilbert space and let $\xi \in \mathcal{H}_0$ be a unit vector. Let ω be the state on $\mathcal{B}(\mathcal{H}_0)$ given by $x \mapsto (x\xi, \xi)$, $x \in \mathcal{H}_0$. Then the GNS triple $(\mathcal{H}, \pi, \Omega)$ is the representation with $\mathcal{H} = \mathcal{H}_0$, $\Omega = \xi$ and $\pi(x) = x$ for all $x \in \mathcal{B}(\mathcal{H}_0)$. (This follows from the uniqueness.)

Definition 8.5. Let $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha \in I}$ be a collection of representations of a C^*-algebra A. Their direct sum is the representation $(\mathcal{H}, \pi)$ with $\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha$, and action $\pi(a) = \bigoplus_{\alpha \in I} \pi_\alpha(a)$ for $\alpha \in A$.

It is easy to see that $(\mathcal{H}, \pi)$ really is a representation.

Theorem 8.6. *Any C^*-algebra A is isometrically *-isomorphic to a C^*-algebra of operators on a Hilbert space.*

Proof. Without loss of generality, we may suppose that A has a unit (if not, we consider $\widetilde{A}$ instead of A). Let $\mathcal{S}_\mathcal{A}$ denote the set of states on A, and, for each $\omega \in \mathcal{S}_\mathcal{A}$, let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the corresponding GNS representation of A. Let $(\mathcal{H}, \pi)$ be their direct sum, $\mathcal{H} = \bigoplus_{\omega \in \mathcal{S}_\mathcal{A}} \mathcal{H}_\omega$ and $\pi = \bigoplus_{\omega \in \mathcal{S}_\mathcal{A}} \pi_\omega$, and let Ω be the vector $\bigoplus_\omega \Omega_\omega$

Suppose that $\pi(a) = \pi(b)$ for some $a, b \in A$. Then $0 = (\pi(a) - \pi(b))\Omega = \bigoplus_\omega (\pi_\omega(a) - \pi_\omega(b))\Omega_\omega$. Hence $\pi_\omega(a-b)\Omega_\omega = 0$ for all $\omega \in \mathcal{S}_\mathcal{A}$. In particular, $(\pi_\omega(a-b)\Omega_\omega, \Omega_\omega) = 0$ iė; $\omega(a-b) = 0$ for all $\omega \in \mathcal{S}_\mathcal{A}$. But $\mathcal{S}_\mathcal{A}$ separates the points of A, and so we conclude that $a = b$. Thus, π is faithful (injective) and so A is isometrically *-isomorphic to $\pi(A)$. ∎

Remark 8.7. $(\mathcal{H}, \pi)$ is called the universal representation of the C^*-algebra A. Notice that every state ω on A is realised as a vector state on $\pi(A)$; $\omega(a) = (\pi(a)\Omega_\omega, \Omega_\omega)$, $a \in A$, $\omega \in \mathcal{S}_\mathcal{A}$. Also, every state on $\pi(A)$ defines a state on A, so that every state on $\pi(A)$ is also given by a vector state.

Chapter 9

Pure States

Definition 9.1. A subset E of a linear space is said to be convex if for any $x, y \in E$, and for all $0 \le \alpha \le 1$, we have $\alpha x + (1-\alpha)y \in E$.

A point $z \in E$, with E convex, is said to be an extreme point of E if $z = \alpha x + (1-\alpha)y$, with $0 < \alpha < 1$, x, $y \in E$, has only the solution $x = y = z$; i.e., z is an extreme point of E if it is not a convex combination of two distinct points of E.

Clearly, if A is a C^*-algebra, then the set of states on A is a convex set in A^*, the dual of A.

Definition 9.2. The extreme points of the set of states of a C^*-algebra are called pure states. If a state is not pure, then it called a mixture. (The terminology is once again taken from theoretical physics.)

One easily sees that a state ω is a mixture if and only if there are states $\omega_1 \neq \omega_2$ such that

$$\omega = \tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2.$$

Indeed, suppose that $\omega = \alpha\omega' + (1-\alpha)\omega''$ for states ω', ω'' and $0 < \alpha < 1$. Without loss of generality, we may assume that $0 < \alpha \le \frac{1}{2}$. Then ω can be written as $\omega = \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$, where $\omega_1 = 2\alpha\omega' + (1-2\alpha)\omega''$ and $\omega_2 = \omega''$. Evidently ω_1 and ω_2 are states on A.

Theorem 9.3. *Let A be a commutative unital C^*-algebra. Then the set of pure states of A is exactly* $\operatorname{Sp} A$*: i.e., a state ω is pure if and only if ω is a character.*

Proof. Let $\omega \in \operatorname{Sp} A$, and suppose that $\omega = \omega_1 + \omega_2$, where ω_1 and ω_2 are states on A. Let $a \in A$ with $a = a^*$. Then

$$\begin{aligned}
\omega(a^2) &= \tfrac{1}{2}\left(\omega_1(a^2) + \omega_2(a^2)\right) \\
&= \omega(a)^2, \text{ since } \omega \in \operatorname{Sp} A, \\
&= \tfrac{1}{4}\left(\omega_1(a) + \omega_2(a)\right)^2.
\end{aligned}$$

Now, for any state ρ,

$$\begin{aligned}\rho(a)^2 &= \rho(\mathbb{1}^*a)^2 \\ &\le \rho(\mathbb{1}^*\mathbb{1})\rho(a^*a)\,, \text{ by Schwarz' inequality,} \\ &= \rho(a^2)\,, \text{ if } a = a^*.\end{aligned}$$

Hence, we have

$$\omega(a^2) = \tfrac{1}{2}\left(\omega_1(a^2) + \omega_2(a^2)\right) \ge \tfrac{1}{2}\left(\omega_1(a)^2 + \omega_2(a)^2\right).$$

Combining this with the earlier inequality gives

$$\tfrac{1}{4}\left(\omega_1(a) + \omega_2(a)\right)^2 \ge \tfrac{1}{2}\left(\omega_1(a)^2 + \omega_2(a)^2\right).$$

This reduces to the inequality

$$0 \ge (\omega_1(a) - \omega_2(a))^2$$

and we conclude that $\omega_1(a) = \omega_2(a)$ for all $a \in A$ with $a = a^*$. But this implies that $\omega_1 = \omega_2 = \omega$ (by **??**), and we see that ω is pure.

Conversely, suppose that ω is a pure state on A. Suppose that $a = a^* \in A$, $0 \le a \le \mathbb{1}$, and $0 \ne \omega(a) \ne 1$. For any $x \in A$, set

$$\begin{aligned}\omega_1(x) &= \omega(ax)/\omega(a)\,, \\ \omega_2(x) &= \omega((\mathbb{1} - a)x)/\omega(\mathbb{1} - a).\end{aligned}$$

Then ω_1 and ω_2 are states on A, and $\omega(a)\omega_1(x) + \omega(\mathbb{1} - a)\omega_2(x) = \omega(x)$, i.e., $\omega = \omega(a)\omega_1 + (1 - \omega(a))\omega_2$. Since ω is pure, it follows that $\omega_1 = \omega_2 = \omega$, i.e., $\omega(x) = \omega_1(x) = \omega(ax)/\omega(a)$. In other words, $\omega(ax) = \omega(a)\omega(x)$ for all $x \in A$ and for $a = a^*$ as above.

Now suppose that $a \ge 0$ and $\omega(a) = 0$. Then

$$\begin{aligned}|\omega(ax)| &= |\omega(a^{1/2}a^{1/2}x)| \\ &\le \omega(a)^{1/2}\omega(x^*ax), \qquad \text{by Schwarz' inequality,} \\ &= 0 \\ &= \omega(a)\omega(x)\end{aligned}$$

for all $x \in A$. Now suppose that $a \le \mathbb{1}$ and $\omega(a) = 1$. Putting $b = \mathbb{1} - a$ and arguing as above, we deduce that

$$\omega(bx) = 0 = \omega(b)\omega(x)\,,$$

and so we obtain

$$\omega(ax) = \omega(x) = \omega(a)\omega(x)\,,$$

for all $x \in A$.

Thus, combining these three situations, we have

$$\omega(ax) = \omega(a)\omega(x)$$

for all $x \in A$ and for all $a \in A$ with $a = a^*$ and $0 \le a \le \mathbb{1}$. By linearity, this also holds for all $0 \le a$, and, again by linearity, for all $a = a^*$, and hence for arbitrary $a \in A$. Thus, ω is a character. ■

Corollary 9.4. *Let X be a compact Hausdorff space. Then $X = \operatorname{Sp} \mathcal{C}(X)$, and X is the set of pure states on $\mathcal{C}(X)$.*

Theorem 9.5. *Let $A \subseteq B$ be C^*-algebras and suppose that ω is a pure state on A. Then ω has an extension to a pure state ρ on B.*

Proof. Let $F = \{\rho : \rho \text{ is a state on } B,\ \rho \restriction A = \omega\}$. Then we know that F is non-empty. Evidently F is convex and w^*-closed. (To see this, suppose that $\rho_0 \notin F$. Then $\rho_0 \restriction A \ne \omega$ so that there is some $a \in A$ such that $\rho_0(a) \ne \omega(a)$ Put $\varepsilon = |\rho_0(a) - \omega(a)|$. Then $\varepsilon > 0$ and the w^*-neighbourhood $N_\omega = \mathfrak{N}\{\rho_0 : \{a\}, \frac{1}{2}\varepsilon\}$ which, we recall, is defined as $\{\rho \in \mathcal{S}_B : |\rho(a) - \rho_0(a)| < \frac{1}{2}\varepsilon\}$, is contained in the complement of F (—every $\rho \in N_\omega$ satisfies $\rho(a) \ne \omega(a)$). Thus the complement of F is w^*-closed.) It follows that F has extreme points (by the Krein-Milman theorem). Let ρ be an extreme point of F. We shall show that ρ is a pure state of B. To see this, suppose the contrary;

$$\rho = \alpha\rho_1 + (1 - \alpha)\rho_2$$

for $0 < \alpha < 1$ and states $\rho_1 \ne \rho_2$ on B.

Since $\rho \in F$, we have $\omega = \alpha\rho_1 \restriction A + (1-\alpha)\rho_2 \restriction A$. But $\rho_1 \restriction A$ and $\rho_2 \restriction A$ are both states on A, and ω is pure (on A). Hence $\rho_1 \restriction A = \rho_2 \restriction A = \omega$, and it follows that ρ_1 and ρ_2 belong to F. But ρ is an extreme point of F, which demands that $\rho_1 = \rho_2$. This contradiction implies that ρ is pure, as claimed. ■

Corollary 9.6. *Suppose that ω is a state on the non-unital C^*-algebra A. Then the unique extension $\widetilde{\omega}$ of ω to a state on $\widetilde{A}$ is pure if and only if ω is pure.*

Proof. Suppose that ω is not pure on A. Then there are states ω_1, ω_2 on A with $\omega_1 \ne \omega_2$ and such that $\omega = \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ on A. Evidently, $\widetilde{\omega} = \frac{1}{2}\widetilde{\omega}_1 + \frac{1}{2}\widetilde{\omega}_2$ on $\widetilde{A}$, and $\widetilde{\omega}_1 \ne \widetilde{\omega}_2$. Thus $\widetilde{\omega}$ is not pure on $\widetilde{A}$.

Conversely, suppose that ω is pure on A. Then, by the theorem, ω has an extension to a pure state on $\widetilde{A}$. But $\widetilde{\omega}$ is the unique extension of ω to a state on $\widetilde{A}$, so that $\widetilde{\omega}$ is this pure extension. ■

Corollary 9.7. *The pure states separate points of a unital C^*-algebra A.*

Proof. By the preceding result, we may assume that A has a unit. Let $a \in A$, with $a = a^*$ and $a \neq 0$. As usual, let $\mathcal{A}(a)$ denote the commutative unital C^*-algebra generated by a, and let $\widehat{\ }$ denote the Gelfand isomorphism between $\mathcal{A}(a)$ and $\mathcal{C}(\operatorname{Sp}\mathcal{A}(a))$. Since $a \neq 0$, $\widehat{a}$ is not the zero function in $\mathcal{C}(\operatorname{Sp}\mathcal{A}(a))$, and so there is some $\kappa_0 \in \operatorname{Sp}\mathcal{A}(a)$ such that $\widehat{a}(\kappa_0) \neq 0$, i.e., $\kappa_0(a) \neq 0$. But κ_0 is a character and so is a pure state on $\mathcal{A}(a)$ and hence has an extension to a pure state ρ, say, on A (or $\widetilde{A}$, if A is non-unital). Hence $\rho(a) = \kappa_0(a) \neq 0$. Thus, for any $a = a^* \neq 0 \in A$, there is a pure state ρ on A such that $\rho(a) \neq 0$. The result follows from this. ∎

Corollary 9.8. *Let $h = h^*$ be a self-adjoint element of a unital C^*-algebra A. Then $\lambda \in \sigma(h)$ implies that there is a pure state ω on A with $\omega(h) = \lambda$.*

Proof. Note that $\lambda \in \sigma(h)$ implies that λ is in the range of the Gelfand transform $\widehat{h}$ of h, i.e., there is $\kappa \in \operatorname{Sp}\mathcal{A}(h)$ such that $\widehat{h}(\kappa) = \lambda$, i.e., $\kappa(h) = \lambda$. By the theorem, there is a pure state ω on A such that $\omega \restriction \mathcal{A}(h) = \kappa$. But then we have $\lambda = \kappa(h) = \omega(h)$. ∎

Example 9.9. The converse to the above result is false. Let A be the C^*-algebra $\mathcal{B}(\mathcal{H})$, and let $p \in A$ be the projection onto the one-dimensional subspace of $\mathcal{H}$ spanned by the unit vector ξ. Clearly, $\sigma(p) = \{0, 1\}$. Let $\eta \in \mathcal{H}$ be a unit vector orthogonal to ξ, and set $\zeta = (\xi + \eta)/\sqrt{2}$. Let ω_ζ denote the vector state determined by ζ. Then ω_ζ is pure on A, since any vector state on $\mathcal{B}(\mathcal{H})$ is pure, (—as we will see later) and we have

$$\omega_\zeta(p) = (p\zeta, \zeta) = \tfrac{1}{2}(\xi, \xi) = \tfrac{1}{2}\,,$$

but $\frac{1}{2} \notin \sigma(p)$.

Theorem 9.10. *Let ω be a state on a unital C^*-algebra A, and let $(\pi, \mathcal{H}, \Omega)$ be the associated GNS triple. Suppose that ρ is a positive linear functional on A with $\rho \leq \omega$ (i.e., $\omega - \rho$ is positive). Then there is a unique operator $t \in \mathcal{B}(\mathcal{H})$ such that $0 \leq t \leq \mathbb{1}$, t commutes with each $\pi(a)$, $a \in A$, and*

$$\rho(b^* a) = (t\pi(a)\Omega, \pi(b)\Omega)$$

for all $a, b \in A$.

Conversely, if $t \in \mathcal{B}(\mathcal{H})$, $0 \leq t \leq \mathbb{1}$ and t commutes with $\pi(a)$, for all $a \in A$, then $a \mapsto \rho(a) = (t\pi(a)\Omega, \Omega)$ is a positive linear functional on A with $\rho \leq \omega$.

Proof. Let ρ be a given positive linear functional on A with $\rho \leq \omega$. Let $\operatorname{cl} a$ denote the class of a in A/N, where N is the left-ideal $N = \{x \in A : \omega(x^* x) = 0\}$. Then we have

$$\begin{aligned} |\rho(b^* a)| &\leq \rho(b^* b)\rho(a^*) \\ &\leq \omega(b^* b)\omega(a^*) \end{aligned}$$

$$= \|\operatorname{cl} b\|_\omega^2 \, \|\operatorname{cl} a\|_\omega^2.$$

Hence ρ defines a bounded sesquilinear form on $K = A/N$ and hence extends to a bounded sesquilinear form, λ, say, on $\mathcal{H}$, the completion of K. By Riesz' lemma, there is a unique $t \in \mathcal{B}(\mathcal{H})$ with

$$\lambda(\xi, \eta) = (t\xi, \eta)$$

for $\xi, \eta \in \mathcal{H}$. But $\lambda(\operatorname{cl} a, \operatorname{cl} b) = \rho(b^*a)$, and so, with $b = a$, we have $(t \operatorname{cl} a, \operatorname{cl} a) = \rho(a^*a) \le \omega(a^*a) = (\operatorname{cl} a, \operatorname{cl} a)$. Thus $((\mathbb{1} - t)\operatorname{cl} a, \operatorname{cl} a) \ge 0$ for all $a \in A$. It follows that $0 \le t \le \mathbb{1}$ (since K is dense in $\mathcal{H}$). Furthermore, using $\operatorname{cl} a = \pi(a)\Omega$, we have

$$\begin{aligned} \rho(b^*a) &= (t \operatorname{cl} a, \operatorname{cl} b) \\ &= (t\pi(a)\Omega, \pi(b)\Omega). \end{aligned}$$

Finally, for a, b, $c \in A$,

$$\begin{aligned} ((t\pi(a) - \pi(a)t)\pi(c)\Omega, \pi(b)\Omega) &= \rho(b^*ac) - (t\pi(c)\Omega, \pi(a^*b)\Omega) \\ &= \rho(b^*ac) - \rho((a^*b)^*c) \\ &= \rho(b^*ac) - \rho(b^*ac) \\ &= 0. \end{aligned}$$

Since Ω is cyclic, it follows that $(t\pi(a) - \pi(a)t)\pi(c)\Omega = 0$ for all a, $c \in A$, and so $t\pi(a) - \pi(a)t = 0$ for all $a \in A$, that is, t commutes with all members of $\pi(A)$.

The proof of the converse is a direct computation. ∎

Definition 9.11. The commutant of a set S of bounded operators on a Hilbert space $\mathcal{H}$ is the set

$$S' = \{y \in \mathcal{B}(\mathcal{H}) : ys = sy \text{ for all } s \in S\}.$$

Theorem 9.12. *Let ω be a state on a unital C^*-algebra A and $(\mathcal{H}, \pi, \Omega)$ the associated cyclic (GNS) representation of A. Then $\pi(A)' = \mathbb{C}\mathbb{1}$ if and only if ω is pure.*

Proof. Suppose first that $\pi(A) = \mathbb{C}\mathbb{1}$, and let $\omega = \alpha\rho_1 + (1 - \alpha)\rho_2$ with $0 < \alpha < 1$. Then $(1 - \alpha)\rho_2 \ge 0$ implies that $\alpha\rho_1 \le \omega$. Hence, there is $t \in \pi(A)'$ such that $0 \le t \le \mathbb{1}$ and $\alpha\rho_1(a) = (t\pi(a)\Omega, \Omega)$ for all $a \in A$. But $\pi(A) = \mathbb{C}\mathbb{1}$ implies that $t = \beta\mathbb{1}$ for some $0 \le \beta \le 1$. Hence $\alpha\rho_1(a) = \beta(\pi(a)\Omega, \Omega) = \beta\omega(a)$, for all $a \in A$. In particular, taking $a = \mathbb{1}$, we get $\alpha = \beta$ and so $\rho_1 = \omega$ and we deduce that ω is pure on A.

Conversely, suppose that ω is pure. Let $0 \le t \le \mathbb{1}$, $t \in \pi(A)'$, and suppose that $t \ne 0$ and $t \ne \mathbb{1}$. Define ρ on A by $\rho(a) = (t\pi(a)\Omega, \Omega)$, $a \in A$.

Then it is easy to see that ρ is a positive linear functional on A and that $\rho \leq \omega$.

Now, $|\rho(a)|^2 \leq \rho(\mathbb{1})\rho(a^*a)$, by Schwarz' inequality, and so $\rho(\mathbb{1}) = 0$ if and only if $\rho(a) = 0$ for all $a \in A$. By cyclicity, this is equivalent to $t = 0$. Hence $\rho(\mathbb{1}) \neq 0$. Furthermore, $(\omega - \rho)(a) = ((\mathbb{1} - t)\pi(a)\Omega, \Omega)$ and so a similar argument (with $(\mathbb{1} - t)$ replacing t) implies that $(\omega - \rho)(\mathbb{1}) = 0$ if and only if $t = \mathbb{1}$. Thus $(\omega - \rho)(\mathbb{1}) \neq 0$. It follows that both $\rho(\,\cdot\,)/\rho(\mathbb{1})$ and $(\omega - \rho)(\,\cdot\,)/(\omega - \rho)(\mathbb{1}))$ are states on A. But then

$$\omega(\,\cdot\,) = \rho(\mathbb{1})\,\frac{\rho(\,\cdot\,)}{\rho(\mathbb{1})} + (\omega - \rho)(\mathbb{1})\,\frac{(\omega - \rho)(\,\cdot\,)}{(\omega - \rho)(\mathbb{1})}$$

expresses ω as a convex combination of states. Since ω is pure, we must have $\rho(\,\cdot\,)/\rho(\mathbb{1}) = \omega(\,\cdot\,)$. Thus

$$(t\pi(a)\Omega, \Omega) = \underbrace{(t\Omega, \Omega)}_{\rho(\mathbb{1})}\,(\pi(a)\Omega, \Omega)$$

and so $((t - \rho(\mathbb{1}))\pi(a)\Omega, \Omega) = 0$, for all $a \in A$. It follows that $t = \rho(\mathbb{1})\mathbb{1}$, and hence we have $\pi(A)' = \mathbb{C}\mathbb{1}$. ∎

Definition 9.13. A representation $(\mathcal{H}, \pi)$ of a C^*-algebra A is called irreducible if $\pi(A)' = \mathbb{C}\mathbb{1}$.

Thus, the previous theorem says that the GNS representation of a unital C^*-algebra A induced by a state ω is irreducible if and only if ω is pure. The point is that if a representation $(\mathcal{H}, \pi)$ is *not* irreducible, then $\pi(A)'$ will contain a non-trivial projection, p, say. In this case, $\pi(A)$ maps the subspace $p\mathcal{H}$ into $p\mathcal{H}$ and $(\mathbb{1} - p)\mathcal{H}$ into $(\mathbb{1} - p)\mathcal{H}$, i.e., $p\mathcal{H}$ and $(\mathbb{1} - p)\mathcal{H}$ are invariant subspaces. If $(\mathcal{H}, \pi)$ is irreducible, then there are no such non-trivial invariant subspaces of $\mathcal{H}$ (under $\pi(A)$).

Example 9.14. Let ρ be the vector state on $\mathcal{B}(\mathcal{H})$ given by a unit vector $\xi \in \mathcal{H}$. Then ξ is a cyclic vector for the identity representation $\pi(a) = a$, $a \in \mathcal{B}(\mathcal{H})$, and $(\pi(a)\xi, \xi) = \rho(a)$, $a \in \mathcal{B}(\mathcal{H})$. By the uniqueness of the GNS representation, this representation is (unitarily equivalent to) the GNS representation. By the remarks above, we conclude that ρ is pure, since $\mathcal{B}(\mathcal{H})$ is irreducible.

Theorem 9.15. *Let A be a unital C^*-algebra. Then A is isometrically isomorphic to a direct sum of irreducible representations of itself.*

Proof. Let $\mathcal{E}$ denote the set of pure states of A, and let $(\mathcal{H}, \pi)$ be the representation with $\mathcal{H} = \bigoplus_{\omega \in \mathcal{E}} \mathcal{H}_\omega$, and $\pi = \bigoplus_{\omega \in \mathcal{E}} \pi_\omega$. Each $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible, and $(\mathcal{H}, \pi)$ is faithful since $\mathcal{E}$ separates points of A. ∎

Theorem 9.16. *Let ω be a state on a unital C^*-algebra A with associated GNS triple $(\mathcal{H}, \pi, \Omega)$. Suppose that α is an automorphism of A such that ω is invariant under α, i.e., $\omega(\alpha(a)) = \omega(a)$ for all $a \in A$. Then there is a unitary operator U on $\mathcal{H}$ such that $U\Omega = \Omega$ and $U\pi(a)U^* = \pi(\alpha(a))$ for all $a \in A$. Moreover, U is unique.*

Proof. Define the representation $(\mathcal{H}, \pi')$ of A by the assignment $\pi'(a) = \pi(\alpha(a))$ for $a \in A$ and consider the triple $(\mathcal{H}, \pi', \Omega)$.

Since $\alpha(A) = A$, Ω is cyclic for π'. Furthermore,

$$\begin{aligned}(\Omega, \pi'(a)\Omega) &= (\Omega, \pi(\alpha(a))\Omega) \\ &= \omega(\alpha(a)) \\ &= \omega(a)\end{aligned}$$

for all $a \in A$. By the uniqueness of the GNS triple, we deduce that there is a unitary U on $\mathcal{H}$ such that $U\Omega = \Omega$ and $U\pi(a)U^* = \pi'(a) = \pi(\alpha(a))$ for all $a \in A$.

Suppose that V is another unitary operator on $\mathcal{H}$ with the same properties. Then

$$\begin{aligned}U\pi(a)\Omega = U\pi(a)U^*U\Omega &= U\pi(a)U^*\Omega = \pi(\alpha(a))\Omega \\ &= V\pi(a)V^*\Omega = V\pi(a)\Omega\end{aligned}$$

for all $a \in A$. Since Ω is cyclic, it follows that $U = V$. ∎

Corollary 9.17. *Let A be a C^*-algebra and G a topological group. Suppose that $g \mapsto \alpha_g$, for $g \in G$, is a representation of G in* Aut A*. Suppose that ω is a state on A which is invariant under each α_g, i.e., $\omega(\alpha_g(a)) = \omega(a)$ for each $a \in A$ and all $g \in G$. Suppose, further, that for any $a, b \in A$ the map $g \mapsto \omega(b^*\alpha_g(a))$ is continuous. Then there is a strongly continuous unitary representation $g \mapsto U(g)$ of G on $\mathcal{H}$, where $(\mathcal{H}, \pi, \Omega)$ is the GNS triple associated with ω, satisfying $U(g)\Omega = \Omega$ for all $g \in G$ and $U(g)\pi(a)U(g)^* = \pi(\alpha_g(a))$ for all $g \in G$ and $a \in A$. Moreover, the $U(g)$ are unique.*

Proof. By the theorem, for each $g \in G$ there is a unique unitary operator $U(g)$ on $\mathcal{H}$ satisfying $U(g)\Omega = \Omega$ and $U(g)\pi(a)U(g)^* = \pi(\alpha_g(a))$ for all $a \in A$. To see that $g \mapsto U(g)$ is a representation of G, we compute

$$\begin{aligned}U(g)U(h)\pi(a)\Omega &= U(g)\pi(\alpha_h(a))\Omega \\ &= \pi(\alpha_g(\alpha_h(a)))\Omega \\ &= \pi(\alpha_{gh}(a))\Omega \\ &= U(gh)\pi(a)\Omega\end{aligned}$$

for all $a \in A$, $g, h \in G$. Since Ω is cyclic, it follows that $U(g)U(h) = U(gh)$.

It remains to show that $U(\cdot)$ is strongly continuous. To see this, let $a, b \in A$. Then

$$(\pi(b)\Omega, U(g)\pi(a)\Omega) = \omega(b^* \alpha_g(a))$$

is continuous in g, by hypothesis. Since each $U(g)$ is unitary, it follows that $U(\cdot)$ is weakly continuous on $\mathcal{H}$ and therefore strongly continuous.

The uniqueness of the $U(g)$ follows as in the theorem. ∎

Definition 9.18. Let A be a C^*-algebra and $g \mapsto \alpha_g \in \operatorname{Aut} A$ a representation of the group G in $\operatorname{Aut} A$. A state ω on A is said to be extremal invariant (with respect to α_g) if ω is an extreme point of the convex set $\{\, \rho \text{ state on } A : \rho \circ \alpha_g = \rho \text{ for all } g \in G \,\}$.

Corollary 9.19. *With the assumptions and notation as above, we have* $\mathfrak{R} \equiv (\{\, U(g) : g \in G \,\} \cup \{\, \pi(A) \,\})' = \mathbb{C}\mathbb{1}$ *if and only if* ω *is extremal invariant.*

Proof. Suppose ω is extremal invariant, but $\mathfrak{R} \neq \mathbb{C}\mathbb{1}$. Let P be a non-trivial projection in $\mathfrak{R}$. Then the cyclicity of ω implies that $P\Omega \neq 0$ and $P\Omega \neq \Omega$. Let ω_1 be the state on A given by

$$\omega_1(a) = \frac{(P\Omega, \pi(a)P\Omega)}{\|P\Omega\|^2}$$

for $a \in A$, and let ω_2 be the state given by

$$\omega_2(a) = \frac{(Q\Omega, \pi(a)Q\Omega)}{\|Q\Omega\|^2}$$

for $a \in A$, where $P + Q = \mathbb{1}$. Evidently, ω is given as the convex combination

$$\omega = \|P\Omega\|^2 \omega_1 + \|Q\Omega\|^2 \omega_2 \,.$$

Moreover, ω_1 and ω_2 are each invariant under every α_g and one readily sees that $\omega_1 \neq \omega_2$. Indeed, for any fixed $\xi \in H$, the cyclicity of Ω implies that there is a sequence $a_n \in A$ such that $\pi(a_n)\Omega \to P\xi$. But then

$$\omega_1(a_n) = (P\Omega, \pi(a_n)P\Omega)/\|P\Omega\|^2 = (P\Omega, \pi(a_n)\Omega)/\|P\Omega\|^2 \to (P\Omega, \xi)/\|P\Omega\|^2 \,.$$

However,

$$\begin{aligned} \omega_2(a_n) &= (Q\Omega, \pi(a_n)Q\Omega)/\|Q\Omega\|^2 \\ &= (Q\Omega, \pi(a_n)\Omega)/\|Q\Omega\|^2 \\ &\to (Q\Omega, P\xi)/\|Q\Omega\|^2 = 0 \,. \end{aligned}$$

The equality $\omega_1 = \omega_2$ would then entail that $(P\Omega, \xi) = 0$ for all $\xi \in \mathcal{H}$, thus giving $P\Omega = 0$, which we know not to be true. We conclude that $\omega_1 \neq \omega_2$ and so we have exhibited ω as a convex combination of invariant states. This contradicts the supposed extremal invariance of ω and so we must have $\mathfrak{R} = \mathbb{C}\mathbb{1}$.

For the converse, suppose that $\mathfrak{R} = \mathbb{C}\mathbb{1}$, but ω is not extremal invariant. Then there are distinct invariant states ω_1 and ω_2 and $0 < \lambda < 1$ such that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. Hence $\omega \geq \lambda\omega_1 = \rho$, say. It follows that there is an operator $T \in \pi(A)'$ such that

$$\rho(b^*a) = (\pi(b)\Omega, T\pi(a)\Omega)$$

for all $a, b \in A$. The invariance of ρ under α_g together with the cyclicity of Ω implies that $U(g)^*TU(g) = T$ which is to say that T commutes with each $U(g)$. Hence $T \in \mathfrak{R}$ and so $T = \mu\mathbb{1}$ for some $\mu \in \mathbb{C}$. But this implies that ω_1 is proportional to ω and hence equal to ω. This in turn means that $\omega_2 = \omega$ giving $\omega_1 = \omega_2$ which is false. We conclude that ω is extremal invariant, as claimed. ∎