Contents

Lecture 1

Basic properties

1.1 Generalities about rational maps and linear systems

Recall that a *rational map* $f : X \dashrightarrow Y$ of algebraic varieties over a field $\mathbb{K}$ is a regular map defined on a dense open Zariski subset $U \subset X$. The largest such set to which f can be extended as a regular map is denoted by $\mathrm{dom}(f)$. Two rational maps are considered to be equivalent if their restrictions to an open dense subset coincide. A rational map is called *dominant* if $f : \mathrm{dom}(f) \to Y$ is a dominant regular map, i.e. the image is dense in Y. Algebraic varieties form a category $\mathrm{Rat}_{\mathbb{K}}$ with morphisms taken to be equivalence classes of dominant rational maps.

From now on we restrict ourselves with rational maps of irreducible varieties over $\mathbb{C}$. We use f_{d} to denote the restriction of f to $\mathrm{dom}(f)$, or to any open subset of $\mathrm{dom}(f)$. A dominant map $f_{\mathrm{d}} : \mathrm{dom}(X) \to Y$ defines a homomorphism of the fields of rational functions $f^* : R(Y) \to R(X)$. Conversely, any homomorphism $R(Y) \to R(X)$ arises from a unique dominant rational map $X \dashrightarrow Y$. If f^* makes $R(X)$ a finite extension of $R(Y)$, then the degree of the extension is the *degree* of f. A rational map of degree 1 is called a *birational map*. It is also can be defined as an invertible rational map.

We will further assume that X is a smooth projective variety. It follows that the complement of $\mathrm{dom}(f)$ is of codimension ≥ 2. A rational map $f : X \dashrightarrow Y$ is defined by a linear system. Namely, we embed Y in a projective space $\mathbb{P}^r$ and consider the complete linear system $\mathcal{H}_Y = |\mathcal{O}_Y(1) := |H^0(Y, \mathcal{O}_Y(1))|$. Its divisors are hyperplane sections of Y. The invertible sheaf $f_{\mathrm{d}}^*\mathcal{O}_Y(1)$ on $\mathrm{dom}(f)$ can be extended to a unique invertible sheaf $\mathcal{L}$ on all of X. Also we can extend the sections $f_{\mathrm{d}}^*(s), s \in V'$, to sections of $\mathcal{L}$ on all of X. The obtained homomorphism $f^* : V' = H^0(Y, \mathcal{O}_Y(1)) \to H^0(X, \mathcal{L})$ is injective and its image is a linear subspace $V \subset H^0(X, \mathcal{L})$. The associated projective space $|V| \subset |\mathcal{L}|$ is the linear system $\mathcal{H}_X$ defining a morphism $f_{\mathrm{d}} : \mathrm{dom}(f) \to Y \hookrightarrow \mathbb{P}^r$.

The rational map f is given in the usual way. Evaluating sections of V at a point, we get a map $\mathrm{dom}(f) \to |V^\vee|$, and by restriction, the map $\mathrm{dom}(f) \to |V'^\vee|$ which factors through $Y \hookrightarrow |V'^\vee|$. A choice of a basis in V and a basis in V' defines a rational map $f : X \dashrightarrow Y \subset \mathbb{P}^r$, where $r = \dim |\mathcal{H}_Y|$.

For any rational map $f : X \dashrightarrow Y$ and any closed reduced subvariety Z of Y we denote by $f^{-1}(Z)$ the closure of $f_{\mathrm{d}}^{-1}(Z)$ in X. It is called the *inverse transform* of Z under the rational map f. Thus the divisors from $\mathcal{H}_X$ are inverse transforms of hyperplane sections Z of Y in the embedding $\iota : Y \hookrightarrow \mathbb{P}^r$ such that $Z \cap f(\mathrm{dom}(f)) \neq \emptyset$.

If $|V'| \subset |\mathcal{L}'|$ is a linear system on Y, then we define its *inverse transform* $f^{-1}(|V'|)$ of $|V'|$ following the procedure from above defining the linear system $\mathcal{H}_X$. The members of $f^{-1}(|V'|)$ are the inverse transforms of members of $|V'|$. When f is a morphism, the inverse transform is equal to the full transform $f^*(|V|) \subset |f^*\mathcal{L}'|$.

Let $\mathcal{L}$ be a line bundle and $V \subset H^0(X, \mathcal{L})$. Consider the natural *evaluation map* of sheaves

$$\mathrm{ev} : V \otimes \mathcal{O}_X \to \mathcal{L}$$

defined by restricting global sections to stalks of $\mathcal{L}$. It is equivalent to a map

$$\mathrm{ev} : V \otimes \mathcal{L}^{-1} \to \mathcal{O}_X$$

whose image is a sheaf of ideals in $\mathcal{O}_X$. This sheaf of ideals is denoted $\mathfrak{b}(|V|)$ and is called the *base ideal* of the linear system $|V|$. The closed subscheme $\mathrm{Bs}(\mathcal{H}_X)$ of X defined by this ideal is called the *base locus scheme* of $|V|$. We have

$$\mathrm{Bs}(|V|) = \cap_{D \in \mathcal{H}_X} D = D_0 \cap \ldots \cap D_r \text{ (scheme-theoretically)},$$

where $D_0, \ldots, D_r$ are the divisors of sections forming a basis of V. The largest positive divisor F contained in all divisors from $|V|$ (equivalently, in the divisors $D_0, \ldots, D_r$) is called the *fixed component* of $|V|$. The linear system without fixed component is sometimes called *irreducible*. Each irreducible component of its base scheme is of codimension ≥ 2.

If $F = \mathrm{div}(s_0)$ for some $s_0 \in \mathcal{O}_X(F)$, then the multiplication by s_0 defines an injective map $\mathcal{L}(-F) \to \mathcal{L}$ and the linear map $H^0(X, \mathcal{L}(-F)) \to H^0(X, \mathcal{L})$ defines an isomorphism from a subspace V' of $H^0(X, \mathcal{L}(-F))$ onto V. The linear system $|V'| \subset |\mathcal{L}(-F)|$ is irreducible and defines a rational map $f' : X \dashrightarrow |V'^\vee|$ equal to the composition of f with the transpose isomorphism $|V^\vee| \to |V'^\vee|$.

The linear system is called *basepoint-free*, or simply *free* if its base scheme is empty, i.e. $\mathfrak{b}(|V|) \cong \mathcal{O}_X$. The proper transform of such a system under a rational map is an irreducible linear system. In particular, the linear system $\mathcal{H}_X$ defining a rational map $X \dashrightarrow Y$ as described in above, is always irreducible.

The morphism

$$U = X \setminus \mathrm{Bs}(\mathcal{H}_X) \to |V^\vee|$$

defined by the linear system $|V|$ is the projection

$$\mathrm{Proj}\ \mathrm{Sym}(V \otimes (\mathcal{L}|U)^{-1}) \cong \mathrm{Proj}\ \mathrm{Sym}(V \otimes \mathcal{O}_U) \cong U \times |V^\vee| \to |V^\vee|.$$

If $|V|$ is an irreducible linear system,

$$\mathrm{dom}(f) = X \setminus \mathrm{Bs}(\mathcal{H}_X)_{\mathrm{red}} = X \setminus \mathrm{Supp}(\mathrm{Bs}(\mathcal{H}_X)).$$

Let $f : X \dashrightarrow Y$ be a rational map defined by the inverse transform $\mathcal{H}_X = |V|$ of a very ample complete linear system $\mathcal{H}_Y$ on Y. After choosing a basis in $H^0(Y, \mathcal{O}_Y(1))$ and a basis $(s_0, \ldots, s_r)$ in V, the map $f : \mathrm{dom}(f) \to Y \hookrightarrow \mathbb{P}^r$ is given by the formula

$$x \mapsto [s_0(x), \ldots, s_r(x)].$$

By definition, this is the formula defining the rational map $f : X \dashrightarrow Y$. Of course, different embeddings $Y \hookrightarrow \mathbb{P}^r$ define different formulas.

Here are some simple properties of the base locus scheme.

(i) $|V| \subset |\mathcal{L} \otimes \mathfrak{b}(|V|)| := |H^0(X, \mathfrak{b}(|V|) \otimes \mathcal{L})|$.

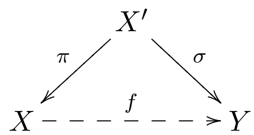
(ii) Let $\phi : X' \to X$ be a regular map, and $V' = \phi^*(V) \subset H^0(X', \phi^*\mathcal{L})$. Then $\phi^{-1}(\mathfrak{b}(|V|)) = \mathfrak{b}(f^{-1}(|V|))$. Recall that, for any ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, its inverse image $\phi^{-1}(\mathfrak{a})$ is defined to be the image of $\phi^*(\mathfrak{a}) = \mathfrak{a} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ in $\mathcal{O}_{X'}$ under the canonical multiplication map.

(iii) If $\mathfrak{b}(|V|)$ is an invertible ideal (i.e. isomorphic to $\mathcal{O}_X(-F)$ for some effective divisor F), then $\mathrm{dom}(f) = X$ and f is defined by the linear system $|\mathcal{L}(-F)|$.

(iv) If $\mathrm{dom}(f) = X$, then $\mathfrak{b}(|V|)$ is an invertible sheaf and $\mathrm{Bs}(\mathcal{H}_X) = \emptyset$.

1.2 Resolution of a rational map

Definition 1.2.1. *A* resolution *of a rational map $f : X \dashrightarrow Y$ of projective varieties is a pair of regular projective morphisms $\pi : X' \to X$ and $\sigma : X' \to Y$ such that $f = \sigma \circ \pi^{-1}$ (in* $\mathrm{Rat}_{\mathbb{K}}$*) and π is an isomorphism over* $\mathrm{dom}(f)$.



We say that a resolution is smooth (normal) if X' is smooth (normal).

Let $Z = V(\mathfrak{a})$ be the closed subscheme defined by $\mathfrak{a}$ and

$$\sigma : \mathrm{Bl}_Z X = \mathrm{Proj} \bigoplus_{k=0}^{\infty} \mathfrak{a}^k \to X.$$

be the *blow-up* of Z (see [Hartshorne]). We will also use the notation $\mathrm{Bl}(\mathfrak{a})$ for the blow up of $V(\mathfrak{a})$. The invertible sheaf $\sigma^{-1}(\mathfrak{a})$ is isomorphic to $\mathcal{O}_{\mathrm{Bl}_Z X}(-E)$, where E is the uniquely defined effective divisor on $\mathrm{Bl}_Z X$. We call E the *exceptional divisor* of σ. Any birational morphism $u : X' \to X$ such that $u^{-1}(\mathfrak{a})$ is an invertible sheaf of ideals factors through the blow-up of $\mathfrak{a}$. This property uniquely determines the blow-up, up to isomorphism. The morphism u is isomorphic to the morphism $\mathrm{Bl}_{Z'} X \to X$ for some closed subscheme $Z' \subset Z$. The exceptional divisor of this morphism contains the pre-image of the exceptional divisor of σ. For any closed subscheme $i : Y \hookrightarrow X$, the blow-up of the ideal $i^{-1}(\mathfrak{a})$ in Y is isomorphic to a closed subscheme of $\mathrm{Bl}_Z X$, called the *proper transform* of Y under the blow-up. Set-theoretically, it is equal to the closure of $\sigma^{-1}(Y \setminus Y \cap Z)$ in $\mathrm{Bl}_Z X$. In particular, it is empty if $Y \subset Z$.

Let $\nu : \mathrm{Bl}_Z^+ X \to X$ denote the normalization of the blow-up $\mathrm{Bl}_Z X$ and E^+ be the scheme-theoretical inverse image of the exceptional divisor. It is the exceptional divisor of ν. We have

$$\nu_* \mathcal{O}_{\mathrm{Bl}_Z X}(-E^+) = \bar{\mathfrak{a}},$$

where $\bar{\mathfrak{a}}$ denotes the integral closure of the ideal sheaf $\mathfrak{a}$ (see [Lazarsfeld], II, 9.6). A local definition of the integral closure of an ideal I in an integral domain A is the set of elements x in the fraction field of A such that $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ for some $n > 0$ and $a_k \in I^k$ (pay attention to the power of I here). If $E^+ = \sum r_i E_i$, considered as a Weil divisor, then locally elements in $\bar{\mathfrak{a}}$ are functions ϕ such that $\mathrm{ord}_{E_i}(\nu^*(\phi)) \geq r_i$ for all i.

We have $\mathrm{Bl}_Z^+ X = \mathrm{Bl}_Z X$ if and only $\mathfrak{a}^m$ is integrally closed for $m \gg 0$. If X is nonsingular, and $\dim X = 2$, then $m = 1$ suffices.

Proposition 1.2.1. *Let $\pi : \mathrm{Bl}_{\mathrm{Bs}(\mathcal{H}_X)} X \to X$ be the blow-up scheme of the base locus scheme of a rational map $f : X \dashrightarrow Y$. Then there exists a unique regular* *map* $\sigma : \mathrm{Bl}_{\mathrm{Bs}(\mathcal{H}_X)}X \to Y$ *such that* (π, σ) *is a resolution of* f*. For any resolution* (π', σ') *of* f *there exists a unique morphism* $\alpha : X' \to \mathrm{Bl}_{\mathrm{Bs}(\mathcal{H}_X)}X$ *such that* $\pi' = \pi \circ \alpha, \sigma' = \sigma \circ \alpha$*.*

Proof. By properties (ii) and (iii) from above, the linear system $\pi^{-1}(\mathcal{H}_X) = |\pi^*(\mathcal{L}) \otimes \pi^{-1}(\mathfrak{b})|$ defines a regular map $\sigma : \mathrm{Bl}_{\mathrm{Bs}(\mathcal{H}_X)} \to Y$. It follows from the definition of maps defined by linear systems that $f = \sigma \circ \pi^{-1}$. For any resolution, (π', σ'), the base locus of the pre-image $\pi^{-1}(\mathcal{H}_X)$ on X' is equal to the pre-image of the base scheme of $\mathcal{H}_X$. The morphism σ' is defined by the linear system $\pi'^{-1}(\mathcal{H}_X)$ and hence its base sheaf is invertible. This implies that π' factors through the blow-up of $\mathrm{Bs}(\mathcal{H}_X)$. □

Note that we also obtain that the exceptional divisor of π' is equal to the pre-image of the exceptional divisor of the blow-up of $\mathrm{Bs}(\mathcal{H}_X)$.

In many applications we will need a smooth resolution of a rational map. The following result follows from Hironaka's theorem on resolutions of singularities.

Definition 1.2.2. *An effective divisor* $D = \sum a_i D_i$ *on a smooth variety of dimension* n *is called a* simple normal crossing *(SNC) divisor if each irreducible component* D_i *is smooth and, at any point* $x \in \mathrm{Supp}(D)$*, the reduced divisor* $\sum D_i$ *is defined by local equations* $\phi_1 \cdots \phi_k = 0$*, where* $(\phi_1, \ldots, \phi_k)$ *is subset of a local system of parameters in* $\mathcal{O}_{X,x}$*.*

Theorem 1.2.2. *Let* X *be an irreducible algebraic variety (over* $\mathbb{C}$ *as always) and let* D *be an effective Weil divisor on* X*.*

(i) There exists a projective birational morphism $\nu : X' \to X$*, where* X' *is smooth and* μ *has divisorial exceptional locus* $\mathrm{Exc}(\nu)$ *such that* $\nu^*(D) + \mathrm{Exc}(\nu)$ *is a SNC divisor.*

(ii) X' *is obtained from* X *by a sequence of blow-ups with smooth centers supported in the singular locus* $\mathrm{Sing}(X)$ *and the singular locus* $\mathrm{Sing}(D)$ *of* D*. In particular, one can assume that* ν *is an isomorphism over* $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(D))$*.*

We will call X' a *log resolution* of (X, D) and will apply this to the case when D is the exceptional divisor of the normalization of the blow-up of a closed subscheme Z in X. We will call it a *log resolution of* Z.

1.3 The base ideal of a Cremona transformation

Theorem 1.3.1. *Assume that* $f : X \dashrightarrow Y$ *is a birational map of normal projective varieties and* f *is given by a linear system* $\mathcal{H}_X = |V| \subset |\mathcal{L}|$ *equal to the inverse* *transform of a very ample complete linear system $\mathcal{H}_Y$ on Y. Let $(\pi, \sigma) : X' \to X \times Y$ be a resolution of f and E be the exceptional divisor of π. Then the canonical map*

$$V \to H^0(X', \pi^*\mathcal{L}(-E))$$

is an isomorphism.

Proof. Set $\mathfrak{b} = \mathfrak{b}(\mathcal{H}_X)$. We have natural maps

$$V \to H^0(X, \mathcal{L} \otimes \mathfrak{b}) \to H^0(X', \pi^*\mathcal{L} \otimes \pi^{-1}(\mathfrak{b}))$$

$$\stackrel{\cong}{\to} H^0(X', (\pi^*\mathcal{L})(-E)) \stackrel{\cong}{\to} H^0(X', \sigma^*\mathcal{O}_Y(1)) \stackrel{\cong}{\to} H^0(Y, \sigma_*\sigma^*\mathcal{O}_Y(1))$$

$$\stackrel{\cong}{\to} H^0(Y, \mathcal{O}_Y(1) \otimes \sigma_*\mathcal{O}_{X'}) \stackrel{\cong}{\to} H^0(Y, \mathcal{O}_Y(1)) \cong V.$$

Here we used the Main Zariski Theorem that asserts that $\sigma_*\mathcal{O}_{X'} \cong \mathcal{O}_Y$ because σ is a birational morphism and Y is normal [Hartshorne Cor. 11.4]. The last isomorphism comes from the assumption of linear normality of Y in $\mathbb{P}^r$ which gives $H^0(Y, \mathcal{O}_Y(1)) \cong H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \cong V$. By definition of the linear system defining f, the composition of all these maps is a bijection. Since each map here is injective, we obtain that all the maps are bijective. One of the compositions is our map $V \to H^0(X', \pi^*\mathcal{L}(-E))$, hence it is bijective. □

Corollary 1.3.2. *Assume, additionally, that the resolution (X, π, σ) is normal. Then the natural maps*

$$V \to H^0(X, \mathcal{L} \otimes \mathfrak{b}(\mathcal{H}_X)) \to H^0(X', \pi^*(\mathcal{L})(-E)) \to H^0(Y, \mathcal{L} \otimes \overline{\mathfrak{b}(\mathcal{H}_X)})$$

are bijective.

We apply Theorem 1.3.1 to the case when $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a birational map, a *Cremona transformation*. In this case $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d \geq 1$, called the *degree* of the Cremona transformation f. We take $\mathcal{H}_Y = |\mathcal{O}_{\mathbb{P}^n}(1)|$. The linear system $\mathcal{H}_X = |\mathfrak{b}(\mathcal{H}_X)(d)|$ defining a Cremona transformation is called a *homaloidal linear system*. Classically, members of $\mathcal{H}_X$ were called *homaloids*. More generally, a k-*homaloid* is a proper transform of a k-dimensional linear subspace in the target space. They were classically called Φ-curves, Φ-surfaces, etc.).

Proposition 1.3.3.

$$H^1(\mathbb{P}^n, \mathcal{L} \otimes \overline{\mathfrak{b}(\mathcal{H}_X)}) = 0.$$

Proof. Let $(\pi, \sigma) : X \to \mathbb{P}^n$ be the resolution of f defined by the normalization of the blow-up of $\mathrm{Bs}(\mathcal{H}_X)$. We know that $\pi^*\mathcal{L}(-E)) \cong \sigma^*\mathcal{O}_{\mathbb{P}^n}(1)$. The exact sequence

$$0 \to H^1(\mathbb{P}^n, \sigma_*\sigma^*\mathcal{O}_{\mathbb{P}^n}(1)) \to H^1(X, \sigma^*\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, R^1\sigma_*\sigma^*\mathcal{O}_{\mathbb{P}^n}(1))$$

defined by the Leray spectral sequence, together with the projection formula, can be rewritten in the form

$$0 \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^1(X, \sigma^*\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, R^1\sigma_*\mathcal{O}_{X'} \otimes \mathcal{O}_{\mathbb{P}^n}(1)). \tag{1.1}$$

Let $\nu : X' \to X$ be a resolution of singularities of X. Then, we have the spectral sequence

$$E_2^{pq} = R^p\sigma_*(R^q\nu_*\mathcal{O}_{X'}) \Rightarrow R^{p+q}(\pi \circ \nu)_*\mathcal{O}_{X'}.$$

It gives the exact sequence

$$R^1\pi_*(\nu_*\mathcal{O}_{X'}) \to R^1(\pi \circ \nu)_*\mathcal{O}_{X'}) \to \pi_*(R^1\nu * \mathcal{O}_{X'}).$$

Since X is normal, $\nu_*\mathcal{O}_{X'} = \mathcal{O}_X$. Since the composition $\pi \circ \nu : X' \to \mathbb{P}^n$ is a birational morphism of nonsingular varieties, $R^1(\pi \circ \nu)_*\mathcal{O}_{X'} = 0$. This shows that

$$R^1\pi_*(\nu_*\mathcal{O}_{X'}) = 0.$$

Together with vanishing of $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, (1.1) implies that

$$H^1(X, \pi^*(\mathcal{L})(-E)) = 0.$$

It remains to use that the canonical map

$$H^1(\mathbb{P}^n, \mathcal{L} \otimes \overline{\mathfrak{b}(\mathcal{H}_X)}) \cong H^1(\mathbb{P}^n, \pi_*(\pi^*(\mathcal{L})(-E))) \to H^1(X, \pi^*(\mathcal{L})(-E))$$

is injective (use Čech cohomology, or the Leray spectral sequence). □

Using the exact sequence

$$0 \to \overline{\mathfrak{b}(\mathcal{H}_X)} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)/\overline{\mathfrak{b}(\mathcal{H}_X)} \to 0,$$

and tensoring it by $\mathcal{O}_{\mathbb{P}^n}(d)$, we obtain

Corollary 1.3.4 (Postulation Formula).

$$h^0(\mathcal{O}_{V(\overline{\mathfrak{b}(\mathcal{H}_X)})}(d)) = \binom{n+d}{d} - n - 1.$$

Remark 1.3.1. We do not know whether

$$H^1(\mathbb{P}^n, \mathfrak{b}(\mathcal{H}_X)(d)) = 0.$$

However, if Bs$(\mathcal{H}_X)$ is locally complete intersection, then Lemma 4.3.16 and Remark 4.3.17 in [Lazarsfeld] implies that

$$H^1(\mathbb{P}^n, \mathfrak{b}(\mathcal{H}_X)(d)) \cong H^1(X, \pi^*\mathcal{O}_{\mathbb{P}^n}(d)(-E))$$

and the rest of the argument in the previous proof imply the vanishing. Under the assumption of Proposition 1.3.3, the exact sequence

$$0 \to \mathfrak{b}(\mathcal{H}_X)(d) \to \overline{\mathfrak{b}(\mathcal{H}_X)}(d) \to (\overline{\mathfrak{b}(\mathcal{H}_X)}/\mathfrak{b}(\mathcal{H}_X))(d) \to 0,$$

implies that

$$h^0(\mathcal{O}_{V(\overline{\mathfrak{b}(\mathcal{H}_X)})}(d)) = h^0(\mathcal{O}_{V(\mathfrak{b}(\mathcal{H}_X))}(d)),$$

or, equivalently, $H^1(\mathbb{P}^n, \mathfrak{b}(\mathcal{H}_X)(d)) = 0$, if and only if

$$h^0((\overline{\mathfrak{b}(\mathcal{H}_X)}/\mathfrak{b}(\mathcal{H}_X))(d)) = 0.$$

For example, it does not hold if the base scheme is 0-dimensional.

Let us recall the following definition from [Lazarsfeld], Definition 9.6.15:

Definition 1.3.1. *Let $\mathfrak{a}$ be an non-zero ideal in $\mathcal{O}_X$. A* reduction *of $\mathfrak{a}$ is an ideal $\mathfrak{r} \subset \mathfrak{a}$ such that $\bar{\mathfrak{r}} = \bar{\mathfrak{a}}$. A reduction is called* minimal *if is not properly contained in any other reduction of $\mathfrak{a}$.*

Proposition 9.6.16 from loc.cit. says that $\mathfrak{r}$ is a reduction of $\mathfrak{a}$ if and only if

$$\nu^{-1}(\mathfrak{r}) = \nu^{-1}(\mathfrak{a}),$$

where $\nu : X' \to X$ is the normalization of the blow-up of $\mathfrak{a}$. Also note that the radical ideals of $\mathfrak{r}$ and $\mathfrak{a}$ coincide. It is known, that a minimal reduction $\mathfrak{r}$ can be locally generated at a closed point $x \in V(\mathfrak{r})$ by $\ell(\mathfrak{r})_x + 1$ elements, where $\ell(\mathfrak{r})_x$ is equal to the dimension of the fibre of Bl$(\mathfrak{r}) \to X$ at x ([Vasconcelos], Theorem 1.77).

Let $\mathfrak{a} \subset \mathcal{O}_X$ be an integrally closed ideal. Then $\mathfrak{a}$ has a canonical primary decomposition

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k, \tag{1.2}$$

where $\mathfrak{q}_i$ are primary integrally closed ideals. If we write the exceptional divisor of the normalized blow-up X' of $\mathfrak{a}$ in the form $E = \sum r_i E_i$, where E_i are irreducible divisors, then

$$\mathfrak{q}_i = \nu_*\mathcal{O}_{X'}(-r_i E_i).$$

Let

$$\mathfrak{B}(\mathcal{H}_X) = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^n, \mathfrak{b}(\mathcal{H}_X)(k))$$

be the graded homogeneous ideal of the closed subscheme $\mathrm{Bs}(\mathcal{H}_X)$.

$$\mathfrak{B}(\mathcal{H}_X)_d = H^0(\mathbb{P}^n, \mathfrak{b}(\mathcal{H}_X)(d)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathbb{C}^{n+1}. \tag{1.3}$$

Also we have

$$\mathfrak{B}(\mathcal{H}_X)_k = 0, \ k < d. \tag{1.4}$$

Indeed, otherwise $|\mathfrak{b}(\mathcal{H}_X)(d)|$ contains $|\mathcal{O}_{\mathbb{P}^n}(d-k)| + |\mathfrak{b}(\mathcal{H}_X)(k)|$ and its dimension is strictly larger than $n+1$ if $k < d-1$, or it has fixed component if $k = d-1$.

The two formulas (1.3) and (1.4) are classically known as *postulation formulas* for homaloidal linear system.

The base locus $\mathrm{Bs}(\mathcal{H}_X)$ could be very complicated, e.g. it could be non-reduced or even contain embedded components. We will see this behavior in later examples. However, there are some special properties they share.

Proposition 1.3.5. *Let* $\mathfrak{B}(\mathcal{H}_X)_{\text{red}}$ *be the homogeneous ideal of* $\mathrm{Bs}(\mathcal{H}_X)_{\text{red}}$. *Then*

$$\mathfrak{B}(\mathcal{H}_X) \neq \mathfrak{B}(\mathcal{H}_X)^q_{\text{red}}$$

for any $q > 1$.

Proof. We set for brevity of notation, $\mathfrak{B}(\mathcal{H}_X) = \mathfrak{B}, \mathfrak{B}(\mathcal{H}_X)_{\text{red}} = \mathfrak{B}_{\text{red}}$. Assume that $\mathfrak{B} = \mathfrak{B}^q_{\text{red}}$ for some $q \geq 2$. Let $G \in \mathfrak{B}_{\text{red}}$, then $G^q = A_0F_0 + \dots A_rF_r$, where $F_0 \dots F_n$ is a basis of V. Hence $\deg\ G \geq \frac{d}{q}$. Now each F_i is a sum of the products $G_{i1} \cdots G_{iq}$, where $G_{ij} \in \mathfrak{B}_{\text{red}}$. Hence each G_{ij} has degree $p := \frac{d}{q}$ and p is an integer. Indeed each G_{ij} has degree $p_{ij} \geq \frac{d}{q}$ and, moreover, we have $p_{i1} + \dots + p_{iq} = d$. Since $\deg G_{ij} = p$, it follows that the multiplication

$$\mu : \mathrm{Sym}^q\ (\mathfrak{B}_{\text{red}})_p \to (\mathfrak{B}_{\text{red}})_{pq}$$

is surjective. Let $s+1 = \dim(\mathfrak{B}_{red})_p$, then $f = u \circ g$, where $g : \mathbb{P}^n \to \mathbb{P}^s$ is the rational map defined by $|\ (\mathfrak{B}_{red})_p\ |$ and $u : \mathbb{P}^s \to \mathbb{P}^{\binom{s+q}{q}-1}$ is the q-th Veronese map. On the other hand, f is birational, hence it follows that $f(\mathbb{P}^n)$ is a linear space contained in the Veronese variety $u(\mathbb{P}^s)$. But this variety does not contain linear spaces unless $q = 1$, and the statement follows. □

Another special property of $\mathrm{Bs}(\mathcal{H}_X)$ is given by the following.

Proposition 1.3.6. $\mathrm{Bs}(\mathcal{H}_X)$ *is not a complete intersection.*

Proof. If $\mathrm{Bs}(\mathcal{H}_X)$ is a complete intersection, then $\mathrm{Bs}(\mathcal{H}_X)$ is equidimensional of codimension $c \leq n$ and its homogeneous ideal $\mathfrak{B}(\mathcal{H}_X)$ is generated by c forms $G_1 \dots G_c$. By (1.3), we must have $\mathfrak{B}(\mathcal{H}_X)_d = n+1$. If G_i has degree $d_i < d$ then

$$G_i H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d - d_i)) \subset V.$$

Then, for dimension reasons, $d - d_i = 1$. But then $V(G_i)$ is a fixed component of $\mathrm{Bs}(\mathcal{H}_X)$, a contradiction. Hence it holds $d_i \geq d$, $i = 1, \dots, c$. Since $c \leq n$ it follows that $\dim \mathfrak{B}(\mathcal{H}_X)_d \leq c \leq n$, contradicting (1.3). □

Remark 1.3.2. Let $X \subset \mathbb{P}^n$ be a smooth irreducible non-degenerate subvariety of $\mathbb{P}^n$. Recall that *Hartshorne's conjecture* says that X is a complete intersection as soon as $\dim X > \frac{2n}{3}$. So, assuming that this conjecture is true, we obtain

- If the base locus scheme $\mathrm{Bs}(\mathcal{H}_X)$ is smooth and irreducible, then

$$\dim \mathrm{Bs}(\mathcal{H}_X) \leq \frac{2n}{3}.$$

In many examples $\mathrm{Bs}(\mathcal{H}_X)_{\mathrm{red}}$ is smooth but $\mathrm{Bs}(\mathcal{H}_X) \neq \mathrm{Bs}(\mathcal{H}_X)_{\mathrm{red}}$.

An additional quite strong condition is that $\mathrm{Bs}(\mathcal{H}_X)_{\mathrm{red}}$ is smooth, integral and moreover, f admits a resolution (π, σ), where π is the blow-up of $\mathrm{Bs}(\mathcal{H}_X)_{\mathrm{red}}$. Such a situation has been studied by Crauder and Katz in [CrauderKatz,Amer. J. Math. (1991)], in particular they show that in this case, assuming Hartshorne's conjecture,

$$\mathrm{Bs}(\mathcal{H}_X) = \mathrm{Bs}(\mathcal{H}_X)_{\mathrm{red}}.$$

They also show that, if Hartshorne's conjecture holds and $n \geq 7$, then $d \leq 4$.

1.4 The graph of a Cremona transformation

We define the *graph* Γ_f of a rational map $f : X \dashrightarrow Y$ as the closure in $X \times Y$ of the graph $\Gamma_{f_{\mathrm{d}}}$ of $f_{\mathrm{d}} : \mathrm{dom}(f) \to Y$. Clearly, the graph, together with its projections to X and Y, defines a resolution of the rational map f. Note that the switching the orders of the projections, we obtain that the same variety is the graph of the inverse transformation f^{-1}. In another way, we do not switch the projections but replace Γ_f with the isomorphic variety obtained by switching the factors of the product $\mathbb{P}^n \times \mathbb{P}^n$.

By the universal property of the graph, we obtain that, for any resolution (X', π, σ) of f, the image of the map $X' \to X \times Y$ contains $\Gamma_{f|_{\mathrm{dom}(f)}}$. Since X' is an irreducible projective variety, the image is closed and irreducible, hence it coincides with Γ_f. Thus the first projection $\Gamma_f \to X$ has the universal property for morphisms which invert $\mathfrak{b}(|V|)$, hence it is isomorphic to the blow-up scheme $\mathrm{Bl}_{\mathfrak{b}((|V|)}X$.

Let us consider the case of a Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If $(F_0, \dots, F_n)$ is a basis of V defining the homaloidal linear system, then the graph is a closed subscheme of $\mathbb{P}^n \times \mathbb{P}^n$ which is an irreducible component of the closure of the subvariety of $\mathrm{dom}(f) \times \mathbb{P}^n$ defined by 2×2-minors of the matrix

$$\begin{pmatrix} F_0(x) & F_1(x) & \dots & F_n(x) \\ y_0 & y_1 & \dots & y_n \end{pmatrix}, \tag{1.5}$$

where $x = (x_0, \dots, x_n)$ are projective coordinates in the first factor, and $y = (y_0, \dots, y_n)$ are projective coordinates in the second factor

In the usual way, the graph Γ_f defines the linear maps of cohomology

$$f_k^* : H^{2k}(\mathbb{P}^n, \mathbb{Z}) \to H^{2k}(\mathbb{P}^n, \mathbb{Z})$$

Since $H^{2k}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$, these maps are defined by some numbers d_k, the vector $(d_0, \dots, d_n)$ is called the *multi-degree* of f. In more details, we write the cohomology class $[\Gamma_f]$ in $H^*(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{Z})$ as

$$[\Gamma_f] = \sum_{k=0}^{n} d_i h_1^k h_2^{n-k},$$

where $h_i = \mathrm{pr}_i^* h$ and h is the class of a hyperplane in $\mathbb{P}^n$. Then

$$f_k^*(h^k) = (\mathrm{pr}_1)_*([\Gamma_f] \cdot (\mathrm{pr}_2^*(h^k)) = (\mathrm{pr}_1)_*(d_k h_1^k) = d_k h^k.$$

The multi-degree vector has a simple interpretation. The number d_k is equal to the degree of the proper transform under f of a general linear subspace of codimension k in $\mathbb{P}^n$. Since f is birational, $d_0 = d_n = 1$. Also $d_1 = d$ is the degree of f. Inverting f, we obtain that

$$\Gamma_{f^{-1}} = \tilde{\Gamma}_f,$$

where $\tilde{\Gamma}_f$ is the image of Γ_f under the self-map of $\mathbb{P}^n \times \mathbb{P}^n$ that switches the factors. In particular, we see that $(d_r, d_{r-1}, \dots, d_0)$ is the multi-degree of f^{-1}.

In the case when f is a birational map, we have $d_0 = d_n = 1$. We shorten the definition by saying that the multii-degree of a Cremona transformation is equal to $(d_1, \dots, d_{n-1})$.

Remark 1.4.1. Let T be an automorphism of $\mathbb{K}(z_1, \dots, z_n)$ identical on $\mathbb{K}$. This is an algebraic definition of a Cremona transformation. The automorphism T can be given by n rational functions $R_1(z_1, \dots, z_n), \dots, R_n(z_1, \dots, z_n)$. Write each

R_i as the ratio of two coprime polynomials $R_i = A_i/B_i$. Let $d_i = \deg A_i, d_i' = \deg B_i$. Making a substitution $z_i = t_i/t_0$, we can write

$$R_i = t_0^{d_i'-d_i}\frac{\bar{A}_i(t_0,\ldots,t_n)}{\bar{B}_i(t_0,\ldots,t_n)}, \quad i = 1,\ldots,n,$$

where $\bar{A}_i(t_0,\ldots,t_n)$ and $\bar{B}_i(t_0,\ldots,t_n)$ are homogeneous polynomials of degree d_i and d_i'. After reducing to the common denominator, we see that the corresponding Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ can be given by the homogeneous polynomials

$$(t_0^D\bar{A}_0\bar{B}_1\cdots\bar{B}_n, t_0^{D-d_1+d_1'}\bar{A}_1\bar{B}_0\bar{B}_2\cdots\bar{B}_n,\ldots,t_0^{D-d_n+d_n'}\bar{A}_n\bar{B}_1\cdots\bar{B}_{n-1}),$$

where $D = d_1+\ldots+d_n$. Of course, we have to divide these polynomials by their greatest common divisor. The degree of the transformation is less than or equal $D + d_1' + \ldots + d_n'$.

The next result due to L. Cremona puts some restrictions on the multi-degree of a Cremona transformation.

Proposition 1.4.1. *For any* $n \geq i, j \geq 0$,

$$1 \leq d_{i+j} \leq d_i d_j, \quad d_{n-i-j} \leq d_{n-i}d_{n-j}.$$

Proof. It is enough to prove the first inequality. The second one follows from the first one by considering the inverse transformation. Write a general linear subspace L_{i+j} of codimension $i+j$ as the intersection of a general linear subspace L_i of codimension i and a general linear subspace L_j of codimension j. Then, we have $f^{-1}(L_{i+j})$ is an irreducible component of the intersection $f^{-1}(L_i)\cap f^{-1}(L_j)$. By Bezout's Theorem,

$$d_{i+j} = \deg f^{-1}(L_{i+j}) \leq \deg f^{-1}(L_i)\deg f^{-1}(L_j) = d_i d_j.$$

□

There are more conditions on the multi-degree which follow from the irreducibility of Γ_f. For example, we have the following Hodge type inequality (see [Lazarsfeld, Cor. 1.6.3):

Theorem 1.4.2. *Let X be an irreducible variety of dimension n, and $\alpha_1,\ldots,\alpha_p$, $\beta_1,\ldots,\beta_{n-p}$ be the classes of nef Cartier divisors. Then, for any integers $0 \leq p \leq n$,*

$$(\alpha_1\cdot\ldots\cdot\alpha_p\cdot\beta_1\cdot\ldots\cdot\beta_{n-p})^p \geq (\alpha_1^p\cdot\beta_1\cdot\ldots\cdot\beta_{n-p})\cdot\ldots\cdot(\alpha_p^p\cdot\beta_1\cdot\ldots\cdot\beta_{n-p}).$$

Here we use the intersection theory of Cartier divisors (more about this in the next lecture).

For example, letting $p = 2$, and $\alpha_1 = \alpha, \alpha_2 = \beta, \beta_1 = \ldots = \beta_{i-1} = \alpha$ and $\beta_i = \ldots = \beta_{n-p} = \beta$, we obtain the inequality

$$s_i^2 \geq s_{i-1} \cdot s_{i+1},$$

where $s_i = \alpha^i \cdot \beta^{n-i}$. If we take $X = \Gamma_f$ and take $\alpha = h_1, \beta = h_2$ restricted to Γ_f, we get the inequality

$$d_i^2 \geq d_{i-1} d_{i+1} \tag{1.6}$$

for the multi-degree of a Cremona transformation f. For example, if $n = 3$, the only non-trivial inequality followed from the Cremona inequlities is $d_0 d_2 = d_2 \leq d_1^2$, and this is the same as the Hodge type inequality. However, if $n = 4$, we get new inequalities besides the Cremona ones. For example, $(1, 2, 3, 5, 1)$ satisfies the Cremona inequalities, but does not satisfy the Hodge type inequality.

The following are natural questions related to the classification of possible multi-degrees of Cremona transformations.

- Let $(1, d_1, \ldots, d_{n-1}, 1)$ be a sequence of integers satisfying the Cremona inequalities and the Hodge type inequalities: Does there exist an irreducible reduced close subvariety Γ of $\mathbb{P}^n \times \mathbb{P}^n$ with $[\Gamma] = \sum d_k h_1^k h_2^{n-k}$?

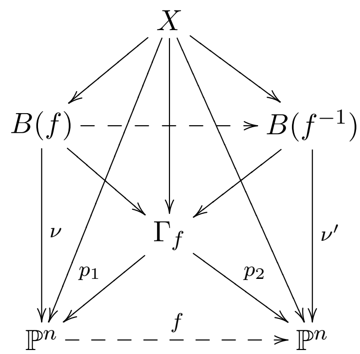
- What are the components of the Hilbert scheme of this class containing an integral scheme?

Note that any irreducible reduced closed subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ with multi-degree $(1, d_1, \ldots, d_{n-1}, 1)$ is realized as the graph of a Cremona transformation.

Remark 1.4.2. The inequalities (1.6) show that the sequence $(d_0, d_1, \ldots, d_n)$ is a *log-concave sequence*. Many other situations where such sequences arise are discussed in June Huh's preprint on the archive.

1.5 F-locus and P-locus

Let (X, π, σ) be any normal resolution of a Cremona transformation $f : \mathbb{P}^n - \rightarrow \mathbb{P}^n$. It factors through the blow-up $B(f)$ of the integral closure of the ideal sheaf of the base scheme of $\mathfrak{b}(\mathcal{H}_X)$ as well as the blow-up $B(f^{-1})$ of the integral closure of the ideal sheaf of the base scheme of the homaloidal linear system $\mathfrak{b}(|W|)$ defining the inverse Cremona transformation f^{-1}. So we have a commutative diagram



Let $E = \sum_{i\in I} r_iE_i$ be the exceptional divisor of $\nu : B(f) \to \mathbb{P}^n$ and $F = \sum_{j\in J} m_jF_j$ be the exceptional divisor of $\nu' : B(f^{-1}) \to \mathbb{P}^n$. Let J' be the largest subset of J such that the proper transform of $F_j, j \in J'$, in X is not equal to the proper transform of some E_i in X. The image of the divisor $\sum_{j\in J'} F_j$ under the composition map $B(f^{-1}) \to \Gamma_f \xrightarrow{p_1} \mathbb{P}^n$ is classically known as the P-*locus* of f (the F-*locus* is $\mathrm{Bs}(\mathcal{H}_X)_{\text{red}}$). It is hypersurface in the source $\mathbb{P}^n$. The image of any irreducible component of the P-locus is blown down under f (after we restrict ourselves to $\mathrm{dom}(f)$) to an irreducible component of the base locus of f^{-1}. If we consider f as a regular map $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by the same formula. Then the P-locus is the image in $\mathbb{P}^n$ of the set of critical points of $\tilde{f}$. It is equal to the set of zeroes of the determinant of the Jacobian matrix of $\tilde{f}$

$$J = \Big(\frac{\partial F_i}{\partial t_j}\Big)_{i,j=0,\ldots,n}.$$

So we expect that the P-locus is a hypersurface of degree $(d-1)^{n+1}$. Some of its components enter with multiplicities. We assign these multiplicities to the corresponding irreducible components of the P-locus.

Example 1.5.1. Consider the *standard quadratic transformation* given by

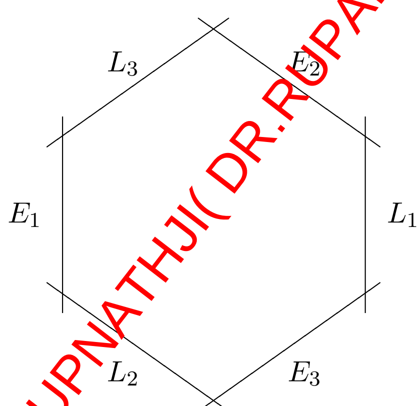
$$T_{\text{st}} : [x_0, x_1, x_2] \mapsto [x_1x_2, x_0x_2, x_0x_1]. \tag{1.7}$$

The P-locus is the union of three coordinate lines $V(t_i)$. The Jacobian matrix is

$$J = \begin{pmatrix} 0 & t_2 & t_1 \\ t_2 & 0 & t_0 \\ t_1 & t_0 & 0 \end{pmatrix}.$$

Its determinant is equal to $2t_0t_1t_2$. We may take $X = B(f)$ as a smooth resolution of f. Let E_1, E_2, E_3 be the exceptional divisors over the base points

$p_1 = [1,0,0], p_2 = [0,1,0], p_3 = [0,0,1]$, and $L_i, i = 1,2,3$, be the proper transform of the coordinate lines $V(t_0), V(t_1), V(t_2)$, respectively. Then the morphism $\sigma : X \to \mathbb{P}^2$ blows down L_1, L_2, L_3 to the points p_1, p_2, p_3, respectively. Note that $f^{-1} = f$, so there is no surprise here. Recall that the blow-up of a closed subscheme is defined uniquely only up to an isomorphism. If we identify $B(f)$ with $B(f^{-1})$, the morphism $B(f) \dashrightarrow B(f^{-1})$ in the above diagram is an automoroph-ism induced by the Cremona transformation. The surface $B(f)$ is a Del Pezzo surface of degree 6, a toric Fano variety of dimension 2. The complement of the open torus orbits is the hexagon of lines $E_1, E_2, E_3, L_1, L_2, L_3$ intersecting each other as in the following picture. We call them lines because they become lines in the embedding $X \hookrightarrow \mathbb{P}^6$ given by the anti-canonical linear system. The automorphism of the surface is the extension of the inversion automorphism $z \to z^{-1}$ of the open torus orbit to the whole surface. It defines the symmetry of the hexagon which exchanges its opposite sides.



Now let us consider the first *degenerate standard quadratic transformation* given by

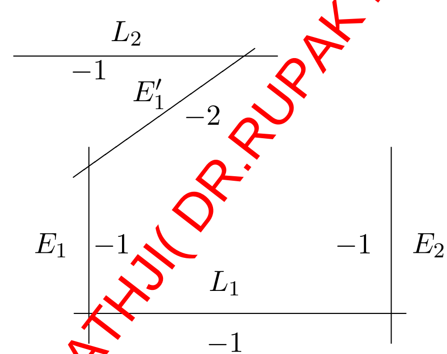
$$T'_{\text{st}} : [x_0, x_1, x_2] \mapsto [x_1^2, x_0x_1, x_0x_2]. \tag{1.8}$$

The P-locus consists of the line $V(t_0)$ blown down to the point $p_1 = [1,0,0]$ and the line $V(t_1)$ blown down to the point $p_2 = [0,0,1]$. The Jacobian matrix is

$$J = \begin{pmatrix} 0 & 2t_1 & 0 \\ t_1 & t_0 & 0 \\ t_2 & 0 & t_0 \end{pmatrix}.$$

Its determinant is equal to $-2t_0t_1^2$. Thus the line $V(t_1)$ enters with multiplicity 2. Let us see what is the resolution in this case. The base scheme is smooth at p_1 and locally isomorphic to $V(y^2, x)$ at the point p_2, where $y = t_1/t_2, x =$

t_0/t_2. The blow-up $B(f)$ is singular over p_2 with the singular point $2'_1$ of type A_1 corresponding to the tangent direction $t_0 = 0$. Thus the exceptional divisor of $B(f) \to \mathbb{P}^2$ is the sum of two irreducible components E_1 and E_2 both isomorphic to $\mathbb{P}^1$ with the singular point p'_2 lying on E_1. The exceptional divisor of $B(f) = B(f^{-1} \to \mathbb{P}^2$ is the union of two components, the proper transform L_1 of the line $V(t_1)$ and the proper transform L_2 of the line $V(t_0)$. When we blow-up p'_2, we get a smooth resolution X of f. The exceptional divisor of $\pi : X \to \mathbb{P}^2$ is the union of the proper transforms of E_1 and E_2 on X and the exceptional divisor E'_1 of the blow-up $X \to B(f)$. The exceptional divisor of $\sigma : X \to \mathbb{P}^2$ is the union of the proper transforms of L_1 and L_2 on X and the exceptional divisor E'_1. Note that the proper transforms of E_1, E_2 and L_1, L_2 are (-1)-curves and the curve E'_1 is a (-2)-curve.[1]

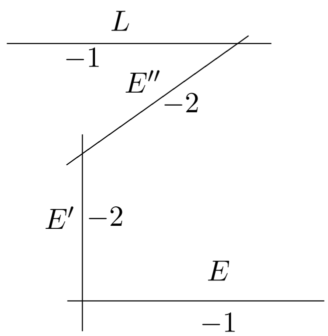


Finally, we can consider the *second degenerate standard quadratic transformation*. Its reduced base scheme consists of one point. The map is given by the formula

$$T''_{\text{st}} : [x_0, x_1, x_2] \mapsto [x_0^2, x_0x_1, x_1^2 - x_0x_2]. \tag{1.9}$$

Its unique base point is $[0, 0, 1]$. In affine coordinates $x = x_0/x_2, y = x_1/x_2$, the base ideal is $(x^2, xy, y^2-x) = (y^3, x-y^2)$. The blow-up of this ideal is singular. It has a singular point of type A_2 on the irreducible exceptional divisor $E \cong \mathbb{P}^1$. The P-locus consists of one line $x_0 = 0$. A smooth resolution of the transformation is obtained by blowing up twice the singular point. The exceptional divisor of this blow-up is the chain of two (-2)-curves. So the picture is as follows:

[1] a $(-n)$-curve is a smooth rational curve with self-intersection $-n$.



Note that the jacobian matrix of transformation T''_{st} is equal to

$$\begin{pmatrix} 2t_0 & 0 & 0 \\ t_1 & t_0 & 0 \\ -t_2 & 2t_1 & -t_0 \end{pmatrix}$$

Its determinant is equal to $-2t_0^3$. So the P-locus consists of one line $V(t_0)$ taken with multiplicity 3.

Lecture 2

Intersection Theory

2.1 The Segre class

The multi-degree of the graph Γ_f can also be computed using intersection theory. Recall some relevant theory from [Fulton].

Let X be a scheme of finite type over a field $\mathbb{K}$. A *k-cycle* is an element of the free abelian group $Z_k(X)$ generated by the set of points $x \in X$ of dimension k (i.e. the residue field $\kappa(x)$ of x is of algebraic dimension k over $\mathbb{K}$). We identify a point x with its closure in X, it is an irreducible reduced closed subscheme V of dimension k (a subvariety). We write $[V]$ for V considered as an element of $Z_k(X)$. Two k-cycles Z and Z' are called *rationally equivalent* if the difference is equal to the projections of a cycle $\mathcal{Z}(0) - \mathcal{Z}(\infty)$ on $X \times \mathbb{P}^1$ for some cycle $\mathcal{Z}$ on the product.

One can give an equivalent definition as follows. A prime divisor $\mathfrak{p}$ of height 1 in an integral domain A defines a function $\mathrm{ord}_{\mathfrak{p}} : A \setminus \{0\} \to \mathbb{Z}$ by setting $\mathrm{ord}_{\mathfrak{p}}(a) = \mathrm{length}(A_{\mathfrak{p}}/(a_{\mathfrak{p}})$, where $a_{\mathfrak{p}}$ is the image of a in the local (one-dimensional) ring $A_{\mathfrak{p}}$. This function is extended to a unique homomorphism $Q(A)^* \to \mathbb{Z}$, where $Q(A)$ is the field of fractions of A. By globalizing, we obtain a function $\mathrm{ord}_x : R(X) \to \mathbb{Z}$, where x is a point on X of codimension 1 and $R(X)$ is the field of rational functions on a variety X[1]. Now we define the subgroup $B_k(X)$ of rationally equivalent k-cycles as the group generated by cycles of the form $\sum_x \mathrm{ord}_x \phi x$, where ϕ is a rational function on a subvariety $Y \subset X$ of dimension $k + 1$. The quotient group $A_k(X) = Z_k(X)/B_k(X)$ is called the *Chow group* of k-cycles on X. We set

$$A_*(X) = \oplus_k A_k(X).$$

[1]From now on a variety over a field $\mathbb{K}$ means an integral algebraic scheme over $\mathbb{K}$

For any $\alpha = \sum n_x x \in Z_0(X)$, we define

$$\int_X \alpha = \sum_x n_x[\kappa(x) : \mathbb{K}].$$

When X is proper, this extends to $A_0(X)$, and, to the whole $A_*(X)$, where, by definition, $\int_X \alpha = 0$ if $\alpha \in A_k(X), k > 0$.

For any scheme X one defines its *fundamental class* by

$$[X] = \sum n_V [V],$$

where V is an irreducible component of X and n_V is its multiplicity, the length of $\mathcal{O}_{X,\eta}$, where η is a generic point of V.

For any proper morphism $\sigma : X \to Y$ of schemes, one defines the push-forward homomorphism

$$\sigma_* : A_*(X) \to A_*(Y)$$

by setting, for any subvariety V,

$$\sigma_*[V] = \deg(V/\sigma(V))[\sigma(V)]$$

and extending the definition by linearity. Note that $\deg(V/\sigma(V)) = 0$ if the map $V \to \sigma(V)$ is not of finite degree. One checks that rationally equivalent to zero cycles go to zero, so the definition is legal. The homomorphism σ_* preserves the grading of $A_*(X)$.

The pull-back $\sigma^* : A_*(Y) \to A_*(X)$ is defined only for flat morphisms, regular closed embeddings, and their compositions. For a flat morphism σ one sets, for any subvariety V, $\sigma^*[V] = [\sigma^{-1}(V)]$. It shifts the degree by increasing it by the relative dimension of σ.

Recall that a *Weil divisor* on a variety X of dimension n is an element of $Z_{n-1}(X)$. A *Cartier divisor* on X is a section of the sheaf $\mathcal{R}_X/\mathcal{O}_X^*$, where $\mathcal{R}_X$ is the constant sheaf of total rings of fractions. The function $\text{ord}_x : \mathcal{R}(X) \to \mathbb{Z}$ factors through $\mathcal{O}_X^*$ and defines a homomorphism

$$\text{CDiv}(X) \to \text{WDiv}(X), \quad D \mapsto [D] = \sum_x \text{ord}_x(D)x,$$

where $\text{CDiv}(X)$ (resp. $\text{WDiv}(X)$) is the group of Cartier (resp. Weil) divisors on X.

Let D be a Cartier divisior on X. One can restrict it to any subvariety V of X, and set

$$D \cdot [V] = [j^*(D)],$$

where $j : V \hookrightarrow X$ is the inclusion morphism of V to X. It is considered as a cycle on V and also as a cycle on X by means of $j_* : A_*(V) \to A_*(X)$. This extends by linearity to the intersection $D \cdot \alpha$ of D with any cycle class $\alpha \in A_*(X)$, so we can consider any divisor D as an endomorphism $\alpha \mapsto D \cdot \alpha$ of $A_*(X)$. It depends only on the linear equivalence class of D. By iterating the endomorphism, we can define, for any Cartier divisors $D_1, \ldots, D_k$ and $\alpha \in A_m(X)$, the intersection

$$D_1 \cdot \ldots \cdot D_k \cdot \alpha \in A_{m-k}(X).$$

For any closed subvariety Y containing $\text{Supp}(D) \cap V$ we can identify $D \cdot [V]$ with an element of $A_*(Y)$. In particular, for any Y as above, any Cartier divisor D on X can be considered as a homomorphism

$$A_k(Y) \to A_{k-1}(\text{Supp}(D) \cap Y), \quad \alpha \mapsto D \cdot \alpha.$$

By definition,

$$D_1 \cdots D_k = D_1 \cdot \cdots \cdot D_k \cdot [X].$$

If X is of pure dimension n, this is an element of $A_{n-k}(X)$. We abbreviate $D^k = D \cdots D$ (k times). The intersection of Cartier divisors is commutative, associative, the projection formula holds, and depends only on the linear equivalence classes.

Let $j : Y \hookrightarrow X$ be a closed subscheme of a scheme Y defined by an ideal sheaf $\mathcal{I}_Y$. Let

$$C = C_Y X = \text{Spec} \bigoplus_{k=0}^{\infty} \mathcal{I}_Y^k / \mathcal{I}_Y^{k+1}$$

be the *normal cone* of Y in X. Let

$$P(C \oplus \mathcal{O}_Y) = \text{Proj}\left(\left(\bigoplus_{k=0}^{\infty} \mathcal{I}_Y^k / \mathcal{I}_Y^{k+1}\right) \otimes \mathcal{O}_Y[z]\right),$$

where the tensor product is the tensor product of graded $\mathcal{O}_Y$-algebras with $\deg z = 1$. This is the *projectivized normal cone* of Y in X. Recall the standard exact sequence of sheaves of differentials:

$$\mathcal{I}_Y / \mathcal{I}_Y^2 \xrightarrow{d} j^*(\Omega^1_{X/\mathbb{K}}) \to \Omega^1_{Y/\mathbb{K}} \to 0. \tag{2.1}$$

Assume that $Y = x$ is a closed point of X. Then, $\Omega^1_{Y/\mathbb{K}} = 0$. By localizing at x, we obtain an isomorphism ([Hartshorne], Chapter 2, Prop. 8.7)

$$\mathfrak{m}_{X,x} / \mathfrak{m}^2_{X,x} \cong \Omega^1_{X/\mathbb{K}}(x).$$

The dual of the first space is, by definition, the Zariski tangent space $T_{X,x}$ of X at x. Going back to the general case, and passing to the fibres of the sheaves, for any closed point $x \in X$, we get an injection $T_{Y,x} \to T_{X,x}$ with cokernel equal to a subspace of $(\mathcal{I}_Y(x)/\mathcal{I}_Y(x)^2)^\vee$. In many cases, the kernel of the map d is trivial (e.g. when X is reduced and a locally complete intersection in characteristic 0, or when X, Y are nonsingular), hence $(\mathcal{I}_Y(x)/\mathcal{I}_Y(x)^2)^\vee$ can be interpreted as the fibre of the normal bundle of Y in X at the point x. More precisely, by definition, the X-scheme

$$\mathbb{V}(\Omega^1_{X/\mathbb{K}}) := \text{Spec Sym}(\Omega^1_{X/\mathbb{K}})$$

is the *tangent vector bundle* of X (its fibres at closed points are Zariski tangent spaces). The Y-scheme Spec Sym$(j^*(\Omega^1_{X/\mathbb{K}})$ is the restriction $\mathbb{V}(\Omega^1_{X/\mathbb{K}})|_Y$ of the tangent bundle $\mathbb{V}(\Omega^1_{X/\mathbb{K}})$ of X to Y. The surjection of sheaves $j^*(\Omega^1_{X/\mathbb{K}}) \to \Omega^1_{Y/\mathbb{K}}$ defines a surjection of the symmetric algebras, and the closed embedding

$$\mathbb{V}(\Omega^1_{Y/\mathbb{K}}) \hookrightarrow \mathbb{V}(\Omega^1_{X/\mathbb{K}})|_Y.$$

In the case when both X and Y are nonsingular, the sheaves of differentials are locally free of ranks equal to the dimensions and the normal cone $C_Y X$ is also a vector bundle isomorphic to $\mathbb{V}(\mathcal{I}_Y/\mathcal{I}_Y^2) = $ Spec (Sym$(\mathcal{I}_Y/\mathcal{I}_Y^2)$. In a general situation, none of these relative affine schemes is a vector bundle, and the normal cone should be considered as the analogue of the normal bundle in a nonsingular situation.

The projectivized normal cone is the analog of the projectivization of the normal vector bundle. Let $\mathcal{G}r_{\mathfrak{a}} = \oplus_{k=0}^\infty \mathcal{I}_Y^k/\mathcal{I}_Y^{k+1}$. The closed subscheme defined by $z = 0$ in $P(C \oplus \mathcal{O}_Y)$ is identified with the exceptional divisor of the blow-up scheme Bl$_Y X$. The complement $D^+(z)$ is equal to $C_Y X$. Thus $P(C \oplus \mathcal{O}_X)$ is the analog of the associated projective bundle, and the exceptional divisor is the analog of the "section at infinity". The canonical projection $q : C_Y X \to X$ admits a section disjoint from the hyperplane at infinity. It corresponds to the surjection $\mathcal{G}r_{X/Y} \to \mathcal{O}_X$ with kernel $\oplus_{k\geq 1}\mathcal{I}_Y^k/\mathcal{I}_Y^{k+1}$.

Recall that, for any graded sheaf of algebras $\mathcal{A} = \oplus_{k=0}\mathcal{A}_k$ on a scheme Z, the $\mathcal{A}$-module $\mathcal{A}[1] = \oplus_{k=0}^\infty \mathcal{A}_{k+1}$ defines a coherent sheaf $\mathcal{O}(1)$ on the projective spectrum Proj $\mathcal{A}$. It is an invertible sheaf if $\mathcal{A}$ is generated by $\mathcal{A}_1$. In our case, the algebra $\mathcal{G}r_{Y/X} \otimes \mathcal{O}_Y[z]$, and the *Rees algebra* $\oplus_{k\geq 0}\mathcal{I}_Y^k$ satisfy this property, so the projective spectrums come equipped with the corresponding invertible sheaf $\mathcal{O}(1)$. We have $\mathcal{O}(1) \cong \mathcal{O}(D)$ for some divisor D. In the case of $\mathbb{P}(C \oplus \mathcal{O}_Y)$, the divisor D can be taken to be the hyperplane at infinity H^∞. In the case of the blow-up scheme Bl$_Y X$, we can take D to be $-E$, where E is the exceptional divisor of the blow-up.

The *Segre class* of Y in X, denoted by $s(Y, X)$, is defined to be

$$s(Y, X) = q_*(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i).$$

In the case when $j : Y \hookrightarrow X$ is a regular embedding, the Segre classes are expressed in terms of the Chern classes of the conormal sheaf $\mathcal{N}_{Y/X}^{\vee}$. Recall that for any locally free sheaf $\mathcal{E}$ of rank r over an n-dimensional variety X, the Chern classes $c_i(\mathcal{E})$ are elements of $A_{n-i}(X)$ defined as follows. Let $P = P(\text{Sym } \mathcal{E} \otimes \mathcal{O}_X[z])$ be be projectivization of the vector bundle $\mathbb{V}(\mathcal{E}) = \text{Spec Sym } \mathcal{E}$. Let $\pi : P \to X$ be the natural projection. It is a flat morphism, so $\pi^* : A_k(X) \to A_{k+r}(P)$ is well-defined. One can show that $A_*(P)$ is generated by $\pi^*(A_*(X))$ and $h = c_1(\mathcal{O}_P(1))$ with one basic relation of the form

$$(-h)^r + a_1(-h)^{r-1} + \cdots + a_r = 0, \quad a_i \in \pi^*(A_{n-i}(X)).$$

By definition, the i-th Chern class $c_i(\mathcal{E})$ is defined by $a_i = \pi^*(c_i(\mathcal{E}))$. Note that here the products are defined in the sense of intersection of Cartier divisor classes h^i with any element of $A_*(P)$.

We set

$$c(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E}) \in A_*(X).$$

The following are the basic properties of the Chern classes (see [Hartshorne, Appendix].

(i) $c_1(\mathcal{O}_X(D)) = [D]$;

(ii) for any morphism $\sigma : Y \to X$, $c_i(\sigma^*(\mathcal{E})) = \sigma^*(c(\mathcal{E}))$; We define

$$c_i(\mathcal{E}) \cap [V] = c_i(j^*\mathcal{E})).$$

This extends to the intersection $c_i(\mathcal{E}) \cdot \alpha$ with any $\alpha \in A_*(X)$. In particular, one can define the value of any integral polynomial at the Chern classes of vector bundles and the intersection of this polynomial with any $\alpha \in A_*(X)$.

(iii) if $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ is an exact sequence of locally free sheaves, then

$$c(\mathcal{E}_2) = c(\mathcal{E}_1) \cdot c(\mathcal{E}_3);$$

(iv) $c_i(\mathcal{E}^{\vee}) = (-1)^i c_i(\mathcal{E})$;

(v) $c_1(\mathcal{E}) = c_1(\bigwedge^r \mathcal{E})$.

We define the *Segre classes* $s_i(\mathcal{E})$ of $\mathcal{E}$ by computing them inductively from the relation

$$c(\mathcal{E}) \cdot (\sum_{i=0}^{\dim X} s_i(\mathcal{E})) = [X].$$

More explicitly, $s_0(\mathcal{E}) = [X]$, and the rest are found by solving inductively the relations

$$\sum_{i+j=k} c_i(\mathcal{E}) s_j(\mathcal{E}) = 0, \quad k = 1, \ldots, \dim X.$$

Now we can state some properties of $s(Y, X)$.

- If $\mathcal{N}_{Y/X} = \mathcal{I}_Y/\mathcal{I}_Y^2$ is locally free and $\bigoplus_{k=0}^{\infty} \mathcal{I}_Y^k/\mathcal{I}_Y^{k+1} \cong \operatorname{Sym} \mathcal{N}_{Y/X}^{\vee}$ (in this case we say that Y is *regularly embedded*), then

$$s(Y, X) = c(\mathcal{N}_{Y/X})^{-1}. \tag{2.2}$$

- If $f : X' \to X$ is a morphism of pure-dimensional schemes and $Y' \subset X'$ is the pre-image of a closed subvariety Y of X with the induced morphism $g : Y' \to Y$, then

$$g_*(s(Y', X')) = \deg(Y'/Y)s(Y, X). \tag{2.3}$$

- If $f : X' \to X$ is flat, and $Y' = f^{-1}(Y)$, then

$$g^*(s(Y, X)) = s(Y', X'). \tag{2.4}$$

- If $\pi : \tilde{X} = \mathrm{Bl}_Y X \to X$ is the blow-up of a proper closed subscheme Y in X, and E is the exceptional divisor with the projection $\sigma : E \to Y$, then

$$s(Y, X) = \sum_{i \geq 1} (-1)^{i-1} \sigma_*(E^i) = \sum_{i \geq 1} (-1)^{i-1} \pi_*(E^i). \tag{2.5}$$

Recall that, by definition, $E^i \in A_{\dim X - i}(\tilde{X})$ which we can also assume to be in $A_{\dim X-i}(E)$. Thus we may also consider $\pi_*(E^i)$ to be in either in $A_*(Y)$ or equal to its image $\pi_*(E^i)$ in $A_*(X)_k$ under the map $i_* : A_*(Y) \to A_*(X)$ corresponding to the inclusion morphism $j : Y \hookrightarrow X$.

Note that the last formula gives

$$\sigma_*(E^i) = (-1)^{i-1} s(Y, X)_{\dim X - i}, \tag{2.6}$$

where $s(Y, X)_k$ is the component of $s(Y, X)$ in $A_*(Y)_k$, or its image in $A_*(X)_k$.

2.2 Self-intersection of exceptional divisors

The properties of Segre classes allow us to compute the self-intersection of exceptional divisors of blow-ups.

Recall that, if X is a nonsingular surface, and Y a closed point on it, the exceptional curve of $\sigma : \mathrm{Bl}_Y X \to X$ is a (-1)-curve[2], i.e. a smooth rational curve E with self-intersection equal to -1. Of course, this agrees with formula $s(Y,X)_0 = -\sigma_*(E^2)$ because $s(Y,X) = c(\mathcal{N}_{Y/X})^{-1} = [Y]$. If we replace X by any smooth n-dimensional variety, and take Y to be a closed point, we obtain, similarly,

$$E^n = (-1)^{n-1}. \tag{2.7}$$

Of course, this can be computed without using Segre classes. We embed X in a projective space, take a smooth hyperplane section H passing through the point Y. Its full transform on $\mathrm{Bl}_Y X$ is equal to the union of E and the proper transform H_0 intersecting E along a hyperplane L inside E identified with $\mathbb{P}^{n-1}$. Replacing H with another hyperplane H' not passing through Y, we obtain

$$H' \cdot E = (H_0 + E) \cdot E = e + E^2 = 0,$$

where e is the class of a hyperplane in E. Thus $E^2 = -e$. This of course agrees with the general theory. The line bundle $\mathcal{O}(1)$ on the blow-up is isomorphic to $\mathcal{O}(-E)$. The conormal sheaf $O_E(-E)$ is isomorphic $\mathcal{O}_E(1)$ on the exceptional divisor.

We also have $H_0 \cdot E^2 = -H_0 \cdot e = -e^2$, hence

$$0 = E^2 \cdot H' = E^2 \cdot (H_0 + E) = E^2 \cdot H_0 + E^3$$

gives $E^3 = e^3$. Continuing in this way, we find

$$E^k = (-1)^{n-1} e^k. \tag{2.8}$$

Now let us take Y to be any smooth subvariety with the normal sheaf $\mathcal{N}_{Y/X}$. Applying (2.6), we find that

$$E^n = (-1)^{n-1} s(Y,X)_0 = (-1)^{n-1} \int_X s(Y,X). \tag{2.9}$$

For example, let $X = \mathbb{P}^n$ and Y be a linear subspace of codimension $k > 1$. We have

$$\mathcal{N}_{Y/X} \cong \mathcal{O}_Y(1)^k,$$

[2] Also known as an exceptional curve of the first kind.

hence

$$s(Y,X) = \frac{1}{(1+h)^k} = \frac{1}{(k-1)!}\Big(\frac{1}{1-x}\Big)^{(k-1)}_{|x=-h} = \sum_{m=k-1}^{n-1} \tbinom{m}{k-1}(-h)^{m-k+1}.$$

(note that $h^i = 0, i > \dim Y = n-k$). In particular, we obtain

$$E^n = (-1)^{k-1}\tbinom{n-1}{k-1}. \tag{2.10}$$

For example, the self-intersection of the exceptional divisor of the blow-up of a line in $\mathbb{P}^n$ is equal to $(-1)^n(n-1)$.

One can use the exact sequence of sheaves of differentials (2.1) to compute $\mathcal{N}_{Y/X}$, where $j : Y \hookrightarrow X$ is a regular embedding of smooth varieties. It gives

$$c(\mathcal{N}_{Y/X}) = c((\Omega^1_X)^\vee)c((\Omega^1_Y)^\vee)^{-1}.$$

For example, when $Y = C$ is a curve of genus g and degree $\deg C$ in $X = \mathbb{P}^n$, we obtain

$$c_1(N_{C/\mathbb{P}^n}) = -K_{\mathbb{P}^n}\cdot C + K_C = (n+1)\deg C + 2g - 2. \tag{2.11}$$

Another useful formula to compute the Chern classes of a regularly embedded subsvariety is the following one.

$$0 \to \mathcal{N}_{Z/Y} \to \mathcal{N}_{Z/X} \to j^*\mathcal{N}_{Y/X} \to 0. \tag{2.12}$$

Passing to the Chern classes and taking the inverse, we get

$$s(Z,X) = c(\mathcal{N}_{Z/Y})^{-1}\cdot j^*c(\mathcal{N}_{Y/X})^{-1}.$$

We will use this formula many times in the following computations.

Recall also from the theory of surfaces that the self-intersection of the proper transform of a curve under the blow-up of a closed point x decreases by the square of the multiplicity of the curve at x. This is generalized as follows (see [Fulton], Appendix B).

First let recall the definition of the proper transform of a closed subscheme $Y \subset X$ under the blow-up of the closed subscheme Z of X (see [Hartshorne], Chap. III, Corollary 7.15).

Lemma 2.2.1. *Let $\sigma : X' = \mathrm{Bl}(\mathfrak{a})$ be the blow-up of an Ideal $\mathfrak{a}$ in a noetherian scheme X, and $f : Y \to X$ be a morphism of noetherian schemes. Let $\nu : Y' \to Y$*

be the blow-up of the inverse image $f^{-1}(\mathfrak{a})$. Then there exists a unique morphism $\tilde{f} : Y' \to X'$ such that the following diagram is commutation

$$\begin{array}{ccc} Y' & \xrightarrow{\tilde{f}} & X' \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \sigma} \\ Y & \xrightarrow{f} & X \end{array}$$

Moreover, if f is a closed embedding, so is $\tilde{f}$.

When $f : Y \hookrightarrow X$ is a closed embedding, the image of $\tilde{f}$ is called the *proper transform* (or *strict transform*) of $f(Y)$. Note that, when $f^{-1}(\mathfrak{a}) = 0$ (i.e. j factors through $V(\mathfrak{a})$), the proper transform is the empty scheme. On the other hand, if Y is reduced, and $U = Y \setminus V(f^{-1}(\mathfrak{a}))$ is dense in Y, then Y' is isomorphic to the closure of $f^{-1}(U)$ in X'.

Note that, in general, the commutative diagram is not a Cartesian diagram, i.e. the natural morphism $Y' \to X' \times_X Y$ is not an isomorphism. When f is a closed embedding, the image of $X' \times_X Y$ in X' is equal, by definition, to $\sigma^{-1}(Y)$. Its ideal sheaf is equal to the product $\mathcal{J}_E \cdot \mathcal{J}_{Y'}$, where E is the exceptional divisor of σ. Also, it follows from the commutativity of the diagram that $\tilde{f}^{-1}(\sigma^{-1}(\mathfrak{a})) = \nu^{-1}(f^{-1}(\mathfrak{a}))$. This means that the pre-image of the exceptional divisor E of σ under $\tilde{f}$ is equal to the exceptional divisor of ν.

For example, take X to be a nonsingular surface, and $\mathfrak{a}$ to be the ideal sheaf of a closed point x. Let $f : Y \hookrightarrow X$ be the inclusion morphism of a reduced curve Y passing through x with multiplicity m. Then $f^{-1}(\mathfrak{a})$ is the ideal of x considered as a closed point of Y. The blow-up of this ideal in Y has the exceptional divisor equal to a subscheme of the exceptional divisor $E \cong \mathbb{P}^1$ of $X' \to X$ of length m. This is the intersection scheme of the proper transform of C with E.

Proposition 2.2.2. *Let $Z \hookrightarrow Y$ and $Y \hookrightarrow X$ be regular embeddings. Let $\sigma : X' = \mathrm{Bl}_Z X \to X$ be the blow-up of Z and $\overline{Y}$ be the proper transform of Y under σ. Then*

$$\mathcal{N}_{\overline{Y}/X'} \cong \nu^*(\mathcal{N}_{Y/X}) \otimes j^*\mathcal{O}_{X'}(-E),$$

where $j : \overline{Y} \hookrightarrow X'$, E is the exceptional divisor of σ and $\nu : \overline{Y} \to Y$ is the restriction of σ to $\overline{Y}$.

In terms of the intersection theory, the formula reads

$$\overline{Y}^2 = \sigma^*(Y^2) - \overline{Y} \cdot E. \tag{2.13}$$

This gives

$$\overline{Y}^3 = \overline{Y} \cdot \sigma^*(Y^2) - \overline{Y}^2 \cdot E = Y^3 - (\sigma^*(Y^2) - \overline{Y} \cdot E) \cdot E = Y^3 + \overline{Y} \cdot E^2,$$

where we used the projection formula $\overline{Y} \cdot \sigma^*(Y^2) = Y^2 \cdot \sigma_*(\overline{Y}) = Y^3$. Continuing in this way, we find, for any $k = 0, \ldots, n$,

$$\overline{Y}^k = \sigma^*(Y^k) + (-1)^{k-1}\overline{Y}^{k-1} \cdot E = \sigma^*(Y^k) + (-1)^{k-1}\overline{Y} \cdot E^k. \qquad (2.14)$$

In the case of surfaces, in order the assumptions of Proposition 2.2.2 are satisfied, we have to take Z to be the ideal sheaf $\mathcal{J}_x^m$ of the "fat point" x, where m is the multiplicity of Y at x. Then the embedding $V(\mathcal{J}_x^m) \hookrightarrow Y$ is regular (check it!). The blow-up of the fat point x is isomorphic to the blow-up $\mathrm{Bl}_x X$ of the point x, but the exceptional divisor is equal to mE, where E is the exceptional divisor of equal to $\mathrm{Bl}_x X$.

We have

$$\deg \mathcal{N}_{Y/X} = \deg \mathcal{O}_Y(Y) = Y^2 = \deg \sigma^*(\mathcal{N}_{Y/X}).$$

Also, $\deg \mathcal{O}_{\overline{Y}}(-E) = -E \cdot \overline{Y} = -\mathrm{mult}_Z Y$. This gives

$$\overline{Y}^2 = \deg \mathcal{N}_{\overline{Y}/X'} = Y^2 - \deg j^*\mathcal{O}_{X'}(-mE) = Y^2 - m^2.$$

Next, let us give a less simple example. Let $X = \mathbb{P}^3$ and Y be a connected union of lines $\ell_1, \ldots, \ell_k$ with $\#\ell_i \cap (\cup_{j\neq i}\ell_j) = n_i$. Let $p_1, \ldots, p_N$ be the intersection points and a_j be the number of lines containing the point p_j.

Let $\sigma : X_1 \to \mathbb{P}^3$ be the blow-up of the intersection points (in any order) and $\nu : X \to X_1$ be the blow-up of the proper transforms of the lines (again in any order). Let E be the exceptional divisor of $\pi = \nu \circ \sigma$. The exceptional divisor of σ consists of the disjoint sum of N divisors $E(p_j) = \sigma^{-1}(p_j)$ isomorphic to $\mathbb{P}^2$. Let $\bar{\ell}_i$ be the proper transform of ℓ_i in X_1. By Proposition 2.2.2,

$$\mathcal{N}_{\bar{\ell}_i/X_1} \cong \mathcal{O}_{\bar{\ell}_i}(1 - n_i) \oplus \mathcal{O}_{\bar{\ell}_i}(1 - n_i).$$

We have

$$s(\bar{\ell}_i, X_1) = c(\mathcal{N}_{\bar{\ell}_i/X_1})^{-1} = (1 + (2 - 2n_i)[\text{point}])^{-1}.$$

Applying (2.10), we obtain

$$\begin{aligned} E_i^2 &= -h_i, \\ E_i^3 &= \int_{X_1} s(\bar{\ell}_i, X_1) = 2n_i - 2, \end{aligned}$$

where $E_i = \nu^{-1}(\bar{\ell}_i)$ and $h_i = c_1(\mathcal{O}_{E_i}(1)) \in A_1(E_i)$. The proper transform $\bar{E}(p_j)$ of $E(p_j)$ in X is isomorphic to the blow-up of $\mathbb{P}^2$ at a_j points, the intersection point of $E(p_j)$ with $\bar{\ell}_i$. The class in $A_*(X)$ of the corresponding exceptional curve is equal to the class f_j of a fibre of the projection $E_i \to \bar{\ell}_i$. The proper transform $\bar{E}(p_j)$ is equal to the full transform $\nu^*(E(p_j))$. We have

$$\bar{E}(p_j)^2 = -e(p_j), \quad \bar{E}(p_j)^3 = E(p_j)^3 = 1,$$

where $e(p_j)$ is the class of a line in $E(p_j)$ which we identify with the class in $A_*(X)$. We have

$$\bar{E}(p_j)^2 \cdot E_i = -e(p_j) \cdot E_i = 0, \quad E_i^2 \cdot \bar{E}(p_j) = -h_i \cdot \bar{E}(p_j) = -1.$$

Here we used that $\bar{E}(p_j)$ is isomorphic to the blow-up of a_j points. It intersects E_i with $p_j \in \ell_i$ at one of the exceptional curves which is a fibre of the projection $E_i \to \bar{\ell}_i$. Now, consider the divisor

$$D = \sum_{i=1}^{k} \alpha_i E_i + \sum_{j=1}^{N} \beta_j \bar{E}(p_j).$$

Collecting all together, we get

$$\begin{aligned} D^3 &= \sum_{i=1}^{k} \alpha_j^3 E_i^3 + \sum_{j=1}^{N} \beta_j^3 \bar{E}(p_j)^3 + 3 \sum_{i=1}^{k} \sum_{j:p_j \in \ell_i} \alpha_i^2 \beta_j E_i^2 \cdot \bar{E}(p_j) \qquad (2.15) \\ &= \sum_{i=1}^{k} 2\alpha_j^3 (n_i - 1) + \sum_{j=1}^{N} \beta_j^3 - 3 \sum_{i=1}^{k} \sum_{j:p_j \in \ell_i} \alpha_i^2 \beta_j. \end{aligned}$$

Let $H \in A_*(X)$ be the class of the pre-image of a hyperplane in $\mathbb{P}^3$. Suppose the linear system $|rH - D|$ has no fixed components and defines a regular map $f : X \dashrightarrow \mathbb{P}^m$. Then $(rH - D)^n$ is equal to the product of the degree of the image of X and the degree of the map. So, if $(rH - D)^3 = 1$, we obtain that the map f is birational onto $\mathbb{P}^3$. Then the composition $f \circ \pi^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is a Cremona transformation.

Taking all of this into account, we obtain

$$(rH - D)^3 = r^3 - D^3 + 3rHD^2 = r^3 - D^3 - 3r \sum_{i=1}^{k} \alpha_i \qquad (2.16)$$

$$= r^3 - 3r \sum_{i=1}^{k} \alpha_i - \sum_{i=1}^{k} 2\alpha_j^3 (n_i - 1) - \sum_{j=1}^{N} \beta_j^3 + 3 \sum_{i=1}^{k} \sum_{j:p_j \in \ell_i} \alpha_i^2 \beta_j.$$

Unfortunately, in many cases, the linear system $|dH - D|$ is not base-point free. So, one needs to blow-up more. The situation is analogous to the planar case, where we have to blow-up infinitely near points (see the next section). For example, suppose $\sigma : X' \to X$ is the blow-up a closed point $x \in X$ and then we want to blow-up a k-codimensional linear subspace L in the exceptional divisor $E = \sigma^{-1}(x) \cong \mathbb{P}^{n-1}$. We use formula (2.12) to obtain

$$c(\mathcal{N}_{L/X'} = c(\mathcal{N}_{L/E}) \cdot j^*(c(\mathcal{N}_{E/X'})),$$

where $j : L \hookrightarrow E$ is the inclusion map. Since $\mathcal{N}_{E/X'} \cong \mathcal{O}_E(-1)$ and $\mathcal{N}_{L/E} = \mathcal{O}_L(1)^{\oplus k}$, we obtain

$$c(\mathcal{N}_{L/X'}) = c(\mathcal{O}_L(1)^{\oplus k}) \cdot c(\mathcal{O}_L(-1)) = (1+h)^k(1-h),$$

where h is the class of a hyperplane in L. This gives

$$s(L, X') = (1+h)^{-k}(1-h)^{-1} = (\sum_{k=0}^{n-1-k} h^k)(\sum_{m=k-1}^{n-2} \tbinom{m}{k-1}(-h)^{m-k+1}).$$

This allows us to compute F^n, where F is the exceptional divisor of the blow-up of L. For example, let $\dim X = 3$ and L be a line in E. We get $s(L, X') = (1 - h^2)^{-1} = 1$. Thus $F^3 = 0$. Exact sequence (2.12) gives $\mathcal{N}_{L/X'} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)$ (it is easy to see that it splits). Thus F is isomorphic to $\mathbb{P}(\mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)) \cong \mathbb{P}(\mathcal{O}_L \oplus \mathcal{O}_L(-2)) \cong \mathbf{F}_2$. We have $F^2 = -(f+e)$, where f is the class of a fibre and e is the class of the exceptional section. Using (2.14), we obtain $\bar{E}^2 = E^2 - \bar{E} \cdot F = -2[\ell]$, where $[\ell]$ is the class of a line in $\bar{E}$. Since $F \cdot [\ell] = 1$, and $F^3 = -F \cdot (f+s) = 1 - F \cdot s = 0$ implies that $F \cap \bar{E} = [s]$ (considered as a cycle on F). Thus the two exceptional divisors F and $\bar{E}$ intersect along the exceptional section of $\mathbf{F}_2$. Now, we get $\bar{E}^3 = E^3 + \bar{E} \cdot F^2 = 1 + s \cdot F = 1 + 1 = 2$.

To compute $(rH - D)^3$, we use that $H^2 D = 0$ (because a general line does not intersect any line ℓ_i). Also, we use that $H \cdot D^2 = -\sum_{i=1}^k \alpha_i$ (because $E_i^2 = -h_i$ and $H \cdot h_i = 1$).

Let us consider some special cases. For example, take $k = 2, N = 1, \alpha_1 = \alpha_2 = \beta_1 = 1$. We get

$$(2H - E)^3 = 8H^3 - E^3 + 12H^2 \cdot E + 6H \cdot E^2 = 8 + 5 - 12 = 1.$$

The linear system $|2H - E|$ has base locus equal to the line ℓ in $\bar{E}(p)$ corresponding to the plane spanned by ℓ_1 and ℓ_2. Using exact sequence (2.12), one checks that the normal bundle of ℓ is isomorphic to $\mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(-1)$. The exceptional divisor of the blow-up is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. One can perform a flop along ℓ to obtain anew 3-fold X' on which our linear system has no based points and defines a regular birational morphism to $\mathbb{P}^3$. To perform the flop, we blow-up ℓ, and then blow-down the exceptional divisor along the other ruling. We will discuss flops later.

We will see that this leads to a quadro-quadratic transformation which we will consider later.

Another example, take ℓ_i to be the six edges of the coordinate tetrahedral $V(t_0t_1t_2t_3)$. Take

$$D = \sum_{i=1}^{6} E_i + 2\sum_{j=1}^{4} \bar{E}(p_j).$$

We get $D^3 = 12 + 32 - 72 = -28$. This gives $(3H - E)^3 = 27 + 28 - 54 = 1$. Thus the linear system $|3H - E|$ defines a birational map $f : X \dashrightarrow \mathbb{P}^3$. It gives an example of a cubo-cubic transformation which we will consider later.

Definition 2.2.1. *A configuration of lines in* $\mathbb{P}^3$ *is called a* homaloidal configuration *if there exists numbers* (r, α_i, β_j) *such that* $(dH - D)^3 = 1$ *and the linear system* $|dH - D|$ *is basepoint-free and dimension* n.

Problem 2.2.1. Find all homaloidal configurations of lines in $\mathbb{P}^3$.

2.3 Computation of the multi-degree

For all rational smooth varieties X over $\mathbb{C}$ we will be dealing with, $A_*(X)$ can be identified with a subgroup of $H_*(X, \mathbb{Z})$ by assigning to any irreducible subvariety V, its fundamental cycle $[V]$, or its dual class in $H^*(X, \mathbb{Z})$ in cohomology if X is smooth. The operations σ_* and σ^* coincide with the corresponding definitions in topology. So, in the definition of the multi-degree of f, we may replace H^* with A_*.

Let us apply the intersection theory to compute the multi-degree of a Cremona transformation.

Let (X, π, σ) be a resolution of a Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Consider the map $\nu = (\pi, \sigma) : X \to \mathbb{P}^n \times \mathbb{P}^n$. We have $\nu_*[X] = [\Gamma_f]$, and, by the projection formula,

$$\nu^*(h_1^k h_2^{n-k}) \cap [X] = [\Gamma_f] \cdot (h_1^k h_2^{n-k}) = d_k.$$

Let $s(Z, \mathbb{P}^n) \in A_*(Z)$ be the Segre class of a closed subscheme of $\mathbb{P}^n$. We write its image in $A_*(\mathbb{P}^n)$ under the canonical map $i_* : A_*(Z) \to A_*(\mathbb{P}^n)$ in the form $\sum s(Z, \mathbb{P}^n)_m h^{n-m}$, where h is the class of a hyperplane.

Proposition 2.3.1. *Let $(d_0, d_1, \dots, d_n)$ be the multi-degree of a Cremona transformation. Let (X, π, σ) be its resolution and Z be the closed subscheme of $\mathbb{P}^n$ such that $\pi : X \to \mathbb{P}^n$ is the blow-up of a closed subscheme Z of $\mathbb{P}^n$. Then*

$$d_k = d^k - \sum_{i=1}^{k} d^{k-i}\binom{k}{i} s(Z, \mathbb{P}^n)_{n-i}.$$

Proof. We know that $\sigma^*\mathcal{O}_{\mathbb{P}^n}(1) = \pi^*\mathcal{O}_{\mathbb{P}^n}(d)(-E) = \mathcal{O}_{X'}(dH - E)$, where $\mathcal{O}_X(H) = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ and E is the exceptional divisor of π. We have $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ for each copy of $\mathbb{P}^n$. Thus

$$d_k = \pi_*(dH - E)^k \cdot h^{n-k} = \sum_{i=0}^{k}((-1)^i d^{k-i}\binom{k}{i}\pi_*(H^{k-i} \cdot E^i)) \cdot h^{n-k}$$

$$= \sum_{i=0}^{k}(-1)^i d^{k-i}\binom{k}{i} h^{k-i} \cdot \pi_*(E^i) \cdot h^{n-k} = \sum_{i=0}^{k}(-1)^i d^{k-i}\binom{k}{i} \cdot \pi_*(E^i) \cdot h^{n-i}$$

$$= d^k + \sum_{i=1}^{k}(-1)^i d^{k-i}\binom{k}{i} \cdot \pi_*(E^i) \cdot h^{n-i} = d^k - \sum_{i=1}^{k} d^{k-i}\binom{k}{i} s(Z, \mathbb{P}^n)_{n-i}.$$

□

Note that one can invert the formulas to express the Segre classes

$$s(Z, \mathbb{P}^n)_k = s_k h^{n-k}$$

in terms of d_k's.

$$s_k = (-1)^{n-k-1} \sum_{0 \le i \le n-k} (-1)^i \binom{n-k}{i} d^{n-k-i} d_i. \tag{2.17}$$

Definition 2.3.1. *A closed subscheme Z in $\mathbb{P}^n$ is called* homaloidal *if there exists a homaloidal linear system $\mathcal{H}_X$ with $\overline{\mathrm{Bs}(\mathcal{H}_X)} = Z$.*

Observe that Proposition 2.3.1 gives a necessary condition for the homaloidal linear system

$$s(Z, \mathbb{P}^n)_0 \equiv 1 \mod d. \tag{2.18}$$

2.4 Homaloidal linear systems in the plane

Let $\mathfrak{b}$ be the base ideal of a homaloidal linear system in the plane. Since $|\mathfrak{b}(d)| = |\overline{\mathfrak{b}}(d)|$, we may assume that it is integrally closed. It is known that in this case the blow-up scheme $\mathrm{Bl}(\mathfrak{b})$ is a normal surface. Let $X \to \mathrm{Bl}(\mathfrak{b})$ be its minimal resolution of singularities and $\sigma : X \to \mathbb{P}^2$ be the composition $X \to \mathrm{Bl}(\mathfrak{b}) \to \mathbb{P}^2$. A birational morphism of nonsingular surfaces is a composition of blow-ups of closed points on smooth surfaces (see [Hartshorne]). Let

$$X = X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbb{P}^2$$

be the sequence of such blow-ups. Here each $\sigma_i : X_i \to X_{i-1}$ is the blow-up of a closed point $x_i \in X_{i-1}$. For any $N \geq b > a \geq 1$, let

$$\sigma_{b,a} = \sigma_a \circ \ldots \circ \sigma_b : X_b \to X_{a-1}.$$

Let $E_i = \sigma_i^{-1}(x_i)$ be the exceptional curve of σ_i. It is a smooth rational curve on X_i with self-intersection equal to -1. We say that a point x_j is *infinitely near* to x_i of order 1 and write $x_j \succ x_i$ if $j > i$ and $\sigma_{j,i+1} : X_j \to X_i$ is an isomorphism in a neighborhood of x_j and maps x_j to a point in E_i. The *Enriques diagram* is an oriented graph whose vertices are point $x_1, \ldots, x_N$ and the arrow goes from x_j to x_i if $x_j \succ x_i$. If a point x_j is connected to x_i by an oriented path of length k, we say that x_j is *infinitely near to* x_i *of order* k and write $x_j \succ_k x_i$. The points from which no edge exits can be identified with points in $\mathbb{P}^2$, they are called *proper points*. However, strictly speaking, only one point lies in $\mathbb{P}^2$, namely the point x_1. Any non-proper point $x_i \in X_{i-1}$ is mapped by $\sigma_{i,1}$ to some proper point.

Let

$$\mathcal{E}_i = \sigma_{N,i}^{-1}(x_i) = \sigma_{N,i-1}^{-1}(E_i), \quad i = 1, \ldots, N.$$

Since we are blowing up points on smooth surfaces, the scheme-theoretical pre-image is a reduced curve. It is reducible if and only if there are no points infinitely near to it. The irreducible components of $\mathcal{E}_i$ consist of proper transforms F_i^j in X of E_i and E_j, where x_j is infinitely near to x_i of some order. We have $\mathcal{E}_j \subset \mathcal{E}_i$ if and only if $x_i = x_j$ or x_j is infinitely near to x_i. The self-intersection of F_i^j is equal to $-1 - a_i$, where a_i is the number of points infinitely near to x_i of order 1.

Since $E_i^2 = -1$, we have $\mathcal{E}_i^2 = E_i^2 = -1$. Also $\mathcal{E}_i \cdot \mathcal{E}_j = 0$ if $i \neq j$. This is because we can always replace the divisor $\mathcal{E}_i$ by a linearly equivalent divisor which does not map to x_j.

The divisor classes $e_0 = \sigma^*(\text{line}), e_i = [\mathcal{E}_i], i = 1, \ldots, N$, form a basis in $A_1(X) = \mathrm{Pic}(X) = H^2(X, \mathbb{Z})$. It is an orthonormal basis in the sense that $e_0^2 = 1, e_i^2 = -1, i > 0, e_i \cdot e_j = 0, i \neq j$.

Let $\mathcal{H} = \mathcal{H}_0 = |\mathfrak{b}(d)|$ be a homaloidal linear system of degree d on $\mathbb{P}^2$. Let m_1 be the multiplicity of a general member D at x_1. The the pre-image of $\mathcal{H}_0$ in X_1 has the fixed component m_1E_1. We subtract it and consider the linear system $\mathcal{H}_1 = |\sigma_1^*(\mathcal{H}_0) - m_1E_1|$. Let m_2 be the multiplicity of a general member of $\mathcal{H}_1$ at $x_2 \in X_1$. Then $|\sigma_2^*(\mathcal{H}_1) - m_2E_2|$ has no fixed components on X_2. Continuing in this way, we find that

$$\mathcal{H}_N = \sigma^*(\mathcal{H}) - \sum_{i=1}^{N} m_i\mathcal{E}_i$$

has no base points and no fixed components. Since it defines a birational morphism to $\mathbb{P}^2$ we must have

$$\sigma^*(\mathcal{H}) - \sum_{i=1}^{N} m_i\mathcal{E}_i = d^2 - \sum m_i^2 = 1. \tag{2.19}$$

Recall that the behavior of the canonical class of a smooth variety X under the blow-up $\sigma : \mathrm{Bl}_Z X \to X$ of a smooth closed subvariety $Z \subset X$ of codimension k

$$K_{\mathrm{Bl}_Z X} == \sigma^*(K_X) + (k-1)E, \tag{2.20}$$

where E is the exceptional divisor. In our case we obtain

$$K_X = -3H + \sum \mathcal{E}_i.$$

Since a general member of a homaloidal linear system is a rational variety, a general member of the linear system $\mathcal{H}_N$ is a smooth rational curve C. We have $C^2 = 1$, and, by the adjunction formula $C^2 + C \cdot K_X = -2$,

$$-3 = C \cdot K_X = -3d + \sum m_i. \tag{2.21}$$

The two equalities (2.19) and (2.21) give necessary conditions for the existence of a homaloidal linear system of plane curves of degree d passing though the points $x_1, \ldots, x_N$ (including infinitely near) with multiplicities $m_1, \ldots, m_N$. They also sufficient, if we check, that the dimension of such linear system is equal to 2[3] and the linear system $\mathcal{H}$ does not fixed components.

Example 2.4.1. Assume that $m_1 = \ldots = m_N = m$. We have

$$d^2 - Nm^2 = 1, \quad 3d - Nm = 3.$$

[3]This is the expected number since passing through a point with multiplicity m gives $(m+1)m/2$ conditions so that the two equalities give $(d+2)(d+1)/2 - 1 - \sum m_i(m_i+1)/2 = 2$.

The only solutions are

$$(d, m, N) = (1, 0, 0), \quad (2, 1, 3), \quad (5, 2, 5), \quad (8, 3, 7), \quad (17, 6, 8).$$

The first transformation is, of course, a projective transformation. The second one is a quadratic transformation defined by the linear system of conics through 3 points. They could be infinitely near, but not collinear (this extends even to infinitely near points meaning that $|H - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3| = \emptyset$. The third linear system is defined by the linear system of 6-nodal plane quintics. If we assume that the base points are proper and in general position, then the blow-up of the six points is isomorphic to a nonsingular cubic surface in $\mathbb{P}^3$ (by the map defined by the linear system of cubics through the six points). The exceptional curves of the blow-up are mapped to six skew lines on the cubic surface. The six skew lines which form a double-six with the set of the exceptional curves are blown down to six points in $\mathbb{P}^2$ by the linear system $|3H - \sum \mathcal{E}_i|$. The P-locus of the Cremona transformation consist of the union of 6 conics passing though the six points except one.

The last two transformations are more elaborate, they are called the *Geiser transformation* and the *Bertini transformation*, respectively.

Remark 2.4.1. The ideal sheaf $\sigma_*(\mathcal{O}_X(-\sum m_i\mathcal{E}_i))$ is an integrally closed ideal sheaf in the plane. According to Zariski, any integrally closed ideal in the plane is obtained in this way. For example, consider the ideal sheaf of the subscheme with support at a point p and given in local coordinates by (x, y^n). Its blow-up is the scheme isomorphic over an affine neighborhood of p with coordinates x, y to $X = \text{Proj } \mathbb{C}[x, y][u, v]/(uy^n - vx) \subset \mathbb{A}^2 \times \mathbb{P}^1$. In a local chart with $v \neq 0$, it is nonsingular. In the local chart $u \neq 0$, it has a singular point locally isomorphic to a singular point $(0, 0, 0)$ on the affine surface $vx - y^n = 0$. The type of this singularity goes under the name A_{n-1}-singularity. The exceptional divisor of its minimal resolution $X' \to X$ consists of a chain of $n - 1$ smooth rational curves with self-intersection -2. The exceptional divisor of the composition $X' \to X \to \mathbb{P}^2$ consists of a chain of n smooth rational curves, with a (-1)-curve at one end, and all other curves are (-2)-curves. These curves form an exceptional configuration $\mathcal{E}_1$ obtained by a sequence of blowing ups of infinitely near points $x_n \succ x_{n-1} \succ \ldots \succ x_1 = x$.

2.5 Smooth homaloidal linear systems

Definition 2.5.1. *A homaloidal linear system $\mathcal{H}$ is called* smooth *if the reduced base scheme is smooth and the base ideal $\mathfrak{b} = \mathfrak{b}(\mathcal{H})$ is a reduction of the product of the powers of the radicals of the primary components of $\mathfrak{b}$.*

If $\mathfrak{b}_i$ are the primary components of $\mathfrak{b}$ and $\mathfrak{b}_i$ is a reduction of $(\sqrt{\mathfrak{b}_i})^{r_i}$ for some $r_i \geq 1$, we write

$$\mathcal{H} = |dH - \sum_{i=1}^{k} r_i Z_i|$$

where H is meant to be the divisor class of a hyperplane and $Z_i = V(\sqrt{\mathfrak{b}_i})$.

Since each Z_i is smooth, the ideals $\sqrt{\mathfrak{b}_i}$ and their powers are integrally closed. Also the blow-up scheme X of the product of the r_i-th powers of $\sqrt{\mathfrak{b}_i}$ is smooth with exceptional divisor equal to $\sum r_i E_i$. Each divisor $E_i \cong \mathbb{P}(\mathcal{N}_{Z_i/\mathbb{P}^n})$. Since $\mathfrak{b}$ becomes invertible on X, X is a smooth resolution of any Cremona transformation f defined by the homaloidal linear system.

Note that the ideal $\mathfrak{b}$ is not necessary integrally closed, so the blow-up of $\mathfrak{b}$ may be non-normal. For example, a primary component $\mathfrak{b}_i$ may look like (x^2, y^2) with integral closure equal to (x^2, y^2, xy).

Let $|dH - r_1Z_1 - \ldots - r_NZ_N|$ be a smooth homaloidal linear system. Let $E = \sum r_iE_i$ be the exceptional divisor of the integral closure of $\mathfrak{b}(\mathcal{H}_X)$. We have

$$s(Z, \mathbb{P}^n)_k = (-1)^{n-k-1}\pi_*(E^{n-k}) = \sum_{i=1}^{N}(-r_i)^{n-k-1}\pi_*(E_i^{n-k}) = \sum_{i=1}^{N} r_i^{n-k} s(Z_i, \mathbb{P}^n)_k. \tag{2.22}$$

We have already explained how to compute $s(Y, \mathbb{P}^n)$ for any smooth subvariety Y of $\mathbb{P}^n$. For example, if C is a curve of degree d in $\mathbb{P}^n$, we get

$$s(C, \mathbb{P}^n) = c(\mathcal{N}_{C/\mathbb{P}^n})^{-1} = 1 - c_1(\mathcal{N}_{C/\mathbb{P}^n}).$$

The exact sequence of sheaves of differentials

$$0 \to \mathcal{N}_{C/\mathbb{P}^n}^{\vee} \to j^*\Omega_{\mathbb{P}^n} \to \Omega_C^1 \to 0$$

gives an isomorphism

$$j^* \bigwedge^{n} \Omega_{\mathbb{P}^n} \cong \bigwedge^{n-1} \mathcal{N}_{C/\mathbb{P}^n}^{\vee} \otimes \Omega_C^1.$$

This yields

$$c_1(\mathcal{N}_{C/\mathbb{P}^n}) = -K_{\mathbb{P}^n} \cdot C + 2g - 2 = (n+1)\deg C + 2g - 2,$$

where g is the genus of C. Hence

$$s(C, \mathbb{P}^n)_k = \begin{cases} 2 - 2g - (n+1)\deg C & \text{if } k = 0, \\ 1 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So, if we assume that $\mathrm{Bs}(\mathcal{H}_X)_{\mathrm{red}}$ consists of M curves $C_1, \ldots, C_M$ and N points $x_1, \ldots, x_N$, formula (3.3) together with Proposition 2.3.1 give

Proposition 2.5.1. *Assume that* $\mathcal{H}_X = |dH - \sum_{i=1}^M r_i C_i - \sum_{i=1}^N m_i x_i|$, *where* $C_i, i = 1, \ldots, M$, *is a smooth curve of degree* $\deg C_i$ *and genus* g_i, *and* x_i *are isolated points. Then*

$$\begin{aligned}
d_n &= d^n + \sum_{i=1}^{M}[(r_i^n(n+1) - dnr_i^{n-1})\deg C_i + r_i^n(2g_i - 2)] - \sum_{i=1}^{N} m_i^n = 1\\
d_{n-1} &= d^{n-1} - \sum_{i=1}^{M} \deg C_i,\\
d_k &= d^k,\ k = 0, \ldots, n-2.
\end{aligned}$$

To find the dimension of a smooth homaloidal system we use some known techniques. Let $\mathcal{J}$ be the ideal sheaf of a smooth closed subscheme Z of $\mathbb{P}^n$. We have a sequence of ideals

$$\mathcal{J}^m \subset \mathcal{J}^{m-1} \subset \ldots \subset \mathcal{J}$$

with quotients $\mathcal{J}^k/\mathcal{J}^{k+1}$ isomorphic to the k-symmetric powers of $\mathcal{N}^\vee_{Y/\mathbb{P}^n} = \mathcal{J}/\mathcal{J}^2$. We also use the exact sequence

$$0 \to (\mathcal{J}/\mathcal{J}^m)(d) \to \mathcal{O}_{mZ}(d) \to \mathcal{O}_Z(d) \to 0,$$

where $mZ = V(\mathcal{J}_Z^m)$. The exact sequence gives

$$h^0(\mathcal{O}_{mZ}(d)) = \mathcal{O}_Z(d) + h^0(\mathcal{J}/\mathcal{J}^m(d)),$$

provided we have checked that $h^1(\mathcal{J}/\mathcal{J}^m(d)) = 0$. Suppose we know $c(\mathcal{N}_{Y/\mathbb{P}^n})$. Then, one can compute the Chern class of the symmetric powers of $\mathcal{N}^\vee_{Z/\mathbb{P}^n}$, also of its twists $\mathcal{N}^\vee_{Z/\mathbb{P}^n}(d)$, and then apply the Riemann-Roch Theorem to compute $\mathcal{O}_Z(d)$ and $h^0((\mathcal{J}/\mathcal{J}^m)(d))$.

Example 2.5.1. Assume that Y is a smooth curve C of genus g and degree $\deg C$. We have $c_1(S^k(\mathcal{N}^\vee_{C/X})) = \frac{1}{2}k(k+1)c_1(\mathcal{N}^\vee_{C/\mathbb{P}^n})$ and by, the additivity of c_1, we know

$$c_1(\mathcal{J}/\mathcal{J}^m) = \tfrac{1}{2}\sum_{k=1}^{m-1} k(k+1)c_1(\mathcal{N}_{C/\mathbb{P}^n}) = \binom{m+1}{3}c_1(\mathcal{N}_{C/\mathbb{P}^n}).$$

Next, we use that, for any locally free sheaf $\mathcal{E}$ of rank r, and an invertible sheaf $\mathcal{L}$, we have

$$c(\mathcal{E} \otimes \mathcal{L}) = \sum_{i=0}^{r} c_{r-i}(\mathcal{E})(1 + c_1(\mathcal{L}))^i. \tag{2.23}$$

Finally, we use Riemann-Roch Theorem which allows us to compute the Euler characteristic $\chi(\mathbb{P}^n, \mathcal{J}/\mathcal{J}^m(d))$.

Since the rank of $\mathcal{J}/\mathcal{J}^m$ is equal to $2 + \ldots + m = \frac{1}{2}(m-1)(m+2)$, we obtain

$$c_1(\mathcal{J}/\mathcal{J}^m(d)) = \binom{m+1}{3}c_1(\mathcal{N}^\vee_{C/\mathbb{P}^3}) + \tfrac{1}{2}(m-1)(m+2)d\deg C.$$

By Riemann-Roch, for any locally free sheaf $\mathcal{E}$ of rank r,

$$\chi(\mathcal{E}) = \deg c_1(\mathcal{E}) + r(1-g).$$

This gives

$$\chi(\mathcal{J}/\mathcal{J}^m(d)) = \binom{m+1}{3}c_1(\mathcal{N}_{C/\mathbb{P}^3}) + \tfrac{1}{2}(m-1)(m+2)d\deg C + \tfrac{1}{2}(m-1)(m+2)(1-g).$$

$$\begin{aligned}\chi(\mathcal{O}_{mC}(d)) &= \chi(\mathcal{J}/\mathcal{J}^m(d)) + \chi(\mathcal{O}_C(d)) = \chi(\mathcal{J}/\mathcal{J}^m(d)) + d\deg C + 1 - g\\ &= \binom{m+1}{3}(-4\deg C + 2 - 2g) + \binom{m+1}{2}(d\deg C + 1 - g)\\ &= \left(-4\binom{m+1}{3} + d\binom{m+1}{2}\right)\deg C + \left(2\binom{m+1}{3} + \binom{m+1}{2}\right)(1-g).\end{aligned}$$

Applying (2.5.1), we have

$$d^3 - 1 = (-4m^3 + 3dm^2)\deg C + 2m^3(1-g). \tag{2.24}$$

Observe that this implies that d is an odd number. Taking (2.24) into account, we can rewrite the previous equality in the form

$$\chi(\mathcal{O}_{mC}(d)) = \frac{1}{6}(d^3 - 1 + m(4+3d)\deg C + m(3m+1)(1-g)).$$

Finally, under assumption, that $\chi(\mathcal{O}_C(d)) = h^0(\mathcal{O}_C(d))$ and $\chi(\mathcal{J}/\mathcal{J}^m(d)) = h^0(\mathcal{J}/\mathcal{J}^m(d))$, we get

$$4 = h^0(\mathcal{J}^m(d)) = \binom{d+3}{3} - \chi(\mathcal{O}_{mC}(d))$$

$$= \frac{1}{6}(6d^2 + 11d + 7 - m(4+3d)\deg C + m(3m+1)(g-1)). \tag{2.25}$$

The equalities (2.24) and (2.25) give necessary conditions for the existence of a homaloidal linear system $|dH - mC|$.

Remark 2.5.1. When $n > 2$, there are no smooth homaloidal linear systems with 0-dimensional base locus. Indeed, suppose such a linear system $|dH - \sum r_i x_i|$ exists. We know that the multi-degree is equal to $(d, d^2, \ldots, d^n)$. Let S be the pre-image of a general plane. This is a surface of degree d^{n-2}. The restriction of the linear system to S is of the form $|dh - \sum r_i x_i|$, where h is a hyperplane section of S. After we blow-up the base points, we get a degree 1 map onto a plane. This implies that $1 = d^2h^2 - \sum m_i^2 = d^n - \sum m_i^2$. On the other hand, we have $1 = d^n - \sum m_i^n$. This gives $\sum r_i^n = \sum m_i^2$, hence all m_i are equal to 1, and the base scheme is reduced and consists of $N = d^n - 1$ points. We know that the pre-image of a general line is an irreducible curve R of degree $d_{n-1} = d^{n-1}$. The linear subsystem of $\mathcal{H}_X$ that consists of divisors passing through R is of dimension $n-2$ (it is isomorphic to the linear system of hyperplanes through a line in the target space). Take two general points on R, then we find a divisor from $\mathcal{H}_X$ that passes through these two points and also $d^n - 1$ base points. Thus it intersects R at $d^n + 1 > d^n$ points contradicting the Bezout Theorem.

Example 2.5.2. Let $n = 3$. We have

$$1 = d^3 + \sum_{i=1}^{M}[(4r_i^3 - 3dr_i^2)\deg C_i + r_i^3(2g_i - 2)] - \sum_{i=1}^{N} m_i^3. \tag{2.26}$$

Take $d = 2$. We have

$$1 = 8 + \sum_{i=1}^{M}[(4r_i^3 - 6r_i^2)\deg C_i + r_i^3(2g_i - 2)] - \sum_{i=1}^{N} m_i^3. \tag{2.27}$$

This leaves us with a few possibilities. By the previous remark, $M \neq 0$. Since each quadric from the linear system passes through C_i with multiplicity r_i, and any the intersection of two quadrics is a connected curve of degree 4, we must have $r_i \leq 2$ and

$$\sum r_i \deg C_i < 4.$$

If $\deg C_1 = 3$, then $M = 1, r_1 = 1, g_1 = 0$, and (2.27) gives $1 = 8 - 6 - 2 - 2\deg C_2 - \sum_{i=1}^N m_i^3$, a contradiction.

If $\deg C_1 = 2$, then $r_1 = 1, g_1 = 0$. If $M = 2$, then $r_2 = 1, \deg C_2 = 1, g_2 = 0$, and we obtain $1 = 8 - 4 - 2 - 2 - \sum_{i=1}^N m_i^3$, a contradiction. If $M = 1$, we get $1 = 8 - 4 - 2 - \sum_{i=1}^N m_i^3$. This gives $N = 1, m_1 = 1$. This case is realized by the linear system of quadrics through a conic and a point lying outside of the plane spanning the conic. The multi-degree is $(2, 2)$. We will discuss these and more general quadro-quadratic transformations of this kind in the next lecture.

If $\deg C_1 = 1$, then $r_1 \leq 2, g_1 = 0$. If $r_1 = 2$, then the quadrics must be of the form $V(aL_1^2 + bL_1L_2 + cL_2^2)$, where C_1 is the intersection of hyperplanes $V(L_1)$ and $V(L_2)$. The dimension of such system is equal to 2, so it is not homaloidal. If $M = 2$, for the same reason, $r_2 = 1$and $\deg C_2 = 1$. We get $1 = 8 - 2 - 2 - 2 - \sum_{i=1}^N m_i^3$, a contradiction. So, the remaining possibility is $M = 1$, and

$N = 3, m_1 = m_2 = m_3 = 1$. This case is realized and leads to a Cremona transformation of multi-degree $(2, 3)$.

Example 2.5.3. It is possible that $\mathrm{Bs}(T)$ is 0-dimensional but the homaloidal linear system is not smooth. There are hidden 1-dimensional infinitely near components. For example, consider the linear system of quadrics in $\mathbb{P}^3$ which pass through 3 non-collinear points and have the same tangent plane at the fourth point. Choosing coordinates, we may assume that the points are $p_1 = [1, 0, 0, 0], p_2 = [0, 1, 0, 0], p_3 = [0, 0, 1, 0], p_4 = [0, 0, 0, 1]$, and the tangent plane at the point p_4 has equation $t_0 + t_1 + t_2 = 0$. After we blow-up the first three points, we obtain that the inverse image of the linear system has the base locus equal to a line in the exceptional divisor E_4 over the point p_4. If we blow-up this line, we resolve the birational map. The exceptional divisor consists of E_1, E_2, E_3, E_4', E_5, where E_4' is the proper transform of E_4 isomorphic to $\mathbb{P}^2$ and and E_5 is the exceptional divisor over the line isomorphic to the minimal ruled surface $\mathbf{F}_2$. The divisors E_4' and E_5 intersect along a curve C which is the exceptional section in E_5 and a line in $\mathbb{P}^2$.

The base ideal in a neighborhood of the point p_4 is isomorphic to the ideal $(xy, yz, xz, x + y + z)$. After we make the change of variables $x \to x + y + z$, it becomes isomorphic to the ideal (x, y^2, yz, z^2). It is easy to see that the blow-up scheme is isomorphic to the projective cone over the blow-up of the maximal ideal (y, z). It has a singular point locally isomorphic to the cone over the Veronese surface. Its exceptional divisor is isomorphic to the quadratic cone. The birational morphism from the resolution in above to the blow-up of the base scheme is the contraction of the divisor E_4' to the singular point of the blow-up.

So the homaloidal linear system can be written in the form $|2H - p_1 - p_2 - p_3 - Z_4|$, where $p_4 = (Z_4)_{\mathrm{red}}$, and Z_4 is locally given by a primary ideal (x, y^2, yz, y^2).

2.6 Special Cremona transformations

A Cremona transformation with smooth connected base scheme is called *special Cremona transformation*. There are no such transformations in the plane and they are rather rare in higher-dimensional spaces and maybe classifiable. We start from one-dimensional base schemes. It follows from our formulas that the following two examples work:

(i) $n = 3, g = 3, \deg C = 6, d = 3$;

(ii) $n = 4, g = 1, \deg C = 5, d = 2$.

The first example is a bilinear (or a cubo-cubic transformation) which we will study in detail later. The second transformation is given by quadrics. The restriction of the homaloidal linear system to a general hyperplane is the linear system of quadrics through 5 points. It is known (see, for example, [Topics]) that the image is a ten-nodal *Segre cubic*. So, the degree of the inverse transformation is equal to 3. The image of a general plane is a Veronese surface of degree 4. So, the multidegree of the transformation is equal to $(2,4,3)$. This is an example of a quadro-cubic transformation in $\mathbb{P}^4$ discussed in Semple-Roth's book. Note that the base locus of the inverse transformation f^{-1} is a surface of degree 5, an elliptic scroll. In fact, the image of a general plane under f is of degree 4 but given by cubics. This implies that the plane meets the base scheme at 5 simple points.

A result of B. Crauder and S. Katz shows that the two cases are the only possible cases with a smooth connected one-dimensional locus.

Next we assume that the dimension of the base locus is equal to 2. The next result also belongs to Crauder and Katz.

Theorem 2.6.1. *A special Cremona transformation with two-dimensional base scheme Z is one of the following:*

(i) $n=4, d=3, Z$ *is an elliptic scroll of degree 5, the base scheme of the inverse of the quadro-cubic transformation from above;*

(ii) $n=4, d=4, Z$ *is a determinantal variety of degree 10 given by* 4×4*-minors of a* 4×5*-matrix of linear forms (a bilinear transformation, see later).*

(iii) $n=5, d=2, Z$ *is a Veronese surface.*

(iv) $n=6, d=2, Z$ *is an elliptic scroll of degree 7;*

(v) $n=6, d=2, Z$ *is an octic surface, the image of a plane under a rational map given by the linear system of quartics through 8 points.*

In cases (ii) and (iii) the inverse transformation is similar, with isomorphic base scheme.

There is no classification for higher-dimensional Z. However, we have the following nice results of L. Ein and N. Shepherd-Barron [Amer. J. Math. 1989].

Recall that a *Severi variety* is a closed a subvariety $Z$of $\mathbb{P}^n$ of dimension $\frac{1}{3}(2n-4)$ such that the secant variety is a proper subvariety of $\mathbb{P}^{n+1}$. All such varieties are classified by F. Zak.

(i) Z is a Veronese surface in $\mathbb{P}^5$;

(ii) Z is the Grassmann variety $G_1(\mathbb{P}^5)$ embedded in the Plücker space $\mathbb{P}^{14}$;

(iii) Z is the Severi variety $s(\mathbb{P}^2\times\mathbb{P}^2)\subset\mathbb{P}^8$;

(iv) Z is the E_6-variety, a 16-dimensional homogeneous variety in $\mathbb{P}^{26}$.

Theorem 2.6.2. *Let f be a quadro-quadratic transformation with smooth base scheme. Then the base scheme is isomorphic to one of the Severi varieties.*

All these transformations are involutions (i.e. $T = T^{-1}$). In particular, their multi-degree vectors are symmetric.

2.7 Double structures

Let $Y \subset X$ be a smooth closed codimension r subvariety of a n-dimensional smooth varierty X and $\mathcal{N}_{Y/X}^\vee = \mathcal{J}_Y/\mathcal{J}_Y^2$ be its conormal locally free sheaf. A *double structure* on Y is a surjection $u : \mathcal{N}_{Y/X}^\vee \to \mathcal{L}$ onto an invertible sheaf $\mathcal{L}$. It defines a section $s : Y \to \mathbb{P}(\mathcal{N}_{Y/X}^\vee)$. Let $\tilde{u} : \mathcal{J}_Y \to \mathcal{L}$ be the composition of u and the canonical surjection $\mathcal{J}_Y \to \mathcal{J}_Y/\mathcal{J}_Y^2$. Its kernel is an ideal sheaf $\mathcal{I}$ that defines a closed subscheme Z containing Y as a closed subscheme and contained in the first infinitesimal neighborhood $Y(1)$ defined by the ideal sheaf $\mathcal{J}_Y^2$. Thus, by definition, we have an exact sequence:

$$0 \to \mathcal{J}_Z \to \mathcal{J}_Y \to \mathcal{L} \to 0. \tag{2.28}$$

We call the scheme Z a *double structure* on Y. In the case when X is a smooth surface, and Y is a closed point, the double structure is a choice of a point on the exceptional curve E of the blow-up of the point. It corresponds to a tangent direction at the point. It is an infinitely near point to Y.

Let E_1 be the exceptional divisor of the blow-up $\sigma_1 : B_1 = \mathrm{Bl}_Y X \to X$ and Y_1 be the image of the section $s : Y \to E_1 = \mathbb{P}(\mathcal{N}_{Y/X})$ defined by $\mathcal{L}$. The exact sequence (2.28) gives a surjection of graded algebras $\oplus \mathcal{J}_Y^k \to \mathrm{Sym}\,\mathcal{L}$. Its kernel is generated by $\sigma_1^{-1}\mathcal{J}_Z$. Recall that the affine cone C_E over E is $\mathrm{Spec}\,\mathrm{Sym}\,\mathcal{N}_{Y/X}^\vee$. The section s corresponds to the surjection $\mathrm{Sym}\,\mathcal{N}_{Y/X}^\vee \to \mathrm{Sym}\,\mathcal{L}$. This shows that the pre-image of the ideal sheaf of Y_1 in C_E is generated by $\mathrm{Ker}(u) \subset \mathcal{N}_{Y/X}^\vee$. This also shows that the ideal $\mathcal{J}_{Y_1}$ of Y_1 in E is equal to $\sigma_1^{-1}(\mathcal{J}_Z) \cdot \mathcal{O}_{E_1}$ and hence

$$\sigma_1^{-1}(\mathcal{J}_Z) = \mathcal{O}_{B_1}(-E_1) \cdot \mathcal{J}_{Y_1}. \tag{2.29}$$

The quotient $\mathcal{O}_{B_1}(-E_1)/\mathcal{O}_{B_1}(-E_1) \cdot \mathcal{J}_{Y_1}$ is isomorphic to $\mathcal{O}_{Y_1}(-E_1)$. It follows from the definition of $\mathcal{O}_E(1) = \mathcal{O}_E(-E_1)$ that $s^*(\mathcal{O}_E(-E_1)) \cong \mathcal{L}$. This shows that we have an exact sequence

$$0 \to \mathcal{O}_{B_1}(-E_1) \cdot \mathcal{J}_{Y_1} \to \mathcal{O}_{B_1}(-E_1) \to \mathcal{L} \to 0.$$

It is obtained from (2.28) by applying σ^{-1}.

Let $\sigma_2 : B_2 \to B_1$ be the blow-up of Y_1. Then the inverse transform of $\mathcal{J}_Z$ in B_2 becomes invertible and isomorphic to $\mathcal{O}_{B_2}(-\mathcal{E}_1' - E_2)$, where $\mathcal{E}_1 = \sigma_2^*(E_1)$ is the full transform of E_1 in B_2 and E_2 is the exceptional divisor of σ_2. We have

$$s(Z,X)_m = (-1)^{n-m-1}\sigma_*(E^{n-m}),$$

where $E = \mathcal{E}_1 + E_2$ and $\sigma = \sigma_1 \circ \sigma_2 : B_2 \to X$. By the projection formula,

$$\sigma_{2*}(\mathcal{E}_1 + E_2)^{n-m} = \sum_{i=0}^{n-m} \binom{n-m}{i} \sigma_{2*}(\mathcal{E}_1^i \cdot E_2^{n-m-i})$$

$$= \sum_{i=0}^{n-m} \binom{n-m}{i} E_1^i \cdot \sigma_{2*}(E_2^{n-m-i}) = \sum_{i=0}^{n-m} \binom{n-m}{i} E_1^i (-1)^{n-m-i-1} s(Y_1, B_1)_{m+i}.$$

Thus

$$s(Z,X)_m = s(Y,X)_m + \sum_{i=0}^{n-m-1} \binom{n-m}{i} (-E_1)^i s(Y_1, B_1)_{m+i}. \tag{2.30}$$

So we need to compute $s(Y_1, B_1) = c(\mathcal{N}_{Y_1/B_1})$. We use the following well-known result about projective bundles.

Lemma 2.7.1. *Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth scheme X and let $\mathbb{P}(\mathcal{E})$ be the projective bundle associated to $\mathcal{E}$. Let $u : \mathcal{E} \to \mathcal{L}$ be a surjection of $\mathcal{E}$ onto an invertible sheaf $\mathcal{L}$ and $s : X = \mathrm{Proj\ Sym}\ \mathcal{L} \to \mathbb{P}(\mathcal{E}) = \mathrm{Proj\ Sym}\ \mathcal{E}$ is the corresponding section. Then*

$$s^*(\mathcal{N}_{s(X)/\mathbb{P}(\mathcal{E})}) \cong \mathrm{Ker}(u)^\vee \otimes \mathcal{L}.$$

Proof. Let $P = \mathbb{P}(\mathcal{E})$ and $S = s(X)$. Consider the exact sequence of sheaves of relative differentials

$$0 \to \mathcal{N}_{S/P}^\vee \to \Omega^1_{P/X} \otimes \mathcal{O}_S \to \Omega^1_{S/P} \to 0.$$

Since the projection $p : P \to X$ induces an isomorphism $S \to X$, we have $\Omega^1_{S/P} = 0$. Thus

$$\mathcal{N}_{S/P}^\vee \cong \Omega^1_{P/X} \otimes \mathcal{O}_S. \tag{2.31}$$

Now consider the Euler exact sequence

$$0 \to \mathcal{O}_P \to p^*(\mathcal{E}^\vee) \otimes \mathcal{O}_P(1) \to (\Omega^1_{P/X})^\vee \to 0. \tag{2.32}$$

It follows from the definition of the sheaf $\mathcal{O}_P(1)$ that $\mathcal{L} \cong s^*(\mathcal{O}_P(1))$. Passing to the duals, and applying s^*, we obtain

$$s^*(\mathcal{N}_{S/P}^{\vee}) \cong \Omega^1_{P/X} \cong \text{Ker}(\mathcal{E} \otimes \mathcal{L}^{-1} \to \mathcal{O}_X) \cong \text{Ker}(u) \otimes \mathcal{L}^{-1}.$$

Taking the dual again, we get

$$s^*(\mathcal{N}_{S/P}) \cong \text{Ker}(u)^{\vee} \otimes \mathcal{L}. \tag{2.33}$$

□

Note that equalities (2.31) and (2.32) give

$$c(\mathcal{N}_{S/P}) = c((\Omega^1_{P/X})^{\vee}) = c(\mathcal{E}^{\vee} \otimes \mathcal{L}). \tag{2.34}$$

Applying (2.23) and (2.31), we find

$$c(\mathcal{N}_{Y_1/E}) = \sum_{i=0}^{r} c_{r-i}(\mathcal{N}_{Y/X})(1+D)^i. \tag{2.35}$$

Since $\mathcal{O}_B(1) = \mathcal{O}_B(-E)$, we get

$$\mathcal{N}_{E/B}|Y_1 = \mathcal{O}_E(E)|Y_1 = \mathcal{L}^{-1} = \mathcal{O}_Y(-D). \tag{2.36}$$

Using exact sequence (2.12), we finally obtain

$$s(Y_1, B) = (\sum_{i=0}^{r} c_i(\mathcal{N}_{Y/X})(1+D)^{r-i})^{-1}(1-D)^{-1}.$$

and, applying (2.30) and (2.36), we get

$$s(Z, X)_m = s(Y, X)_m + \sum_{j=0}^{n-m-1} \binom{n-m}{j} D^j \cdot s(Y_1, B)_{m+j}. \tag{2.37}$$

Example 2.7.1. Let Y be a closed point. We have $s(Y, X)_0 = s(Y_1, B)_0 = [Y]$, so

$$s(Z, X) = 2[Y].$$

Or let Y be a smooth curve on a n-fold X. Then

$$s(Y, X) = ([Y] + c_1(\mathcal{N}_{Y/X}))^{-1} = [Y] - c_1(\mathcal{N}_{Y/X}),$$

$$s(Y_1, B) = ([Y] + (n-1)D + c_1(\mathcal{N}_{Y/X}))^{-1}(1-D)^{-1}$$

$$= [Y] - (n-1)D - c_1(\mathcal{N}_{Y/X}))(1+D)$$
$$= [Y] - (n-2)D - c_1(\mathcal{N}_{Y/X}).$$

This gives

$$\begin{aligned} s(Z,X)_0 &= -2c_1(\mathcal{N}_{Y/X}) + 2D \\ s(Z,X)_1 &= 2[Y]. \end{aligned} \tag{2.38}$$

Let us apply this to our situation when $X = \mathbb{P}^n$ and Z is the base scheme of a homaloidal linear system. We have

$$d_k = d^k - \sum_{i=1}^{k} \binom{k}{i} d^{k-i} s(Z, \mathbb{P}^n)_{n-i}.$$

Example 2.7.2. Let Y be a curve C of genus g in $\mathbb{P}^n$. Consider a double structure on C defined by an invertible sheaf $\mathcal{O}(D)$ of degree $a = \deg D$. From the previous computations $c_1(N_{C/\mathbb{P}^n}) = 2g - 2 + (n+1)\deg C$. Applying (2.38), we get

$$s(Z,\mathbb{P}^n)_0 = 4 - 4g - 2(n+1)\deg C + 2a, \quad s(Z,\mathbb{P}^n)_1 = 2\deg Y,$$

hence

$$\begin{aligned} d_n &= d^n - (4 - 4g - 2(n+1)\deg C + 2a) - 2nd\deg C, \\ d_{n-1} &= d^{n-1} - 2\deg C, \\ d_k &= d^k, \ k < n-1. \end{aligned} \tag{2.39}$$

Consider a special case when $n = 3$ and C is a line. By taking $d = 2, a = -1$, we obtain $d_3 = 2$. Thus adding one isolated base point, we obtain a candidate for a homaloidal linear system of quadrics through the double structure of a line and an isolated point. We will see later that this corresponds to a degeneration of a quadratic transformation through a smooth conic in $\mathbb{P}^3$ and a point outside the plane spanned by the conic.

Another special case when C is a twisted cubic in $\mathbb{P}^3$ and $a = -4$. We take $d = 3$, and get $d_3 = 1$. We will see later that this is a special case of a cubo-cubic Cremona transformation.

By considering a double structure on the section Y_1 of the blow-up of Y, we can introduce a *triple structure* on Y, and continuing in this way, we can define a k*-multiple structure* on Y. It is given by a sequence of the blow-ups of smooth closed subvarieties $Y_i \subset B_{i-1}$

$$B = B_k \xrightarrow{\sigma_k} B_{k-1} \xrightarrow{\sigma_{k-1}} \cdots \xrightarrow{\sigma_2} B_1 \xrightarrow{\sigma_1} B_0 = X$$

such that $Y_0 = Y$, and $Y_i, i > 0$, is a section of the exceptional divisor $E_i \to Y_{i-1}$ of σ_i defined by a surjection $\mathcal{N}_{Y_{i-1}/B_{i-1}} \to \mathcal{L}_i$, where $\mathcal{L}_i$ is an invertible sheaf on Y_{i-1}. Note that the restriction of the composition $\sigma_1 \circ \ldots \circ \sigma_i : B_i \to B_0 = X$ to Y_i defines an isomorphism $Y_i \cong Y$. So we can identify $\mathcal{L}_i$ with an invertible sheaf on Y. For any $k \geq j \geq i \geq 1$, let

$$\sigma_{j,i} = \begin{cases} \sigma_i \circ \ldots \circ \sigma_j : B_j \to B_{i-1} & \text{if } j > i, \\ \sigma_j & \text{if } j = i. \end{cases}$$

Also set

$$\mathcal{E}_{j,i} = \sigma_{j,i}^{-1}(Y_{i-1}).$$

Consider the inclusion of invertible sheaves on B_k

$$\mathcal{O}_{B_k}(-\sum_{i=1}^{k}\mathcal{E}_{ki}) \subset \mathcal{O}_{B_k}(-\sum_{i=1}^{k-1}\mathcal{E}_{ki}) \subset \ldots \subset \mathcal{O}_{B_k}(-\mathcal{E}_{k1} - \mathcal{E}_{k2}) \subset \mathcal{O}_{B_k}(-\mathcal{E}_{k1}).$$

Let Z_j be the subscheme of X defined by the ideal sheaf $(\sigma_{k1})_*\mathcal{O}_{B_k}(-\sum_{i=1}^{j}\mathcal{E}_{ki})$. Then we have a chain of closed subschemes

$$Y = Z_1 \subset Z_2 \subset \ldots \subset Z_k \tag{2.40}$$

with $\mathcal{J}_{Z_j}/\mathcal{J}_{Z_{j+1}} \cong \mathcal{L}_j, \quad j = 1, \ldots, k-1$. By the projection formula,

$$(\sigma_{k1})_*\mathcal{O}_{B_k}(-\sum_{i=1}^{j}\mathcal{E}_{ki}) = (\sigma_{j1})_*(-\sum_{i=1}^{j}\mathcal{E}_{ji}).$$

Thus, each Z_j is a j-th multiple structure on Y.

Finally, we may consider the scheme

$$Z_k(m_1, \ldots, m_k) = V((\sigma_{k1})_*\mathcal{O}_{B_k}(-\sum_{i=1}^{k} m_i\mathcal{E}_{ki})).$$

Its proper transform on B_k is the linear system $|dH - \sum_{i=1}^{k} m_i\mathcal{E}_{ki}|$. For example, $Z_k = Z_k(1, \ldots, 1)$. By definition, $mZ_k = Z_k(m, \ldots, m)$.

Generalizing the computations for the Segre class of a double structure given in (2.30), we can compute, by induction, the Segre classes of the base scheme Z of $|dH - \sum_{i=1}^{j} m_i\mathcal{E}_{ki}|$. We get

$$s(Z_k(m_1, \ldots, m_k), X)_c = m_1^{n-c}s(Y, X)_c + \sum_{j=0}^{n-c-1} \binom{n-c}{j} m_1^j m_2^{n-c-j} D_1^j s(Y_1, B_1)_{c+j}$$

$$+\sum_{j=0}^{n-c-1}\binom{n-c}{j}m_2^j m_3^{n-c-j}D_2^j s(Y_2,B_2)_{c+j}+\ldots+$$
$$\sum_{j=0}^{n-c-1}\binom{n-c}{j}m_{k-1}^j m_k^{n-c-j}D_{k-1}^j s(Y_{k-1},B_{k-1})_{c+j}.$$

where $s(Y_j, B_j)$ are computed by induction using (2.12),

$$s(Y_j,B_j)=(\sum_{i=0}^{r}c_i(\mathcal{N}_{Y_{j-1}/B_{j-1}})(1+D_j)^{r-i})^{-1}(1-D_j)^{-1}. \tag{2.41}$$

Example 2.7.3. Assume that the base scheme is a k-multiple structure Z_k on a smooth curve C. Applying (2.41), we find

$$s(Y_j,B_j)_0=-(n-2)D_j+s(Y_{j-1},B_{j-1})_0=$$
$$[Y_j]-(n-2)D_j-(n-2)D_{j-1}-s(Y_{j-2},B_{j-2})_0$$
$$=-(n-2)(D_1+\ldots+D_j)-c_1(\mathcal{N}_{C/\mathbb{P}^n}),$$

This gives

$$\begin{aligned}
s(Z_k(m_1,\ldots,m_k),\mathbb{P}^n)_0 &= \sum_{j=1}^{k-1}m_j^n s(Y_j,B_j)_0+n\sum_{j=1}^{k-1}m_j m_{j+1}^{n-1}D_j,\\
s(Z_k(m_1,\ldots,m_k),\mathbb{P}^n)_1 &= \sum_{j=1}^{k}m_j^{n-1}[C].
\end{aligned}$$

For a concrete example, let us take all $m_j = m$. Then we obtain

$$\begin{aligned}
s(mZ_k,\mathbb{P}^n)_0 &= -m^n[(n-2)(\sum_{i=1}^{k-1}(k-i)D_i)+kc_1(\mathcal{N}_{C/\mathbb{P}^n})-n\sum_{i=1}^{k-1}D_i],\\
s(Z_k,\mathbb{P}^n)_1 &= km^{n-1}[C].
\end{aligned}$$

As soon as we know the Segre classes of $Z_k(m_1,\ldots,m_k)$ we can compute the multi-degrees of the transformation and check whether $d_n = 1$, the necessary condition for a homaloidal linear system.

Another condition to check is $\dim|dH - Z_k(m_1,\ldots,m_k)| = n$. For this we use the Riemann-Roch Theorem (see [Hartshorne], Appendix]. We state it only in the case $n = 3$.

$$\chi(X,\mathcal{O}_X(D))=\frac{1}{12}(2D^3-3K_X\cdot D^2+D\cdot K_X^2)+\frac{1}{12}D\cdot c_2(\Omega_X^1)+\chi(\mathcal{O}_X). \tag{2.42}$$

In our situation, $X = B_k, D = dH - E$, where $E = \sum_{j=1}^k m_j \mathcal{E}_{kj}$,

$$K_X = -4H + \sum_{i=1}^k \mathcal{E}_{ki}.$$

We also use the following formula for $c_2(X) := c_2(\Omega_X^1)$ from [Fulton], Example 15.4.3. It is obtained by successive application of the formula

$$c_2(B_j) = \sigma_j^*(c_2(B_{j-1})) + (2-2g)f_j - E_j^2, \tag{2.43}$$

where f_j is the class of a fibre of $E_j \to Y_{j-1}$. We skip the computations which show that

$$c_2(B_k) \cdot D = 24d + (d-4)\deg C(\sum_{j=1}^k m_j.$$

Note that $c_2(B_k) \cdot D = 24$ if $d = 4$. This reflects the fact that, for any nonsingular 3-fold X

$$c_2(X) \cdot c_1(X) = 24\chi(X, \mathcal{O}_X).$$

Note that $c_1(B_k) = 4H - \sum_{j=1}^k \mathcal{E}_{kj}$.

To compute the other ingredients in the Riemann-Roch formula, we use that, for any $i > j$,

$$\mathcal{E}_{ki} \cdot \mathcal{E}_{kj} = 0$$

because, replacing E_j by an algebraically equivalent cycle, we can assume that it does not intersect Y_{j-1}, hence $\sigma_{ki}^*(E_j) \cdot \mathcal{E}_{kj'}$ must be zero. Also we have $\mathcal{E}_{kj}^3 = -c_1(\mathcal{N}_{Y_{j-1}/B_{j-1}})$. Using this we obtain

$$\begin{aligned}
D^3 &= d^3 + \sum_{j=1}^k m_j^3 c_1(\mathcal{N}_{Y_{j-1}/B_{j-1}}) - 3d\deg C \sum_{j=1}^k m_j^2, \\
K_{B_k} \cdot D^2 &= -4d^2 + \sum_{j=1}^k (4m_i^2 + 2dm_i)\deg C - \sum_{j=1}^k m_j^2 c_1(\mathcal{N}_{Y_{j-1}/B_{j-1}}), \\
K_{B_k}^2 \cdot D &= 16d - \sum_{j=1}^k (d + 8m_i)\deg C + \sum_{j=1}^k m_j c_1(\mathcal{N}_{Y_{j-1}/B_{j-1}}), \\
c_2(B_k) \cdot D &= 6d + (d-4)\deg C \sum_{j=1}^k m_j.
\end{aligned}$$

Finally, we get

$$12\chi(B_k, \mathcal{O}_X(dH-E)) = 12\binom{d+3}{3} + \sum_{j=1}^{k}(2m_j^3 + 3m_j^2 + m_j)c_1(\mathcal{N}_{Y_{j-1}/B_{j-1}})$$

$$-((6d+12)\sum_{j=1}^{k} m_j^2 + (5d+12)\sum_{j=1}^{k} m_j + kd)\deg C.$$

If $|dH - E|$ is a homaloidal system, the right-hand side must be equal to 36 (provided that $H^1(B_k, \mathcal{O}_{B_k}(dH-E)) = 0$).

To convince the reader that we have not made a mistake let us make a few tests. First, we take $\deg C = 2, k = m_1 = 1$. The right-hand side is equal to $120+36-2(24+22+2) = 60$. Since $\dim |2H-C| = 4$, the formula is correct.

Next, we take $\deg C = 3, k = 2, \deg D_1 = -4, m_1 = m_2 = 1$. The right-hand side is equal to $54+108+66+6(10+18)-6(30+27+3) = 36$. We know that the linear system is homaloidal, so the formula is correct.

The conditions $(dH-E)^3 = 1$ and $\chi(B_k, \mathcal{O}_X(dH-E)) = 4$ are not the only necessary conditions for the existence of a homaloidal linear system. Note that a general member D of $|dH-E|$ is a smooth rational surface. This implies that its canonical linear system is empty. By the adjunction formula,

$$K_D = ((d-4)H - \sum_{j=1}^{k}(m_i - 1)\mathcal{E}_{kj}) \cdot D.$$

For example, if all $m_j = 1$, this implies that $d < 4$. We have seen already an example with $d = 3, k = 2$ and $\deg C = 3$. One more possible example is $d = 3, k = 3, \deg C = 2$ and the triple multiple structure Z on the conic is defined by line bundles of degree -2 and -3. The arithmetic genus of Z is equal to 3, the same as the genus of the double structure on a twisted cubic. In the next lecture we will study cubo-cubic transformations with basis scheme an arbitrary arithmetical Cohern-Macaulay one-dimensional schme of degree 6 and arithmetic genus 3. Our examples are special cases of such transformations.

2.8 Dilated transformations

Starting from a Cremona transformation T in $\mathbb{P}^{n-1}$ we seek to extend it to a Cremona transformation in $\mathbb{P}^n$. More precisely, if $p_{\mathfrak{o}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is a projection map from a point $\mathfrak{o}$, we want to find a Cremona transformation $\overline{T} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$

such that $p_{\mathfrak{o}} \circ \bar{T} = T \circ p_{\mathfrak{o}}$. Suppose that T is given by a sequence of degree d homogeneous polynomials $(G_1, \ldots, G_n)$. Composing with a projective transformation in $\mathbb{P}^n$, we may assume that $\mathfrak{o} = [1, 0, \ldots, 0]$. Thus the transformation $\overline{T}$ must be given by $(F_0, QG_1, \ldots, QG_n)$, where Q and F_0 are coprime polynomials of degrees r and $d + r$.

Proposition 2.8.1. *(I. Pan) Let $(G_1, \ldots, G_n)$ be homogeneous polynomials of degree d in $t_1, \ldots, t_n$. Let $F_0 = t_0 A_1 + A_2, Q = t_0 B_1 + B_2$, where A_1, A_2, B_1, B_2 are homogeneous polynomials of degrees $d + r - 1, d + r, r, r - 1$, respectively. Assume that F_0 and Q are coprime and $A_1 B_2 \neq A_2 B_1$. Then the polynomials $(F_0, QG_1, \ldots, QG_n)$ define a Cremona transformation of $\mathbb{P}^n$ if and only if $(G_1, \ldots, G_n)$ define a Cremona transformation of $\mathbb{P}^{n-1}$.*

Proof. We will give two proofs, a geometric and purely algebraic one. The first one is geometric. Assume that the conditions are satisfied. Take a general point y in the target space. The linear system of hyperplanes through y contains a codimension 1 subsystem of hyperplanes passing through the point $\mathfrak{o}$. Thus we can write y as the intersection of $n + 1$ hyperplanes $H_1, \ldots, H_n, H_{n+1}$, where $H_1, \ldots, H_n$ contains $\mathfrak{o}$ and H_{n+1} does not. The pre-image of y under $\overline{T}$ is equal to the intersection of n hypersurfaces $D_1, \ldots, D_n$ from the linear span of the polynomials $QG_1, \ldots, QG_n$ and a hypersurface $D = V(F_0 + \sum a_i QG_i)$. The intersection of the first n hypersurfaces is the union of the line $\ell = \langle x, \mathfrak{o} \rangle$ and the hypersurface $V(Q)$, where $x \in V(t_0)$ is the unique pre-image of y under T. The intersection $D \cap (V(Q) \cup \ell)$ consists of the union of $D \cap V(Q)$ and $D \cap \ell$. The first part belongs to the base locus of $\overline{T}$. The conditions on the multiplicities of Q and F_0 imply that D has a singular point of multiplicity $d + r - 1$, one less than its degree. Thus the second intersection consists of one point outside the base point. It is easy to see that the conditions are necessary for this.

Our second proof is algebraic. Let $F'(z_1, \ldots, z_n)$ denote the dehomogenization of a homogeneous polynomial $F(t_0, \ldots, t_n)$ in the variable t_1. It is obvious that $(F_0, \ldots, F_n)$ define a Cremona transformation if and only if the field

$$\mathbb{C}(F_1/F_0, \ldots, F_n/F_0) := \mathbb{C}(F_1'/F_0', \ldots, F_n'/F_0') = \mathbb{C}(z_1, \ldots, z_n).$$

Consider the ratio $F_0/QG_1 = \frac{t_0 A_1 + A_2}{t_0 GB_1 + GB_2}$. Dehomogenizing with respect to t_1, we can write the ratio in the form $\frac{az_1 + b}{cz_1 + d}$, where $a, b, c, d \in \mathbb{C}(z_2, \ldots, z_n)$. By our assumption, $ad - bc \neq 0$. Then

$$\mathbb{C}(F_1/F_0, \ldots, F_n/F_0) = \mathbb{C}(F_0/QG_1, G_2/G_1, \ldots, G_n/G_1)$$

$$= \mathbb{C}(G_2/G_1, \ldots, G_n/G_1)(F_0/QG_1) = \mathbb{C}(G_2/G_1, \ldots, G_n/G_1)(\frac{az_1 + b}{cz_1 + d}).$$

This field coincides with $\mathbb{C}(z_1, \ldots, z_n)$ if and only if $\mathbb{C}(G_2/G_1, \ldots, G_n/G_1)$ coincides with $\mathbb{C}(z_2, \ldots, z_n)$.

□

Definition 2.8.1. *The Cremona transformation $\bar{T}$ in $\mathbb{P}^n$ obtained from a Cremona transfromation T in $\mathbb{P}^{n-1}$ by the above construction is called a* dilation *of T. It depends on the choice of a point $\mathfrak{o}$ and polynomials F_0, Q of degrees $d+r$ and r satisfying the conditions on their multiplicities at $\mathfrak{o}$ stated in the assertion of the previous proposition.*

The first geometric proof in Proposition 2.8.1 gives some information about the possible multi-degree $(\tilde{d}_1, \ldots, \tilde{d}_{n-1})$ of the dilated transformation. Let $(d_1, \ldots, d_{n-2})$ be the multi-degree of T. Let L be a general linear subspace of $\mathbb{P}^n$ of dimension k. We can write L as the intersection of $n-k-1$ hyperplanes $H_1, \ldots, H_{n-k-1}$ containing L and a hyperplane H_{n-k} containing L but not containing $\mathfrak{o}$. Let D_i be the divisors of the homaloidal linear system defining $\bar{T}$ which corresponds to H_i. The intersection $D_1 \cap \ldots \cap D_{n-k-1}$ is equal to the union of the cone over the base scheme of T and the cone over C_X over a k-dimensional variety $T^{-1}(L \cap V(t_0))$ of degree d_{n-k-1}. The intersection $D_1 \cap \ldots \cap D_{n-k-1} \cap D_{n-k}$ is equal to the union of the base locus of $\bar{T}$ and the intersection of X with D_{n-k}. The latter is of degree $(d+r)d_{n-k-1}$. Thus we obtain

$$\tilde{d}_{n-k} \leq (d+r)d_{n-k-1}. \tag{2.44}$$

When $k = n-1$, we have the equality. For smaller k, the equality occurs when $D \cap C_X$ has no components of dimension k belonging to the base scheme of $\overline{T}$. It is easy to see that the latter consists of the union of $V(F_0) \cap V(Q)$ and $V(F_0) \cap C_{\mathrm{Bs}(T)}$.

Example 2.8.1. Let $n = 3$ and T be defined by a smooth homaloidal linear system $|dh - \sum_{i=1}^N m_i p_i|$. Let $F_0 = t_0 A_1 + A_2$ and $Q = t_0 B_1 + B_2$ as in Proposition 2.8.1. and let k_i be the minimal of the multiplicities of A_1 and A_2 at the point p_i and similar n_i be for B_1, B_2. This implies that $V(F_0)$ pass through the lines $\langle \mathfrak{o}, p_i \rangle$ with multiplicity $\geq k_i$ and $V(Q)$ passes through these lines with multiplities n_i. Let $s_i = \min\{m_i + n_i, k_i\}$. Then the dilated homaloidal linear system contains the lines $\langle \mathfrak{o}, p_i \rangle$ with multiplicities s_i, we obtain

$$\tilde{d}_2 = (d+r)d - \sum s_i.$$

In a special case, take T to be the standard Cremona transformation T_{st} given by (t_2t_3, t_1t_3, t_1t_2) and $F_0 = t_0t_1$. Thus two of the numbers s_i, corresponding to the base points $(0,1,0)$ and $(0,0,1)$ are equal to one, and one is equal to 0. We obtain $\tilde{d}_2 = 2$. On the other hand, if we take $F_0 = t_0t_1 + t_2$, we obtain $\tilde{d}_2 = 3$ and, if we take $F_0 = t_0(t_1 + t_2)$, we get $\tilde{d}_2 = 4$.

Lecture 3

First examples

3.1 Quadro-quadric transformations

Let us show that any vector $(2,\dots,2)$ is realized as the multi-degree of a Cremona transformation. For $n=2$, we take the homaloidal linear system of conics through three non-collinear points. For $n=3$, we can take the homaloidal linear system of quadrics through a nonsingular conic discussed in the previous section. For arbitrary n we do a similar construction as is explained below.

Consider the linear system of quadrics in $\mathbb{P}^n$ containing a fixed smooth quadric Q_0 of dimension $n-2$. It maps $\mathbb{P}^n$ to a quadric Q in $\mathbb{P}^{n+1}$. We may choose coordinates such that

$$Q_0 = V(z_0) \cap V(\sum_{i=1} z_i^2).$$

so that the hyperplane $H=V(z_0)$ is the linear span of Q_0. Then the linear system is spanned by the quadrics $V(\sum z_i^2), V(z_0z_i), i=0,\dots,n$. It maps the blow-up of $\mathbb{P}^n$ along Q_0 to the quadric Q in $\mathbb{P}^{n+1}$ with equation $t_0t_{n+1} - \sum_{i=1}^n t_i^2 = 0$. The rational map $g: \mathbb{P}^n \dashrightarrow Q$ defined by a choice of a basis of the linear system, can be given by the formula

$$[x_0,\dots,x_n] \mapsto [\sum_{i=1}^n x_i^2, x_0x_1,\dots,x_0x_n, x_0^2].$$

Observe that the image of H is equal to the point $a=[1,0,\dots,0]$. The inverse of g is the projection map

$$\mathrm{pr}_a : Q \dashrightarrow \mathbb{P}^n, \quad [z_0,\dots,z_{n+1}] \to [z_0,\dots,z_n]$$

from the point a. It blows down the hyperplane $V(z_{n+1}) \subset \mathbb{P}^{n+1}$ to the quadric Q_0. Now consider the projection map $\mathrm{pr}_b : Q \dashrightarrow \mathbb{P}^n$ from a point $b \neq a$ not lying in the hyperplane $V(t_{n+1})$. Note that this hyperplane is equal to the embedded tangent hyperplane $\mathbb{T}_a Q$ of Q at the point a. The composition $f = \mathrm{pr}_b \circ \mathrm{pr}_a^{-1}$ of the two rational maps is a quadratic transformation defined by the homaloidal linear system of quadrics with the base locus equal to the union of Q_0 and the point $\mathrm{pr}_a(b)$. For example, if we choose $b = [0, \ldots, 0, 1]$ so that $\mathrm{pr}_a(b) = [1, 0, \ldots, 0]$, then the Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ can be given by the formula

$$[x_0, \ldots, x_n] \mapsto [\sum_{i=1}^{n} x_i^2, x_0x_1, \ldots, x_0x_n]. \tag{3.1}$$

Note that $f^{-1} = \mathrm{pr}_a \circ \mathrm{pr}_b^{-1}$ must be given by similar quadratic polynomials. So the degree of f^{-1} is equal to 2. This is the reason for the name *quadro-quadratic transformation*.

For example, if $n = 2$, if we rewrite the equation of Q_0 in the form $z_0 = z_1z_2 = 0$ and obtain the formula (1.7) for the standard quadratic transformation.

Let us compute the multi-degree. For any general linear subspace L of codimension $k > 0$, its pre-image under the projection $p_b : Q \dashrightarrow \mathbb{P}^n$ is the intersection of Q with the subspace $L' = \langle L, b\rangle$. It is a quadric in this subspace. Since the point a does not belong to L', the projection of this quadric from the point a is a quadric in the projection of L' from the same point. Thus $d_k = 2$. This shows that the multi-degree of the transformation is equal to $(2, \ldots, 2)$.

Let us confirm it by the "high-tech" computations using the Segre classes. We have the exact sequence of normal sheaves

$$0 \to \mathcal{N}_{Q_0/H} \to \mathcal{N}_{Q_0/\mathbb{P}^n} \to \mathcal{N}_{H/\mathbb{P}^n}|Q_0 \to 0.$$

This gives

$$s(Q_0, \mathbb{P}^n) = c(\mathcal{N}_{Q_0/\mathbb{P}^n})^{-1} = c(\mathcal{N}_{Q_0/H})^{-1} c(\mathcal{N}_{H/\mathbb{P}^n}) \cap [Q_0]$$

$$= (1+2h_0)^{-1}(1+h_0)^{-1} = \Big(\sum_{i\ge 0}(-2h_0)^i\Big)\Big(\sum_{i\ge 0}(-h_0)^i\Big) = \sum_{k\ge 0}(2^{k+1}-1)(-1)^k h_0^k.$$

where h_0 is the class of a hyperplane section of Q_0 and 1 stands for $[Q_0]$. Note that under the homomorphism $i_* : A_*(Q_0) \to A_*(\mathbb{P}^n)$, the image of $h_0^k \in A_{n-3}(Q_0)$ is equal to $2h^{k+2}$. Thus we obtain, for $k < n$,

$$d_k = 2^k - \sum_{i=1}^{k} 2^{k-i}\tbinom{k}{i} s(\mathrm{Bs}(\mathcal{H}_X), \mathbb{P}^n)_{n-i} h^{n-i}$$

$$= 2^k - \sum_{i=1}^{k} 2^{k-i}\tbinom{k}{i}[s(Q_0, \mathbb{P}^n)_{n-i} + s(x_0, \mathbb{P}^n)_{n-i}]h^{n-i}$$

$$= 2^k - \sum_{i=1}^{k} 2^{k-i}\binom{k}{i}(-1)^{i-2}2(2^{i-1}-1) + [x_0]h^{n-k}$$

$$= 2^k - 2^k(\sum_{i=1}^{k}(-1)^i\binom{k}{i}) + 2\sum_{i=1}^{k} 2^{k-i}(-1)^i\binom{k}{i} = 2^k + 2^k + 2(1-2^k) = 2.$$

Let us consider some degenerations of the transformation given by (3.1). Let us take two nonsingular points a, b on an arbitrary irreducible quadric $Q \subset \mathbb{P}^{n+1}$. We assume that b does not lie in the intersection of Q with the embedded tangent space $\mathbb{T}_aQ$ of Q at a. Let $f = \mathrm{pr}_a \circ \mathrm{pr}_b^{-1}$. The projection p_a blows down the intersection $\mathbb{T}_aQ \cap Q$ to a quadric Q_0 in the hyperplane $H = \mathrm{pr}_a(\mathbb{T}_aQ)$. If $r = \mathrm{rank}\, Q$ (i.e. $n+1-r$ is the dimension of the singular locus of Q), then rank $Q \cap \mathbb{T}_aQ = r-1$. Its singular locus is spanned by the singular locus of Q and the point a. The projection Q_0 of $Q \cap \mathbb{T}_aQ$ is a quadric with singular locus of dimension $n+1-r$, thus, it is a quadric of rank equal to $n-1-(n+1-r) = r-2$ in H. The inverse transformation $\mathrm{pr}_a^{-1} : \mathbb{P}^n \dashrightarrow Q$ is given by the linear system of quadrics in $\mathbb{P}^n$ which pass through Q_0. So, taking $a = [1, 0, \ldots, 0]$ and $b = [0, \ldots, 0, 1]$ as in the non-degenerate case, we obtain that f is given by

$$f : [x_0, \ldots, x_n] \mapsto [\sum_{i=1}^{r-2} x_i^2, x_0x_1, \ldots, x_0x_n]. \tag{3.2}$$

Note the special cases. If $n = 2$, and Q is an irreducible quadric cone, then $r = 3$ and we get the formula for the first degenerate standard quadratic transformation (1.8). To get the second degenerate standard quadratic transformation, we should abandon the condition that $b \notin \mathbb{T}_aQ$. We leave the details to the reader.

Example 3.1.1. Consider the Cremona transformation given by the quadric polynomials

$$(t_3^2 - t_1t_2, t_0(t_1-t_2), t_0(t_2-t_3), t_0(\lambda t_3 - t_0).$$

When $\lambda \neq 0$, the base scheme is reduced and is equal to the union of the smooth conic $C : t_3^2 - t_1t_2 = t_0 = 0$ and the point $[\lambda, 1, 1, 1]$. When $\lambda = 0, u \neq 0$, the reduced base scheme is equal to C, however, it is not reduced. It contains an embedded point $[0, 1, 1, 1]$ on C. It corresponds to the point b lying on $\mathbb{T}_aQ$.

Example 3.1.2. Another example of a degenerate quadratic transformation is obtained by the homaloidal linear system of quadrics through a double structure Z of codimension 2 linear subspace L of $\mathbb{P}^n$. We consider only the case $n = 3$ and leave the computations in other cases to the reader. Consider the double structure on a line L defined by the invertible sheaf $\mathcal{O}_L(-1)$. Consider the exact sequences, obtained from exact sequence (2.28) after twisting by $\mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{O}_{\mathbb{P}^3}(2)$,

$$0 \to \mathcal{J}_Z(2) \to \mathcal{J}_L(2) \to \mathcal{O}_L(1) \to 0,$$

$$0 \to \mathcal{J}_Z(1) \to \mathcal{J}_L(1) \to \mathcal{O}_L \to 0.$$

It is easy to see that the homomorphism $H^0(\mathbb{P}^3, \mathcal{J}_L(i)) \to H^0(\mathbb{P}^3, \mathcal{O}_L(i-1))$ is surjective. Indeed, it assigns to a section s defining the plane (quadric) containing L the section $\bar{s}$ obtained by dividing s by the equation of the line and restricting $\bar{s}$ to L. This gives $h^0(\mathcal{J}_Z(1)) = 1, h^0(\mathcal{J}_Z(2)) = 5$. Using computations from Example 2.7.2 we obtain that the linear system of quadrics through Z and a point outside the plane corresponding to a non-zero section of $\mathcal{J}_Z(1)$ is homaloidal. The corresponding Cremona transformation is the composition of the transformation

$$[x_0, x_1, x_2, x_3] \mapsto [x_1^2, x_0x_1, x_0x_2, x_0x_3]$$

and a projective transformation. The equation of the line L here is $x_0 = x_1 = 0$, and the equation of the plane containing the double structure is $x_0 = 0$. The transformation is the composition of a rational map to a quadric of rank 3 in $\mathbb{P}^4$, and the projection from a nonsingular point of the quadric.

It is not true that the multi-degree of a quadro-quadratic transformation in $\mathbb{P}^n, n > 3$ is always of the form $(2, \ldots, 2)$. For example, if $n = 4$, applying Cremona's inequalities, we obtain $d_2 \leq 4$. A transformation of multi-degree $(2, 3, 2)$ can be obtained by taking the homaloidal linear system of quadrics with the base scheme equal to a plane and two lines intersecting the plane at one point.

3.2 Quadro-quartic transformations

Suppose we have a Cremona transformation with multi-degree $(2, m)$. An Φ-curve, the proper transform of a line in the target space, is of degree m. It is contained in the intersection of two quadrics, the proper transforms of planes. This shows that $m \leq 4$. This can be also deduced from the log-concavity of the multi-degree. If $m = 4$, the formula for d_2 in Proposition 2.3.1 shows that the reduced base scheme consists of isolated points. The image of a general plane under the transformation is given by a 3-dimensional linear system of conics without base points. Thus the image must be a quartic surface, a projection of a Veronese surface to $\mathbb{P}^3$.

We know that there are no smooth homaloidal systems with isolated base points. Thus one of the base conditions must be that the proper transform of the homaloidal linear system to the blow-up of four points has base points on one of the exceptional locus. Since the intersection of two general quadrics from the linear system is a curve of arithmetic genus 1, and, on the hand it must be an irreducible rational curve (the pre-image of a line under the Cremona transformation), we see that the base condition must be that all quadrics are tangent at one point. Counting dimension, we see that there must be four base points $p_1, \ldots, p_4$ with a tangency condition at p_4. If we assume that neither of the first three points is infinitely near, after composing the transformation with a projective transformation, we can give it by a formula

$$[x_0, x_1, x_2, x_3] \mapsto [x_0 l(x_1, x_2, x_3), x_2x_3, x_1x_3, x_1x_2], \tag{3.3}$$

where l is a linear form that vanishes at $p_4 = [1, 0, 0, 0]$ but does not not vanish at any of the first base points $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. All quadrics in the linear system have the same tangent plane at $p_4 = [1, 0, 0, 0]$ equal to $V(l)$. The conditions on l imply that

$$l = at_1 + bt_2 + ct_3, \quad a, b, c \neq 0.$$

We recognize in formula (3.3) a formula for a dilated quadratic transformation.

Let us find the P-locus of the dilated quadratic transformation. Consider the planes spanned by two of the first three points p_i, p_j and $p_4 = [1, 0, 0, 0]$. The restriction of the transformation to this plane is given by the linear system of conics through the points p_i, p_j, p_4 with a fixed tangent direction at p_4. It is a one-dimensional linear system which maps this plane to a line ℓ_{ij} in the target space. Also consider the restriction of the plane to the plane $V(l)$. All quadrics are tangent to this plane at p_4, so they restrict to conics with a singular point at p_4. These conics consist of two lines passing through p_4 and map the plane to a conic C in the target space. Thus the P-locus consists of four planes. Computing the jacobian of the transformation, we see that the expected degree of the P-locus is equal to 4, so there is nothing else in the P-locus.

The reduced base locus of the inverse transformation T^{-1} must contain the image of the P-locus under T. We see that it consists of the union of three lines intersecting at one point (the image of the intersection point of the three planes $\langle p_i, p_j, p_4 \rangle$). The image of the conic intersects all these lines.

Let us see this in formulas. In homogeneous coordinates, our transformation is given by the polynomials $(t_0(at_1 + bt_2 + ct_3), t_2t_3, t_1t_3, t_2t_3)$. Dividing by the first coordinate, and setting $z_i = t_i/t_0$, we obtain a formula for the transformation in inhomogeneous coordinates

$$(z_1, z_2, z_3) \mapsto (u_1, u_2, u_3) = (\frac{z_2z_3}{az_1 + bz_2 + cz_3}, \frac{z_1z_3}{az_1 + bz_2 + cz_3}, \frac{z_1z_2}{az_1 + bz_2 + cz_3}).$$

It is easy to invert it. We find

$$au_2u_3 + bu_1u_3 + cu_1u_2 = \frac{z_1z_2z_3}{az_1 + bz_2 + cz_3}.$$

This gives

$$(z_1, z_2, z_3) = (\frac{au_2u_3 + bu_1u_3 + cu_1u_2}{u_1}, \frac{au_2u_3 + bu_1u_3 + cu_1u_2}{u_2}, \frac{au_2u_3 + bu_1u_3 + cu_1u_2}{u_3}).$$

Homogenizing again, we find the inverse is given by the polynomials

$$(t_0t_1t_2t_3, (at_2t_3+bt_1t_3+ct_1t_2)t_1t_2, (at_2t_3+bt_1t_3+ct_1t_2)t_1t_3, (at_2t_3+bt_1t_3+ct_1t_2)t_2t_3).$$

We see that the transformation is dilated from the standard Cremona transformation. In notation of section 2.8, we have $F_0 = t_0t_1t_2t_3, Q = at_2t_3 + bt_1t_3 + ct_1t_2$. The formula exhibits the base locus equal to the union of the conic $V(at_2t_3 + bt_1t_3 + ct_1t_2, t_0)$ and the three lines $V(t_i, t_j),\ i, j \neq 0)$. The lines intersect at the point $(1, 0, 0, 0)$. The conic intersects the three lines.

The inverse map is given by the homaloidal linear system of quartics passing through the conic and passing with multiplicity 2 through the three lines. These surfaces are known to be *Steiner quartic surfaces*. They are projections of a Veronese surface to $\mathbb{P}^3$.

3.3 Quadro-cubic transformations

Now lets us consider some examples of transformations of type $(2, 3)$, *quadro-cubic transformations*. The formula for d_2 in Proposition 2.3.1 shows that the one-dimensional part of the base locus is a line. Counting the dimension, we see that the rest of the base locus must consist of three simple points, maybe infinitely near. We have seen such an example in Example 2.5.2. The transformation is given by a smooth homaloidal linear system with base scheme equal to the union of a line ℓ and three isolated points p_1, p_2, p_3. The restriction of the linear system to a general plane is a linear system of conics passing through a fixed point. It maps the plane onto a cubic scroll in $\mathbb{P}^3$. This confirms that the inverse transformation is of degree 3.

The P-locus consists of the union of four planes, the spans $\langle \ell, p_i\rangle$ and the span $\langle p_1, p_2, p_3\rangle$. The moving part of the restriction of the linear system to $\langle \ell, p_i\rangle$ is the linear system of lines through the point p_i. The restriction of the linear system to the fourth plane is the linear system of conics through 4 points (the fourth point p_4 is the intersection of ℓ with the forth plane). The planes in the P-locus are blown down to four lines in the target space. The first three lines are skew, and the fourth line intersects the first three lines at the points equal to the images of the lines $\langle p_4, p_i\rangle, i = 1, 2, 3$. The four lines form the base locus of the inverse transformation T^{-1}. Using (2.26) we see that the homaloidal system of cubics defining the inverse transformation is not smooth. Formula (2.39) from Example 2.7.2 shows that the linear system of cubics through the forth line with double structure defined by a line bundle $\mathcal{L}$ of degree 3. This gives 7 conditions for containing the double structure and 9 more conditions to contain the the three lines.

All of this can be checked by formulas. Without loss of generality, we may assume that the base locus of T consists of the line $t_0 - t_1 = t_1 - t_2 = 0$ and the three points $[1,0,0,0],[0,1,0,0],[1,0,0,0]$. The homaloidal linear system is generated by quadrics

$$Q : at_0(t_1 - t_2) + b(t_1 - t_2)t_3 + c(t_0 - t_1)t_2 + (t_0 - t_1)t_3 = 0.$$

A Cremona transformation T is given by choosing a basis in the space of such quadrics. For example, we can take

$$T : [x_0, x_1, x_2, x_3] \mapsto [x_0(x_1 - x_2), (x_1 - x_2)x_3, (x_0 - x_1)x_2, (x_0 - x_1)x_3].$$

Dividing by the first coordinate and using affine coordinates $z_1 = t_1/t_0, z_2, t_2/t_0, z_3 = t_3/t_0$, the transformation is given by the formula

$$(z_1, z_2, z_3) \mapsto (u_1, u_2, u_3) = (z_3, (1 - z_1)z_2/(z_1 - z_2), (1 - z_1)z_3/(z_1 - z_2)).$$

Its inverse is given by the formula

$$(z_1, z_2, z_3) = (u_1 + u_1u_2)/(u_1 + u_3), u_1u_2/u_3, u_1).$$

In homogeneous coordinates, we get the formula

$$[t_0, t_1, t_2, t_3] \mapsto [t_0t_3(t_1 + t_3), t_1(t_0 + t_2)t_3, t_1t_2(t_1 + t_3), t_0t_1(t_1 + t_3)]$$

We see that the degree of this transformation is equal to 3. Its base locus consists of four lines

$$t_0 = t_1 = 0, \quad t_0 = t_2 = 0, \quad t_1 + -t_3 = t_0 + t_2 = 0, \quad t_1 = t_3 = 0.$$

The first three lines are skew. They all intersect the fourth line. All cubics from the homaloidal linear system are tangent along the last line. This defines a double structure on this line.

3.4 Bilinear Cremona transformations

These transformations generalize to any dimension cubic-cubic transformations in $\mathbb{P}^3$, one of which is the standard Cremona transformation T_{st} given by polynomials

$$(t_1 \cdots t_n, t_0t_2 \cdots t_n, \ldots, t_0 \cdots t_{n-1}).$$

as well as the transformation with base scheme equal to a double structure on a twisted cubic considered in Example 2.7.2. All of them belong to the same irreducible family of transformations with base locus in the Hilbert scheme of purely one-dimensional connected schemes Z with Hilbert polynomial $P(t) = 6t - 2$

and the additional condition that $H^0(Z, \mathcal{O}_Z(2)) = 0$. In particular, the degree of Z is equal to 6 and the arithmetic genus equal to 3. The last condition requires that Z does not lie on a quadric. These subschemes are examples of *arithmetical Cohen-Macaulay schemes* (ACM-schemes, for short).

Let Z be a closed subscheme of $\mathbb{P}^n$. Recall that the homogenous ideal of Z is the graded ideal

$$I_Z = \Gamma_*(\mathcal{J}_Z) = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{J}_Z(k))$$

in the ring

$$\Gamma_*(\mathcal{O}_{\mathbb{P}^n}) = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{J}_Z(k)) \cong S := \mathbb{C}[t_0, \ldots, t_n].$$

The graded quotient ring $A(Z) = S/I_Z$ is called the homogeneous coordinate ring of Z.

Definition 3.4.1. *A closed subscheme Z of $\mathbb{P}^n$ of pure dimension r is called arithmetically Cohen-Macaulay (ACM for short) if the homogeneous coordinate ring A_Z is Cohen-Macaulay ting. This is equivalent to the following conditions:*

(i) the canonical restriction map

$$A(Z)_k = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))/H^0(\mathbb{P}^n, \mathcal{J}_Z(k)) \to H^0(Z, \mathcal{O}_Z(k))$$

is bijective.

(ii) $H^i(Z, \mathcal{O}_Z(j)) = 0$ for $1 \leq i \leq r-1$ and $j \in \mathbb{Z}$.

Assume that $\mathrm{codim} Z = 2$. It follows from the standard facts in commutative algebra that the projective dimension of the ring $A(Z)$ is equal to 2. This means that I_Z admits a resolution of length 2 of projective graded S-modules. Since each such module is free and it is isomorphic to the direct sum of graded S-modules $S[a_i]$[1] Since $\mathcal{J}_Z = \tilde{\Gamma}_*(\mathcal{J}_Z)$ is the associated sheaf of the module I_Z, we obtain a locally free resolution

$$0 \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \bigoplus_{j=1}^{m+1} \mathcal{O}_{\mathbb{P}^n}(-b_j) \to \mathcal{J}_Z \to 0. \tag{3.4}$$

for some sequences of integers (a_i) and (b_j).

[1]Recall that $S[a] = S$ with shifted grading $S[a]_k = S_{a+k}$.

It is easy to see that the existence of such a resolution implies that Z is an ACM subscheme of pure codimension 2. The numbers (a_i) and (b_j) are determined from the Hilbert polynomials of Z.

We will consider a special case of resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1)^n \to \mathcal{O}_{\mathbb{P}^n}(-n)^{n+1} \to \mathcal{J}_Z \to 0. \tag{3.5}$$

It implies that

$$\chi(\mathcal{O}_Z(k)) = \chi(\mathcal{O}_{\mathbb{P}^n}(k)) - \chi(\mathcal{J}_Z(k)) =$$
$$= \chi(\mathcal{O}_{\mathbb{P}^n}(k)) + (n+1)\chi(\mathcal{O}_{\mathbb{P}^n}(k-n)) - n\chi(\mathcal{O}_{\mathbb{P}^n}(k-n-1))$$
$$= \binom{n+k}{n} - (n+1)\binom{k}{n} + n\binom{k-1}{n}.$$

When $n = 3$, we get $\chi(\mathcal{O}_Z(k)) = 6k - 2$, as we expect.

Twisting (3.6) by $\mathcal{O}_{\mathbb{P}^n}(n)$, we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^n \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{J}_Z(n) \to 0. \tag{3.6}$$

Taking cohomology, we obtain canonical isomorphisms

$$H^0(\mathbb{P}^n, \mathcal{J}_Z(n)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}) \cong \mathbb{C}^{n+1},$$

$$H^{n-1}(\mathbb{P}^n, \mathcal{J}_Z) \cong H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1-n))^n) \cong \mathbb{C}^n.$$

The latter isomorphism follows from Serre's Duality since $\mathcal{O}_{\mathbb{P}^n}(-1-n) \cong \omega_{\mathbb{P}^n}$. Let

$$\begin{aligned} V &= H^0(\mathbb{P}^n, \mathcal{J}_Z(n)), \\ W &= H^{n-1}(\mathbb{P}^n, \mathcal{J}_Z). \end{aligned}$$

Then we can rewrite (3.6) in the form

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \otimes W \to \mathcal{O}_{\mathbb{P}^n} \otimes V \to \mathcal{J}_Z(n) \to 0. \tag{3.7}$$

Let $\mathbb{P}^n = \mathbb{P}(E^\vee) = |E|$. Passing to the maps of stalks, we see that the exact sequence is defined by a linear map

$$W \otimes E \to V.$$

or, equivalently by a tensor

$$\mathfrak{a} \in W^\vee \otimes E^\vee \otimes V.$$

We can also view it as a linear map

$$\mathfrak{a} : E \to \mathrm{Hom}(W, V) = W^\vee \otimes V. \tag{3.8}$$

Choosing coordinates $t_0, \ldots, t_n$ in E, a basis in W and a basis in V, the map $\mathfrak{a}$ is given by a matrix $A = (a_{ij}(t))$ of size $(n+1) \times n$ with entries linear functions $a_{ij}(t)$. For any point $x = [x_0, \ldots, x_n]$, we denote by $A(x)$ the matrix $(a_{ij}(x))$. Tensoring exact sequence (3.7) by the residue field of a closed point, we find that

$$\{x \in \mathbb{P}^n : \mathrm{rank} A(x) < n\} = \mathrm{Supp}(Z).$$

We can do better. Recall that exact sequence (3.6) comes from a projective resolution

$$0 \to \bigoplus S[-n-1]^n \xrightarrow{A} S[-n]^{n+1} \to I_Z \to 0,$$

where the map A is given by our matrix $A(t)$ of linear forms. By *Hilbert-Birch Theorem* (see [Eisenbud, Com. Algebra,Theorem 20.15]), this implies that I_Z is generated by maximal minors D_i of the matrix A. Since the ideal I_Z is saturated, it determines uniquely the scheme Z (see [Hartshorne], Chap. 2, Exercise 5.10]. Thus Z is equal to the base scheme of a rational map

$$T_{\mathfrak{a}} : |E| \to |V^\vee|$$

defined by the n-dimensional linear system $|V| = |\mathcal{J}_Z(n)|$ generated by the maximal minors of A. Explicitly, the map $T_{\mathfrak{a}}$ is defined by

$$T_{\mathfrak{a}}([x]) = |\mathrm{Coker}(\mathfrak{a}(x))| \in |V^\vee| = \mathrm{Ker}({}^t\mathfrak{a}(x)),$$

where ${}^t\mathfrak{a}(x) : V^\vee \to W$ is the transpose of $\mathfrak{a}(x)$.

Remark 3.4.1. The Hilbert scheme of ACM subschemes Z of $\mathbb{P}^n$ admitting a resolution (3.7) is isomorphic to an open subset of the projective space of $(n+1) \times n$ of matrices $A(t)$ of linear forms such that the rank of $A(t)$ is equal to n for an open non-empty subset of $\mathbb{P}^n$. Modulo the action by $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ by left and right multiplication. It is a connected smooth variety of dimension $n(n^2-1)$ (see [Peskine-Szpiro], Inventiones, 1974], or [Ellingsrud, Ann. Ec. Norm. Sup. 8 (1975)]).

Proposition 3.4.1. *The map $T_{\mathfrak{a}}$ is a birational map. The multi-degree is equal to $(d_k) = (\binom{n}{k})$.*

Proof. Let us view the tensor $\mathfrak{a}$ as as a bilinear map

$$E \otimes V^\vee \to W^\vee. \tag{3.9}$$

If we fix a basis in E, a basis in $V^\vee$, and a basis in W, then the bilinear map is defined by n square matrices $B_1, \ldots, B_n$ of size $n+1$. If we write $A(t)$ as $t_0A_0 + \ldots + t_nA_n$, and write A_j in terms of columns $A_j = [A_{j1}, \ldots, A_{jn}]$, then

$$B_i = [A_{0i}, \ldots, A_{ni}], \quad i = 1, \ldots, n.$$

The linear map $\mathfrak{a}$ can be considered as a bilinear map (or, as a set of n bilinear forms)

$$\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^n, \quad (\vec{x}, \vec{y}) \mapsto (\vec{x} \cdot B_1 \cdot \vec{y}, \ldots, \vec{x} \cdot B_n \cdot \vec{y}).$$

The map $T_\mathfrak{a}$ is given by assigning to $[\vec{x}] \in |E|$ the intersection of the kernels of the matrices $\vec{x} \cdot B_i$. The common kernel is one-dimensional if $[\vec{x} \notin \text{Supp}(Z)$. This is equivalent to that the n vectors $\vec{x} \cdot B_i$ are linearly independent. The inverse map is defined by assigning to $[\vec{y}]$, the intersection of the kernels of the matrices $B_i \cdot \vec{y} = \vec{y} \cdot {}^tB_i$. Since the rank of a matrix and its transpose are the same, we obtain, that the inverse map is well-defined on the complement of $\text{Supp}(Z)$.

Note that $T_\mathfrak{a}^{-1} = T_\mathfrak{a}$ if we can choose coordinates in V, E, W such that the matrices B_i are symmetric.

The graph of the map $T_\mathfrak{a}$ is the closed subset of $\mathbb{P}^n \times \mathbb{P}^n = |E| \times |V^\vee|$ given by n divisors of type $(1,1)$ corresponding to the matrices B_i considered as bilinear forms on $E \times V^\vee$. The matrices $B_1, \ldots, B_n$ are linearly independent since

$$\sum_{i=1}^n \lambda_i B_i = [\sum_{i=1}^n \lambda_i A_{0i}, \ldots, \sum_{i=1}^n \lambda_i A_{ni}] = 0$$

for some $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \neq 0$ implies that $A_0 \cdot \vec{\lambda} = \ldots = A_0 \cdot \vec{\lambda} = 0$, hence $A(x) \cdot \vec{\lambda} = (x_0A_0 + \ldots + x_nA_n) \cdot \vec{\lambda} = 0$. This means that $\text{rank}A(t) < n$ for all $[x] \in \mathbb{P}^n$ contradicting our assumption. Now we can compute the cohomology class of the graph. It is equal to

$$(h_1 + h_2)^n = \sum_{k=0}^n \binom{n}{k} h_1^i h_2^{n-i}.$$

$\square$

Consider the tensor $\mathfrak{a}$ as a linear map

$$\psi : W \to E \otimes V^\vee = (E^\vee \otimes V)^\vee. \tag{3.10}$$

We view the target space as the space of bilinear forms on $E^\vee \times V$, or as a space of linear maps $E^\vee \to V^\vee$. By choosing bases in E and V, it can be identified with the linear space of square matrices of size $n+1$. The injectivity of the map $E \to \mathrm{Hom}(W,V)$ implies that ψ is injective, so we can identify W with its image. It is a linear n-dimensional subspace of the space of bilinear forms. Let $D_k \subset W$ be the closed subvariety of bilinear forms of rank $\le k$. Its equations in the affine space $W \cong \mathbb{C}^{n+1}$ are $k \times k$-minors of the matrix representing the bilinear form. Let D_k be the image of D_k in the projective space $|W|$. We have a regular map

$$\mathfrak{l} : \mathsf{D}_n \setminus \mathsf{D}_{n-1} \to |E| \quad (\text{resp. } \mathfrak{r} : \mathsf{D}_n \setminus \mathsf{D}_{n-1} \to |V^\vee|)$$

which assigns to $z = [w]$ the left (resp. the right) kernel of the bilinear form $\psi(w)$. By definition, the image of the map $\mathfrak{l}$ (resp. $\mathfrak{r}$) is contained in the base locus of the rational map $T_\mathfrak{a}$ (resp. $T_\mathfrak{a}^{-1}$).

Note the special case when $E = V^\vee$ and $T_\mathfrak{a} = T_\mathfrak{a}^{-1}$. In this case W lies in the space of symmetric bilinear forms and the two maps $\mathfrak{l}$ and $\mathfrak{r}$ coincide.

Example 3.4.1. Consider the *standard Cremona transformation* of degree n in $\mathbb{P}^n$ given by

$$T_{\mathrm{st}} : [x_0, \ldots, x_n] \mapsto [\frac{x_0 \cdots x_n}{x_0}, \ldots, \frac{x_0 \cdots x_n}{x_n}]. \tag{3.11}$$

In affine coordinates, $z_i = t_i/t_0$, it is given by the formula

$$(z_1, \ldots, z_n) \mapsto (z_1^{-1}, \ldots, z_n^{-1}).$$

The transformation T_{st} is an analogue of the standard quadratic transformation of the plane in higher-dimension.

The base ideal of T_{st} is generated by the polynomials $t_1 \cdots t_n, \ldots, t_0 \cdots t_{n-1}$. It is equal to the ideal generated by the maximal minors of the $n \times n$ matrix

$$A(t) = \begin{pmatrix} t_0 & 0 & \cdots & 0 \\ 0 & t_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_{n-1} \\ -t_n & -t_n & \ldots & -t_n \end{pmatrix}$$

The matrix $A(t)$ defines a resolution (3.7) of the base scheme of T_{st} equal to the union of the coordinate subspaces of codimension 2.

It follows from the proof of Proposition 3.4.1 that the graph of T_{st} is isomorphic to the closed subvariety X of $\mathbb{P}^n \times \mathbb{P}^n$ given by n bilinear equations

$$x_i y_i - x_n y_n = 0, \quad i = 0, \ldots, n-1.$$

It is a smooth subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ isomorphic to the blow-up of the union of coordinate subspaces of codimension 2. The action of the torus $(\mathbb{C}^*)^{n+1}$ on $\mathbb{P}^n$

(by scaling the coordinates) extends to a biregular action on X. The corresponding toric variety is a special case of a toric variety defined by a fan formed by fundamental chambers of a root system of a semi-simple Lie algebra. In our case the root system is of type A_n, and the variety X is denoted by $X(A_n)$. In the case $n = 2$, the toric surface $X(A_2)$ is a Del Pezzo surface of degree 6 isomorphic to the blow-up of 3 points in the plane, no three of which are collinear.

It is classically known and, it is an easy fact, that a Del Pezzo surface of degree 6 embeds by the anti-canonical linear system into $\mathbb{P}^8$ as a complete intersection of the Segre variety $s(\mathbb{P}^2 \times \mathbb{P}^2)$ and two hyperplanes. The previous proof gives the analogous fact for any $X(A_n)$. Namely, $X(A_n)$ embeds in $\mathbb{P}^{n^2+2n}$ as a complete intersection of the Segre variety $s(\mathbb{P}^n \times \mathbb{P}^n)$ and a linear space of codimension n. By computing the canonical class of $X(A_n)$, one can show that the embedding is given by the anti-canonical linear system.

Example 3.4.2. Let Z be a connected closed one-dimensional subscheme of $\mathbb{P}^3$ with Hilbert function $6t - 2$ and $h^0(\mathcal{O}_Z) = 1$. In particular, $[Z] = 6h^2$ and the arithmetic genus $h^1(\mathcal{O}_Z) = 3$. Assume Z is arithmetically Cohen-Macaulay. Then the canonical map

$$\phi_j : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(j)) \to H^0(\mathbb{P}^3, \mathcal{O}_C(j))$$

is surjective for all $j \in \mathbb{Z}$. Since $\deg \mathcal{O}_C(j) = 6j > 2p_a(Z) - 2 = 4$, we have $h^1(\mathcal{O}_C(j)) = 0$. Thus $h^0(\mathcal{O}_C(j)) = \chi(\mathcal{O}_C(j)) = 6j - 2$. Taking $j = 1$, this implies that Z is linearly normal, i.e. it is not a projection of any scheme from a higher-dimensional space. Taking $j = 2$, this implies that $h^0(\mathcal{J}_Z(2)) = 0$, i.e. Z does not lie on a quadric. When Z is reduced and irreducible, these two conditions imply that Z is ACM (see [Ellingsrud, Ann. Ec. Norm. Sup., 1974], p. 430).

Taking the cohomology of the exact sequence

$$0 \to \mathcal{J}_Z \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Z \to 0,$$

we obtain

$$h^2(\mathcal{J}_Z) = h^1(\mathcal{O}_Z) = 3.$$

Using resolution (3.6), we obtain

$$h^2(\mathcal{J}_Z) = \sum_{j=1}^{3} h^3(\mathcal{O}_{\mathbb{P}^3}(-b_j) = \sum_{j=1}^{3} h^0(\mathcal{O}_{\mathbb{P}^3}(b_j - 4) = 3.$$

The only solution is $b_1 = b_2 = b_3 = 4$. Applying Riemann-Roch, we see that $h^0(\mathcal{J}_Z(3)) = 4$, hence

$$4 = \sum_{i=1}^{4} h^0(\mathcal{O}_{\mathbb{P}^3}(3 - a_i)) - \sum_{j=1}^{3} h^0(\mathcal{O}_{\mathbb{P}^3}(3 - b_j)) = \sum_{i=1}^{4} h^0(\mathcal{O}_{\mathbb{P}^3}(3 - a_i)).$$

Since Z does not lie on a quadric, $0 = \sum_{i=1}^{4} h^0(\mathcal{O}_{\mathbb{P}^3}(2 - a_i))$, hence all $a_i > 2$. Together, this easily implies that $a_1 = \ldots = a_4 = 3$. Thus the resolution (3.4) equals the resolutiion (3.7). So the linear system of cubics through R is a homoloidal linear system and the multi-degree of the Cremona transformation is equal to $(1, 3, 3, 1)$. Because the degree of the transformation and its inverse are equal to 3, such transformation are classically known as *cubo-cubic transformations*.

Assume Z is a smooth curve C and let us describe the P-locus of the corresponding Cremona transformation. Obviously, any line intersecting C at three distinct points (a *trisecant line*) must be blown down to a point (otherwise a general cubic in the linear system intersects the line at more than 3 points). Consider the *trisecant locus* Tri(C) of C, the closure in $\mathbb{P}^3$ of the union of lines intersecting C at three points. Note that no line intersects C at > 3 points because the ideal of C is generated by cubic surfaces. Consider the linear system of cubics through C. If all of them are singular, by Bertini's Theorem, there will be a common singular point at the base locus, i.e. at C. But this easily implies that C is singular, contradicting our assumption. Choose a nonsingular cubic surface S containing C. By adjunction formula, we have $C^2 = -K_S \cdot C + \deg K_Z = 6 + 4 = 10$. Take another cubic S' containing C. The intersection $S \cap S'$ is a curve of degree 9, the residual curve A is of degree 3 and $C + A \sim -3K_S$ easily gives $C \cdot A = 18 - 10 = 8$. Note that the curves A are the proper transforms of lines under the Cremona transformation. So they are rational curves of degree 3. We know that the base scheme of the inverse transformation f^{-1} is a curve of degree 6 isomorphic to C. Replacing f with f^{-1} we know that the image of a general line ℓ under f is a rational curve of degree 3 intersecting C' at 8-points. These points are are the images of 8 trisecants intersecting ℓ. This implies that the degree of the trisecant surface Tri(C) is equal to 8. Since the degree of the determinant of the Jacobian matrix of a transformation of degree 3 is equal to 8, we see that there is nothing else in the P-locus.

The linear system of planes containing ℓ cuts out on C a linear series of degree 6 with moving part of degree 3. It is easy to see, by using Riemann-Roch, that any g_3^1 on a curve of genus 3 must be of the form $|K_C - x|$ for a unique point $x \in C$. Conversely, for any point $x \in C$, the linear system $|\mathcal{O}_C(1) - K_C + x|$ is of dimension 0 and of degree 3 (here we use that $|\mathcal{O}_C(1)| = |K_C + \mathfrak{a}|$, where $\mathfrak{a}$ is not effective divisor class of degree 2). Thus it defines a trisecant line (maybe tangent at some point). This shows that the curve R parameterizing trisecant lines is isomorphic to C. This agrees with the fact that R must be isomorphic to the base curve of the inverse transformation. The Cremona transformation can be resolved by blowing up the curve C and then blowing down the proper transform of the surface Tri(C). The exceptional divisor is isomorphic to the minimal ruled surface with the base C. It is the universal family of lines parameterized by C. Its image in the target $\mathbb{P}^3$ is surface Tri(C'), where C' is the base locus of of the inverse transformation (the same curve only re-embedded by the linear system $|K_C + a'|$, where $a' \in |K_C - a|$).

Let $W = H^0(\mathbb{P}^3, \mathcal{J}_C) \cong \mathbb{C}^3$. The hypersurface D_2 is a plane quartic curve. The images of maps $\mathfrak{l} : |W_2 \to \mathbb{P}^3$ and $\mathfrak{r} : |W_2 \to \mathbb{P}^3$ are isomorphic to C. The two embeddings of D_2 in $\mathbb{P}^3$ are given by $|K_C + \mathfrak{a}|$ and $|2K_C - \mathfrak{a}|$. This gives an equation of a plane quartic curve as the determinant of a square matrix of linear forms. Up to a natural equivalence of such representations (replacing the matrix of linear forms by row and column transformations) the set of such representations is parameterized by $\mathrm{Pic}^2(C) \setminus \Theta$, where Θ is the hypersurface of effective divisor classes of degree 2). If $\mathfrak{a} = '$, then $2\mathfrak{a} = K_C$, so that $\mathfrak{a}$ is a *theta characteristic*. Since $\mathfrak{a}$ is not effective, it is an even theta characteristic. It is known that the number of even theta characteristics on a nonsingular curve of genus 3 is equal to 36. In this case we can identify the spaces $|K_C - \mathfrak{a}|^\vee = |U|$ and $|K_C - \mathfrak{a}'|^\vee = |V^\vee|$, and the matrix defining D_3 can be chosen to be a symmetric matrix.

Example 3.4.3. Let Z be the union of 4 skew lines in $\mathbb{P}^3$ and their two *transversals*, i.e. lines intersecting all the four lines. It is easy to compute that Z is a curve of arithmetic genus 3, obviously of degree 6. Assume that the four lines do not lie on a quadric. Then Z is arithmetically Cohen-Macualay and can be taken as the base scheme of a cubo-cubic Cremona transformation.

The construction of two transversals to 4 skew lines $\ell_1, \ldots, \ell_4$ is as follows. Choose a set of three lines among the given 4 lines, say ℓ_1, ℓ_2, ℓ_3. By counting constants, we see that there is a unique quadric Q containing these lines. Since the lines are skew, the quadric is nonsingular, and the three lines belong to one of its rulings by lines. The line ℓ_4 intersects the quadric Q at two points p, q. The two lines passing through p and q from the other ruling are the two transversals. When the line ℓ_4 is tangent to Q, we have $p = q$, so the two transversals degenerate to a double structure of one transversal.

Obviously, any line on Q from the ruling containing the first three lines belongs to the P-locus. Thus Q belongs to a P-locus. Choosing four subsets of three lines among $\ell_1, \ldots, \ell_4$, we obtain that the union of 4 quadrics belongs to P-locus. Since the degree of the determinant of the jacobian matrix is equal to 8, there is nothing else in the P-locus.

Example 3.4.4. Consider the double structure Z on a twisted cubic curve C in $\mathbb{P}^3$ defined by an invertible sheaf $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(-4)$. Exact sequence (2.28) gives an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-4) \to \mathcal{O}_Z \to \mathcal{O}_C \to 0.$$

Computing cohomology, we find that, for $n \geq 1$ $h^0(\mathcal{O}_Z(n)) = h^0(\mathcal{O}_{\mathbb{P}^1}(3n)) + h^0(\mathcal{O}_{\mathbb{P}^1}(3n-4)) = 6n - 2$. Thus the Hilbert polynomial of Z is the same as for the subscheme defined by a resolution (3.7). Tensoring the exact sequence

$$0 \to \mathcal{J}_Z \to \mathcal{J}_C \to \mathcal{O}_{\mathbb{P}^1}(-4) \to 0$$

by $\mathcal{O}_{\mathbb{P}^3}(3)$, we get $h^0(\mathcal{O}_Z(3)) = h^0(\mathcal{O}_{\mathbb{P}^1}(9)) - h^0(\mathcal{O}_{\mathbb{P}^1}(5)) = 4$. Here we used that $h^1(\mathcal{J}_Z(3)) = 0$. This can be verified by using the exact sequences

$$0 \to \mathcal{J}_Z(3) \to \mathcal{J}_C(3) \to \mathcal{O}_{\mathbb{P}^3}(-1) \to 0,$$

and the fact that $h^1(\mathcal{J}_C(3)) = 0$. It follows from calculations in Example 2.7.2 that the linear system $|\mathcal{J}_Z(3)|$ is homaloidal.

Let us write $\mathbb{P}^3$ as $|U| = \mathbb{P}(U^\vee)$ for some two-dimensional linear space U and identify C with the image of the Veronese map

$$v_3 : \mathbb{P}^1 = |U| \to \mathbb{P}^3 = |S^3U|, [u] \mapsto [u^3].$$

Binary cubic forms $\phi \in S^3U$ with three distinct zeros in $|U^\vee|$ form an orbit with respect to $\mathrm{SL}(U)$. Each element of this orbit can be uniquely written as a sum of two forms u^3 and v^3, hence lies on a unique secant of C. A form ϕ with two distinct zeros form another orbit. The closure of this orbit is the tangential surface of C, the closure of the union of tangent lines to C. Finally, the curve C itself is the third orbit, the unique closed orbit. If we choose a basis η_0, η_1 in U and view an element of S^3U as a binary form

$$\phi = a_0\eta_0^3 + 3a_1\eta_0^2\eta_1 + 3a_2\eta_0\eta_1^2 + a_3\eta_1^3 \tag{3.12}$$

on the dual space $U^\vee$, then the equation of the tangential scroll $\mathrm{Tan}(C)$ of C is given by the discriminant of the binary cubic form

$$D_3 = 3t_1^2t_2^2 - 4t_0t_2^3 - 4t_1^3t_3 - t_0^2t_3^2 + 6t_0t_1t_2t_3 = 0. \tag{3.13}$$

For this reason surface $\mathrm{Tan}(C)$ is also called the *discriminant quartic surface*. The curve C is its cuspidal curve. Consider a natural bilinear map of $\mathrm{SL}(U)$-modules

$$S^3U \otimes S^3U \to S^2U \tag{3.14}$$

equal to the projection in the Clebsch-Gordan decomposition of the representations of $\mathrm{SL}(U)$:

$$S^3U \otimes S^3U \to S^6U \oplus S^4U \oplus S^2U \oplus \mathbb{C}.$$

In coordinates, it is given by the second transvectant $\mathfrak{t}_2$. If we view an element of S^3U as a binary form

$$\phi = a_0\eta_0^3 + 3a_1\eta_0^2\eta_1 + 3a_2\eta_0\eta_1^2 + a_3\eta_1^3 \tag{3.15}$$

on the dual space $U^\vee$ with coordinates η_0, η_1, then (3.14) is given by the formula

$$(\phi, \psi) \mapsto \phi_{00}\psi_{11} - 2\phi_{01}\psi_{10} + \phi_{11}\psi_{00},$$

where the subscripts indicate the second partial derivatives with respect to η_0, η_1. Explicitly, (ϕ, ψ) is mapped to $c_0\eta_0^2 + c_1\eta_0\eta_1 + c_2\eta_1^2$, where

$$\begin{aligned} c_0 &= a_0b_2 - 2a_1b_1 + a_2b_0 \\ c_1 &= a_0b_3 - a_1b_2 - a_2b_1 + a_3b_0 \\ c_2 &= a_1b_3 - 2a_2b_2 + a_3b_1. \end{aligned}$$

The pairing defines our tensor $\mathfrak{a} \in S^2U \otimes (S^3U)^\vee \otimes (S^3U)^\vee$. In previous notations $W = S^2U^\vee, E = S^3U, V = S^3U^\vee$. Note that the $\mathrm{SL}(U)$-representations U and $U^\vee$ are canonically isomorphic via the determinant map $U \otimes U \to \Lambda^2 U \cong \mathbb{C}$.

The three 4×4-matrices B_i defining the three bilinear forms are the matrices of the bilinear forms c_0, c_1, c_2. The matrix A of linear forms defining the resolution (3.7) is the following

$$A(t) = \begin{pmatrix} t_2 & t_3 & 0 \\ -2t_1 & -t_2 & t_3 \\ t_0 & -t_1 & -2t_2 \\ 0 & t_0 & t_1 \end{pmatrix}$$

Computing the maximal minors we find them to be proportional to the partial derivatives of the quartic form D_4

$$3t_1t_2t_3 - 2t_2^3 - t_0t_3^2,\ t_0t_2t_3 + t_1t_2^2 - 2t_1^2t_3,\ t_0t_1t_3 + t_1^2t_2 - 2t_0t_2^2,\ 3t_0t_1t_2 - 2t_1^3 - t_0^2t_3.$$

The linear system formed by the partial derivatives is our homaloidal linear system $|\mathcal{J}_Z(3)|$. Since $E \cong V^\vee$, we can canonically identify the source space $|E|$ with the target space $\mathbb{P}^3 = |V^\vee|$.

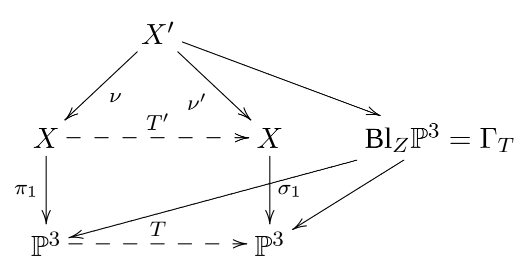
The columns of the matrix $A(t)$ give us the relations between the partial derivatives. As in the proof of Theorem 3.4.1, we find that the blow-up $\mathrm{Bl}_Z\mathbb{P}^3$ is isomorphic to the subscheme of $\mathbb{P}^3 \times \mathbb{P}^3$ given by three bilinear equations

$$\begin{aligned} t_3u_0 - 2t_1u_1 + t_0u_2 &= 0, \\ t_0u_3 - t_1u_2 - t_2u_1 + t_3u_0 &= 0, \\ t_1u_3 - 2t_2u_2 + t_3u_1 &= 0. \end{aligned}$$

Its intersection with the diagonal of $\mathbb{P}^3 \times \mathbb{P}^3$ is isomorphic to the closed subscheme Y of $\mathbb{P}^3$ given by the equations, obtained from the previous equations by setting $u_i = t_i, i = 0, 1, 2, 3$. It is easy to see that these equations define a twisted cubic. By taking the affine pieces $U_{ij} : u_i = t_j = 1$ in $\mathbb{P}^3 \times \mathbb{P}^3$, and computing the matrix of the partial derivatives at each point of Y, we find that that Y is a singular curve on $\mathrm{Bl}_Z\mathbb{P}^3$. Note that the exceptional divisor is isomorphic to the minimal rational ruled surface $\mathbf{F}_2$. Its exceptional section is the curve of singularities of $\mathrm{Bl}_Z\mathbb{P}^3$. The proper transform of $\mathrm{Tan}(C)$ under the first projection on the blow-up scheme is its resolutions of singularities. It is isomorphic to a quadric. Under the second projection to $\mathbb{P}^3$, it is blown-down to C. The restriction of the linear system $|\mathcal{J}_Z(3)|$ to a general plane H is the linear system of cubic curves in H passing through the double structure on the set of three points $H \cap C$. It consists of three points with given tangent directions. A Cremona transformation T defined by the homaloidal linear system $|\mathcal{J}_Z(3)|$ maps H to a cubic surface with 3 double points projectively isomorphic to the surface $w^3 + xyz = 0$.

The P-locus of T is the discriminant surface $\mathrm{Tan}(C)$. It should be thought as a degeneration of the trisecant octic surface of a nonsingular curve of genus 3 and degree 6 in $\mathbb{P}^3$. It is easy to see that the normal bundle of C in $\mathbb{P}^3$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$. Using the computations in Example 2.7.2, we obtain that the exceptional divisor E_1 of the blow-up of $X = \mathrm{Bl}_C\mathbb{P}^3$ is isomorphic to $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. The section R of E_1 defined by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(-4)$ defining the double structure is a curve on E_1 of bi-degree $(1, 1)$. The restriction of the blowing down morphism $\pi_1 : X \to \mathbb{P}^3$ to E_1 is defined by one of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The proper transform F of $\mathrm{Tan}(C)$ on X is isomorphic to $\mathbf{F}_0$. The curve R on $\mathbf{F}_0$ is also of type $(1, 1)$. The two surfaces E_1 and F are tangent to each other along the curve R (recall that C is a cuspidal curve on $\mathrm{Tan}(C)$).

The exceptional divisor E_2 of the blow-up $\nu : X' = \mathrm{Bl}_R X \to X$ is also isomorphic to $\mathbf{F}_0$. The proper transform E_1' of E_1, the proper transform F' of F intersect E_2 along the same curve isomorphic to R. The proper transform of the homaloidal linear system on X' is equal to the linear system $\mathcal{H} = |3H - E_1' - 2E_2|$, where H is the full inverse image of a plane in $\mathbb{P}^3$. One checks that the restriction of $\mathcal{H}$ to E_1' is the linear system $|3f|$, where f is a fibre of one of the projections $\mathbf{F}_0 \to \mathbb{P}^1$. The variety X' defines a smooth resolution of the Cremona transformation. The morphism $\pi : X' \to \mathbb{P}^3$ is the composition $\pi_1 \circ \nu$. The morphism $\sigma : X' \to \mathbb{P}^3$ is the composition $\sigma_1 \circ \nu'$, where $\nu' : X' \to X$ is the blow-down of the proper transform of F' and $\sigma_1 : X \to \mathbb{P}^3$ is the blowing down of the image of E_1' on X to a twisted cubic C' in the target $\mathbb{P}^3$. The image of E_2 is the tangent surface of C'.



Note that the blow-up $\mathrm{Bl}_Z\mathbb{P}^3$ is obtained from X' by blowing down E_1' to the curve of singularities of $\mathrm{Bl}_Z\mathbb{P}^3$. It resolves the singularities of $\mathrm{Bl}_Z\mathbb{P}^3$. One should think about E_1' as an analog of a (-2)-curve, and F' as analog of (-1)-curve.

Finally observe that the determinantal quartic hypersurface D_3 is equal to the double conic. The 2-dimensional linear system of quadrics on $\mathbb{P}^3$ is equal to the net $|\mathcal{J}_R(2)|$. We may think that the double structure on R is the scheme of singular points of quadrics in the net. The situation is very similar to the resolution of the planar quadratic map with one infinitely near point.

Example 3.4.5. Let $n = 4$. The Hilbert function of Z is equal to $5(k-1)^2$. Thus it is a surface of degree 10 and $\chi(\mathcal{O}_Z) = 5$. It admits a degeneration into the union of 10 coordinate planes. The base ideal is generated by five polynomials of degree 4, the maximal minors of a matrix $A(t)$ of size 5×4. If Z is reduced, it is birationally isomorphic to the discriminant surface $\mathsf{D}_4 \subset \mathbb{P}^3$ of degree 5. It is known that, for a general matrix $A(t)$, the discriminant surface is smooth.

3.5 Monomial birational maps

Let X_Σ be a toric variety defined by a fan Σ in a lattice $N \cong \mathbb{Z}^n$. Let $g \in \mathrm{GL}_n(N)$ be an automorphism of N. It defines an isomorphism of the toric varieties X_Σ and $X_{\Sigma'}$, where $\Sigma' = g(\Sigma)$. Let us identify these two toric varieties by means of this isomorphism. Let Π be the minimal common subdivision of Σ and Σ'. Then projections $\pi : X_\Pi \to X_\Sigma$ and $\sigma : X_\Pi \to X_{\Sigma'}$ define a smooth resolution of the monomial Cremona transformation $f = g \circ \pi^{-1}$. On the dense open torus orbit identified with $(\mathbb{C}^*)^n$ with coordinates $z_1, \ldots, z_n$, the map f given by monomials whose exponent vectors are the rows of the matrix g

$$f : [z_1, \ldots, z_n] \mapsto [\mathbf{z}^{\mathbf{m}_1}, \ldots, \mathbf{z}^{\mathbf{m}_n}].$$

Let us take X_Σ to be equal to $\mathbb{P}^n$ with the toric structure defined by the standard fan with the 1-skeleton consisting of rays $\mathbb{R}_{\geq 0}\mathbf{e}_1, \ldots, \mathbb{R}_{\geq 0}\mathbf{e}_n, \mathbb{R}_{\geq 0}\mathbf{e}_0$, where $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is the standard basis of $\mathbb{R}^n$ and $\mathbf{e}_0 = -(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$. Recall that ample torus-linearized invertible sheaves on X_Π corresponds to convex lattice polytopes in the space $M_\mathbb{R}$, where $M = N^\vee$ is the dual lattice. The sheaf $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ corresponds to the simplex Σ_n spanned by the vectors $0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ and the sheaf $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1)$ corresponds to the simplex $g(\Sigma_n)$ spanned by the vectors $0, g(\mathbf{e}_1), \ldots, g(\mathbf{e}_n)$. The intersection $h_1^k h_2^{n-k}$ on X_Π is equal to the mixed volumes

$$d_k = \mathrm{Vol}(\Sigma_n, k; g(\Sigma_n), n-k).$$

If g is the linear map $v \mapsto -v$, we get the standard Cremona transformation of degree n. From our computation, we see that $d_k = \binom{n}{k}$. We do not know how to compute these numbers for an arbitrary $g \in \mathrm{GL}_n(\mathbb{Z})$. However, formulas in Remark 1.4.1 allow us to compute the degree of a monomial map.

The group $\mathrm{GL}(n,\mathbb{Z})$ is generated by the transformations

$$\begin{aligned} g_1 &: (m_1,\ldots,m_n) \mapsto (m_2,m_3,\ldots,m_n,m_1), \\ g_2 &: (m_1,\ldots,m_n) \mapsto (m_1,m_2+m_3,m_3,\ldots,m_n), \\ g_3 &: (m_1,\ldots,m_n) \mapsto (-m_1,m_2,m_3,\ldots,m_n). \end{aligned}$$

(see [Coxeter], 7.2). The corresponding monomial Cremona transformations are

$$\begin{aligned} f_1 &: (z_1,\ldots,z_n) \mapsto (z_2,z_3,\ldots,z_n,z_1), \\ f_2 &: (z_1,\ldots,z_n) \mapsto (z_1,z_2z_3,z_3,z_4,\ldots,z_n), \\ f_3 &: (z_1,\ldots,z_n) \mapsto (z_1^{-1},z_2,z_3,\ldots,z_n). \end{aligned}$$

The first transformation is a projective automorphism of $\mathbb{P}^n$, the other ones are quadratic transformations. Thus the subgroup of the group $\mathrm{Cr}(n)$ of Cremona transformations of $\mathbb{P}^n$ (the *Cremona group of degree* n) generated by monomial transformations is generated by projective and quadratic transformations.

Lecture 4

Involutions

4.1 De Jonquières involutions

We will be often using the following classical notion of *harmonically conjugate* pairs of points on the projective line.

Let $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ be an involution of $\mathbb{P}^1$. It is given by the deck transformation of a degree two map $u : \mathbb{P}^1 \to \mathbb{P}^1$ which, in its turn, is given by a linear series g_2^1 and a choice of its basis. Clearly, it is uniquely defined by a choice of two members $a+b$ and $c+d$ of g_2^1. In particular, we can choose the two members to be the two divisors of the form $2a$ and $2c$ defined by the two ramification points of u. Another choice is to choose a reduced divisor $a+b$ and one non-reduced divisor $2c$. Then the second non-reduced divisor $2d$ is defined uniquely. This suggests the two ways to define an involution on $\mathbb{P}^1$. Choose two distinct points a, b which will be in involution, and assign to a point c the unique point $\gamma(c)$ such that $2c$ and $2\gamma(c)$ are the two non-reduced divisors of g_2^1. Another way, choose two distinct points a, b and define the unique g_2^1 containing $2a$ and $2b$.

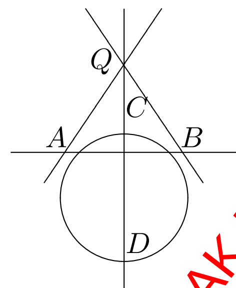
Definition 4.1.1. *Two pairs $\{a, b\}$ and $\{c, d\}$ points on $\mathbb{P}^1$ are called harmonically conjugate, if there exists a g_2^1 such that $2a, 2b, c+d \in g_2^1$.*

Lemma 4.1.1. *Let a, b be the zeros of a binary form $\alpha t_0^2 + 2\beta t_0 t_1 + \gamma t_1^2$ and c, d be the zeros of a binary form $\alpha' t_0^2 + 2\beta' t_0 t_1 + \gamma' t_1^2$. Then the pairs $\{a, b\}$ and $\{c, d\}$ are harmonically conjugate if and only if*

$$\alpha\gamma' + \alpha'\gamma - 2\beta\beta' = 0. \tag{4.1}$$

Proof. Consider the Veronese map $[t_0, t_1] \mapsto [t_0^2, t_0 t_1, t_2^2]$ with the image equal to the conic $C = V(x_0 x_2 - x_1^2)$. The points a, b are mapped to the intersection points A, B of the line $V(\alpha x_0 + 2\beta x_1 + \gamma x_2)$ with the conic. The linear series g_2^1

containing $2a$ and $2b$ is the pre-image of the linear series on C cut out by the pencil of lines through the intersection point Q of the tangent lines of C at the points p_1 and p_2. This is the point $[u_0, u_1, u_2]$ such that its polar line with respect to C is equal to the line $\langle p_1, p_2\rangle$. The equation of the polar line is $u_2x_0 - 2u_1x_1 + u_0x_2 = 0$. Thus $[u_0, u_1, u_2] = [\alpha, -2\beta, \gamma]$. A divisor $c + d$ belongs to g_2^1 if and only it is mapped to the intersection points C, D of the conic with a line passing through Q. This happens if and only if $\alpha'\alpha - 2\beta\beta' + \gamma\gamma' = 0$. □



Note that, as a corollary, we obtain that the definition is symmetric with respect to the pairs $\{a, b\}$ and $\{c, d\}$. Also, if we choose the coordinates t_0, t_1 in $\mathbb{P}^1$ such that $\{a, b\} = V(t_0t_1)$, then a member $c + d$ of the pencil generated by $2a$ and $2b$ are zeros of a binary form $\lambda t_0^2 + \mu t_2^2$. In affine coordinates $a = 0, b = \infty, c = z, d = -z$, and the cross-ratio

$$R(c, a, b, d) = \frac{(c-a)(b-d)}{(c-b)(a-d)} = \frac{(z-0)(\infty+z)}{(z-\infty)(0+z)} = -1.$$

This gives another equivalent definition of harmonically conjugate pairs. Another equivalent definition is that the double cover of $\mathbb{P}^1$ ramified at the union of two pairs is an elliptic curve with complex multiplication by $\sqrt{-1}$.

Let X be a reduced irreducible hypersurface of degree m in $\mathbb{P}^n$ which contains a point $\mathfrak{o}$ of multiplicity $m - 1$. We call such a hypersurface *submonoidal*. For example, every smooth hypersurface of degree ≤ 3 is submonoidal.

Let us choose the coordinates such that $\mathfrak{o} = [1, 0, \ldots, 0]$. Then X is given by an equation

$$F_m = t_0^2 a_{m-2}(t_1, \ldots, t_n) + 2t_0 a_{m-1}(t_1, \ldots, t_n) + a_m(t_1, \ldots, t_n) = 0, \tag{4.2}$$

where the subscripts indicate the degrees of the homogeneous forms. For a general point $x \in X$ consider the intersection of the linear span $\ell_x = \langle \mathfrak{o}, x\rangle$ with X. It contains $\mathfrak{o}$ with multiplicity $m - 2$ and the residual intersection is a set of two points a, b in ℓ_x. Consider the involution on ℓ_x with fixed points a, b. Define $T(x)$ to be the point on ℓ_x such that the pair x and $T(x)$ are conjugate with respect to the involution. In other words, the pairs $\{a, b\}$ and $\{x, T(x)\}$ are harmonically conjugate. We call it a *De Jonquières involution* (observe that $T = T^{-1}$).

Let us find an explicit formula for the De Jonquières involution which we have defined. Let $x = [x_0, \ldots, x_n]$ and let $[u + vx_0, vx_1, \ldots, vx_n]$ be the parametric equation of the line ℓ_x. Plugging in (4.2), we find

$$(u + vx_0)^2 v^{m-2} a_{m-2}(x_1, \ldots, x_n) + 2(u + vx_0)v^{m-1} a_{m-1}(x_1, \ldots, x_n)$$
$$+ v^m a_m(x_1, \ldots, x_n) = 0.$$

Canceling v^{m-2}, we see that the intersection points of the line ℓ_x with X are the two points corresponding to the zeros of the binary form $Au^2 + 2Buv + Cv^2$, where

$$(A, B, C) = (a_{m-2}(x), x_0 a_{m-2}(x) + a_{m-1}(x), F_m(x)).$$

The points x and $T(x)$ corresponds to the parameters satisfying the quadratic equation $A'u^2 + 2B'uv + C'v^2 = 0$, where $AA' + CC' - 2BB' = 0$. Since x corresponds to the parameters $[0, 1]$, we have $C' = 0$. Thus $T(x)$ corresponds to the parameters $[u, v] = [-C, B]$, and

$$T(x) = [-C + Bx_0, Bx_1, \ldots, Bx_n].$$

Plugging the expressions for C and B, we obtain the following formula for the transformation T

$$\begin{aligned} x'_0 &= -x_0 a_{m-1}(x_1, \ldots, x_n) - a_m(x_1, \ldots, x_n), \\ x'_i &= x_i(a_{m-2}(x_1, \ldots, x_n)x_0 + a_{m-1}(x_1, \ldots, x_n)), \quad i = 1, \ldots, n. \end{aligned}$$

Observe that T is a dilation of the identity transformation of $\mathbb{P}^{n-1}$. The divisors from the homaloidal linear system are hypersurfaces of degree m which have singular points of multiplicity $\geq m - 1$ at $\mathfrak{o}$. Such hypersurfaces are classically known as *monoidal hypersurfaces*. In the notation of section 2.8, we have $G_i = t_i, i = 1, \ldots, n, F_0 = -x_0 a_{m-1}(x_1, \ldots, x_n) - a_m(x_1, \ldots, x_n), Q = a_{m-2}(x_1, \ldots, x_n)x_0 + a_{m-1}(x_1, \ldots, x_n)$. So, $d = 1, r = m - 1$. Since the identity transformation has no base points, we obtain that the multi-degree is equal to $(m, m, \ldots, m)$. In particular, we see that any such vector is realized as the multi-degree of a Cremona transformation.

Assume that $m > 2$. The base scheme of the homaloidal linear system $\mathcal{H}$ defining T is equal to

$$\text{Bs}(\mathcal{H}) = V(t_0 a_{m-1} + a_m, t_1(a_{m-2}t_0 + a_{m-1}), \ldots, t_n(a_{m-2}t_0 + a_{m-1})).$$

Note that $V(a_{m-2}t_0 + a_{m-1}) = V(\frac{\partial F_m}{\partial t_0})$ is the polar hypersurface $P_{\mathfrak{o}}(X)$ of X with respect to the point x. Also, $V(t_0 a_{m-1} + a_m) = V(-F_m + \frac{\partial F_m}{\partial t_0})$. Thus the base scheme contains the intersection $X \cap P_{\mathfrak{o}}(X)$. Set-theoretically, this intersection is equal to the locus of points $x \in X$ such that the embedded tangent hyperplane $\mathbb{T}_x X$ contains the point $\mathfrak{o}$. In particular, all singular points of X are contained in the base locus. In other words, the intersection $X \cap P_{\mathfrak{o}}(X) \setminus \{\mathfrak{o}\}$ consists of the points where the projection map $\mathrm{pr}_{\mathfrak{o}} : X \setminus \{\mathfrak{o}\} \to \mathbb{P}^{n-1}$ is not smooth.

Consider the *enveloping cone* $\mathrm{EC}_{\mathfrak{o}}(X)$ of X at the point $\mathfrak{o}$. It is the union of lines through $\mathfrak{o}$ and a point on $X \cap P_{\mathfrak{o}}(X)$. Each such line either contained in X, or intersects it only at one point a besides $\mathfrak{o}$. In the first case, the line is contained in the base locus of T. In the second case, it is blown down to the point a. This easily follows from the formula for the transformation (use that $AC = B^2$ and we can take $A' = 2C, B' = B$). In our case $\mathrm{EC}_{\mathfrak{o}}(X) = V(a_{m-2}^2 - a_{m-1}a_m)$. Its degree is equal to $2(m-1)$. The other part of the P-locus is the polar hypersurface $V(\frac{\partial F_m}{\partial t_0})$. It is blown down to the point $\mathfrak{o}$. Its degree is equal to $m-1$. There is nothing else in the P-locus. When $n > 2$, we have to take the polar hypersurface with multiplicity $n-1$.

Example 4.1.1. Assume that $n = 2$. Then the variety X is a hyperelliptic curve of genus $g = m-2$ (a rational or elliptic curve if $d = 2$ or 3). If we choose the coordinates of $\mathfrak{o}$ to be $[1, 0, 0]$, then the equation of X can be given in the form

$$t_0^2 a_g(t_1, t_2) + 2t_0 a_{g+1}(t_1, t_2) + a_{g+2}(t_1, t_2) = 0. \tag{4.3}$$

The transformation is given by the formula

$$\begin{aligned} x_0' &= -x_0 a_{g+1}(x_1, x_2) - a_{g+2}(x_1, x_2), \\ x_1' &= x_1(a_g(x_1, x_2)x_0 + a_{g+1}(x_1, x_2)), \\ x_2' &= x_2(a_g(x_1, x_2)x_0 + a_{g+1}(x_1, x_2)). \end{aligned}$$

In affine coordinates $z_1 = x_2/x_1, z_2 = x_0/x_1$ it is given by the formula

$$(z_1, z_2) \mapsto \Big(z_1, \frac{-a_{g+1}(1, z_1)z_2 - a_{g+2}(1, z_1)}{a_g(1, z_1)z_2 + a_{g+1}(1, z_1)}\Big).$$

The base locus consists of $2g+2$ Weierstrass points on X and the singular point p of X taken with multiplicity $g+1$. We check

$$d_2 = (g+2)^2 - (2g+2) - (g+1)^2 = 1.$$

The P-locus consists of $2g+2$ lines joining $\mathfrak{o}$ with one of the Weierstrass points and the curve $p_{\mathfrak{o}}(X)$ of degree $g+1$ that passes through the Weierstrass points and intersects X at $\mathfrak{o}$ with multiplicity $g(g+1)$.

Note that the De Jonquières involution associated to a submonoidal hypersurface X has X in the closure of its fixed points. As we discussed in the beginning of this section, there are two involution on each line ℓ_x. One fixes the intersection point of ℓ_x with X and another interchanges them. This allows us to define another Cremona involution that commutes with the De Jonquières involution. It is not defined uniquely.

We assume that X is given by equation

$$t_0^2 a_{m-2}(t_1,\ldots,t_n) + 2t_0 a_{m-1}(t_1,\ldots,t_n) + a_m(t_1,\ldots,t_n) = 0. \tag{4.4}$$

and look for the hypersurface Y in the form

$$t_0^2 b_{m'-2}(t_1,\ldots,t_n) + 2t_0 b_{m'-1}(t_1,\ldots,t_n) + b_{m'}(t_1,\ldots,t_n) = 0. \tag{4.5}$$

The singular point $p = [1,0,\ldots,0]$. A line $up + vx = 0$ passing through a point $x \neq p$ intersects X (resp. Y) at two residual points satisfying the quadratic equation

$$u^2 a_{m-2}(x_1,\ldots,x_n) + 2uv a_{m-1}(x,\ldots,x_n) + a_m(x_1,\ldots,x_n) = 0$$

(resp.

$$u^2 b_{m'-2}(x_1,\ldots,x_n) + 2uv b_{m'-1}(x,\ldots,x_n) + b_{m'}(x_1,\ldots,x_n) = 0)$$

The condition that the two pairs are harmonically conjugate is

$$a_{m-2}b_{m'} - 2a_{m-1}b'_{m'-1} + a_m b_{m'-2} = 0. \tag{4.6}$$

If we choose Y satisfying this condition and define the corresponding De Jonquières involution T', then this involution will restrict to an involution on X. To find the coefficients b_{m-2}, b_{m-1}, b_m satisfying (4.6), we have to solve a system of linear equations. Obviously, a solution is an element of the kernel of a linear map

$$\mathbb{C}[t_1,\ldots,t_n]_{m'} \oplus \mathbb{C}[t_1,\ldots,t_n]_{m'-1} \oplus \mathbb{C}[t_1,\ldots,t_n]_{m'-2} \to \mathbb{C}[t_1,\ldots,t_n]_{m+m'-2}.$$

We expect to find a non-trivial solution when

$$\tbinom{m'+n-1}{n-1} + \tbinom{m'+n-2}{n-1} + \tbinom{m'+n-3}{n-1} > \tbinom{m+m'+n-3}{n-1}.$$

For example, when $n = 2$, the inequality is $2m' > m$.

Note that the two commuting Cremona involutions T and T' defined by two submonoidal hypersurfraces whose equations are related by (4.6) define the third involution $T'' = T \circ T'$. One can show that it is defined by the third submonoidal hypersurface of degree $m + m'$ with equation

$$\det \begin{pmatrix} a_{m-2} & a_{m-1} & a_m \\ b_{m'-2} & b_{m'-1} & b_{m'} \\ 1 & -t_0 & t_0^2 \end{pmatrix} = 0.$$

It describes the locus of common harmonically conjugate pairs of points.

4.2 Planar Cremona involutions

Recall that a Cremona transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is called an involution if $T = T^{-1}$. All involutions in the planar Cremona group have been classified (up to conjugacy). We have already see one example of an involution, the standard Cremona transformation

$$T_{\text{st}} : [x_0, x_1, x_2] \mapsto [x_1x_2, x_0, x_1, x_0x_1].$$

It is generalized to all dimensions. In affine coordinates, this transformation is given by $(z_1, z_2) \mapsto (z_1^{-1}, z_2^{-1})$. The Moebius transformation $z \mapsto z^{-1}$ of $\mathbb{P}^1$ is conjugate to the transformation $z \mapsto -z$ (by means of the map $z \mapsto \frac{z-1}{z+1}$). This shows that the standard Cremona transformation T_{st} is conjugate in $\text{Cr}(2)$ to a transformation $(z_1, z_2) \mapsto (-z_1, -z_2)$. In projective coordinates, it is given by $(x_0, x_1, x_2) \mapsto (x_0, -x_1, -x_2)$.

Another example of a planar involution is a De Jonquières involution associated to a plane hyperelliptic curve of degree d. Its set of fixed points in the domain of the definition is an open subset of the hyperelliptic curve. It is easy to see that its birational type is an invariant of conjugacy class of the transformation. So, non birationally isomorphic hyperelliptic curves define non-conjugate involutions.

There are two more examples of involutions: a Geiser involution and a Bertini involution which we now define.

The classical definition of a *Geiser involution* is as follows. Fix seven points $p_1, \ldots, p_7$ in $\mathbb{P}^2$ (maybe infinitely near) in general position (in the sense that no four points are collinear, not all points lie on a conic). The linear system $\mathcal{H} = |3H - p_1 - \ldots - p_7|$ of cubic curves through the seven points is two-dimensional. Take a general point p and consider the pencil of curves from $\mathcal{H}$ passing through p. Since a general pencil of cubic curves has 9 base points, we can define $\gamma(p)$ as the ninth base point of the pencil.

Another way to see a Geiser involution is as follows. The linear system $\mathcal{H}$ defines a rational map of degree 2

$$f : \mathbb{P}^2 \dashrightarrow |\mathcal{H}|^* \cong \mathbb{P}^2.$$

The points p and $\gamma(p)$ lie in the same fibre. Thus γ is a birational deck transformation of this cover. Blowing up the seven points, we obtain a Del Pezzo surface S of degree 2, and a regular map of degree 2 from S to $\mathbb{P}^2$. The Geiser involution γ becomes an automorphism of the surface S.

It is easy to see that the fixed points of a Geiser involution lie on the ramification curve of f. When the points $p_1, \ldots, p_7$ are in general position, this curve is a curve of geometric genus 3, of degree 6 with double points at the points $p_1, \ldots, p_7$. It is birationally isomorphic to a canonical curve of genus 3. In this way, one can see that a Geiser involution is not conjugate to any De Jonquières involution.

The degree of a Geiser involution is easy to compute. A general line is mapped by the linear system $|3H - p_1 - \ldots - p_7|$ to a cubic curve. The pre-image of this cubic curve consists of the line and a curve of degree 8, the image of the line under the Geiser involution. Thus the degree of the Geiser involution is equal to 8. One can show that the homaloidal linear system consists of curves of degree 8 with triple points at $p_1, \ldots, p_7$.

To define a *Bertini involution* one considers the linear system of plane sextics with 8 double points $p_1, \ldots, p_8$. We assume that the linear system of such sextics has no fixed components. Then one can show that it defines a degree 2 map of the blow-up surface $\mathrm{Bl}_{p_1,\ldots,p+8}$ onto a quadric cone in $\mathbb{P}^3$. The branch divisor is cut out by a cubic surface. If the points are in general position, then the curve is a smooth curve of genus 4 with a vanishing theta characteristic. The deck transformation of this cover is a Bertini involution.

The compute the degree of a Bertini involution, we restrict the double cover to a general line. One can show that its image is a singular curve of degree 6 cut out by a cubic. Its pre-image is equal to the union of the line and a curve of degree 17. This shows that the degree of the Bertini involution is equal to 17. One can show that the homaloidal linear system consists of curves of degree 17 passing through the points $p_1, \ldots, p_8$ with multiplicities 6.

One can also describe ta Bertini involution in the following way. Consider the pencil of cubic curves through the eight points $p_1, \ldots, p_8$. It has the ninth base point p_9. For any general point p there will be a unique cubic curve $C(p)$ from the pencil which passes through p. Take p_9 for the zero in the group law of the cubic $C(p)$ and define $\beta(p)$ as the negative $-p$ with respect to the group law. This defines a birational involution on $\mathbb{P}^2$, a Bertini involution. One can show that the fixed points of a Bertini involution lie on a canonical curve of genus 4 with vanishing theta characteristic (isomorphic to a nonsingular intersection of a cubic surface and a quadric cone in $\mathbb{P}^3$). So, a Bertini involution is not conjugate to a Geiser involution or a De Jonquières involution. It can be realized as an automorphism of the blowup of the eight points (a Del Pezzo surface of degree 1), and the quotient by this involution is isomorphic to a quadratic cone in $\mathbb{P}^3$.

The following theorem due to E. Bertini shows that there is nothing else.

Theorem 4.2.1. *Any element of order 2 in* $\mathrm{Cr}(2)$ *is conjugate to either a projective involution, or a De Jonquières involution, or a Geiser involution, or a Bertini involution.*

4.3 De Jonquières subgroups

Let K be a subfield of $\mathbb{C}(z) = \mathbb{C}(z_1, \ldots, z_n)$ such that $\mathbb{C}(z)$ is a pure transcendental extension of K. Consider the subgroup $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(z), K)$ of $\mathrm{Cr}_{\mathbb{C}}(n)$ which leave the subfield K invariant. Then we have an exact sequence of groups

$$1 \to \mathrm{Aut}_K(\mathbb{C}(z)) \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(z), K) \to \mathrm{Aut}_{\mathbb{C}}(K) \to 1.$$

Since any automorphism of K can be uniquely extended to an automorphism of $\mathbb{C}(z)$. This sequence splits and defines an isomorphism

$$\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(z), K) \cong \mathrm{Aut}_K(\mathbb{C}(z)) \rtimes \mathrm{Aut}_{\mathbb{C}}(K).$$

In the case when K is pure transcendental extension of $\mathbb{C}$ of algebraic dimension k, the subgroup $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(z), K)$ of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(z))$ is called a *De Jonquières subgroup* of *level* k.

Geometrically, the inclusion $K \hookrightarrow \mathbb{C}(X)$ defines a rational dominant map $\mathbb{P}^n \dashrightarrow X$, where K is the field of rational functions of X. Its general fibre is a rational variety over K. The birational automorphisms from $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(z), K)$ preserves this rational fibration.

A Cremona transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a called a *De Jonquières transformation* if there exists a rational fibration with generic fibre isomorphic to $\mathbb{P}^{n-k}$ that is left invariant by T.

We have already seen examples of De Jonquières transformations of order 2. It follows from the proof of Proposition 2.8.1 that a dilated transformation is a De Jonquières transformation of level $n-1$.

Let us see when an involution in $\mathbb{P}^{n-1}$ can be dilated to an involution in $\mathbb{P}^n$.

Suppose $T : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ be an involution given by homogeneous polynomials $G = (G_1, \ldots, G_n)$ of degree d in variables $t_1, \ldots, t_n$. We are looking for a transformation $\tilde{T}$ given by $(F_0, QG_1, \ldots, QG_n)$, where $F_0(t_0, \ldots, t_n)$ is a monoidal polynomial of degree d with multiplicity $\geq d-1$ at the point $[1, 0, \ldots, 0]$ and Q is a monoidal polynomial of degree r with singular point of multiplicity $\geq r-1$. Write $F_0 = t_0A_1 + A_2, Q = t_0B_1 + B_2$. The transformation $\tilde{T}^2$ is given by

$$(F_0(F_0, QG_1, \ldots, QG_n), (QG_1)(F_0, QG_1, \ldots, QG_n), \ldots, (QG_n)(F_0, QG_1, \ldots, QG_n))$$
$$= (F_0A_1(G) + QA_2(G), \tilde{Q}G_1(G), \ldots, \tilde{Q}G_n(G)),$$

where $\tilde{Q} = B_1(G)F_0 + B_2(G)Q$. Since T^2 is the identity, we obtain

$$G_i(G_1, \ldots, G_n) = t_iP, \quad i = 1, \ldots, n,$$

where $P(t_1, \ldots, t_n)$ is a homogeneous polynomial of degree $d^2 - 1$. Now, we must have

$$F_0 A_1(G) + QA_2(G) = t_0(B_1(G)F_0 + QB_2(G))P.$$

Comparing the coefficients at t_0, we find

$$A_1(G)A_1 + A_2(G)B_1 = B_1(G)A_2P + B_2(G)B_2P,$$

$$B_1(G)A_1 + B_2(G)B_1 = 0.$$

For example, if $Q = 1$, we must have

$$A_1(G_1, \ldots, G_n)A_1 = P. \tag{4.7}$$

Example 4.3.1. Suppose $T = T_{\text{st}}$ is the standard Cremona transformation of degree $d = n - 1$ in $\mathbb{P}^{n-1}$. Then $G_i = t_1 \cdots t_n/t_i, i = 1, \ldots, n$. It is easy to see that $P = (t_1 \cdots t_n)^{n-2}$. Thus, if we take $A_1 = t_1^{n-2}$, we obtain (4.7). Therefore, taking $F_0 = t_0 t_1^{n-2}$, we obtain a Cremona involution of degree $n - 1$ in $\mathbb{P}^n$. It is given by the polynomials

$$(t_0 t_1^{n-2}, t_2 \cdots t_n, \ldots, t_1 \cdots t_{n-1}).$$

4.4 Linear systems of isologues

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a Cremona transformation of algebraic degree d and $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^n$ be its graph. Consider the natural rational map $\iota : \Gamma_f \dashrightarrow G_1(\mathbb{P}^n)$ to the Grassmann variety of lines in $\mathbb{P}^n$ which assigns to a point $(x, f(x)) \in \Gamma_f$ the line $\langle x, f(x)\rangle$ spanned by the points x and $f(x)$. It is not defined on the intersection of Γ_f with the diagonal Δ. Let $C(f)$ be the closure of $f(\Gamma_f \setminus \Delta)$. We call it the *complex of lines* associated to f. Note that

$$C(f) = C(f^{-1}).$$

We say that f is *non-degenerate* if $C(f)$ is not contained in a proper subspace of $\mathbb{P}^n$. In terms of Plücker coordinates p_{ij} in $\Lambda^2(\mathbb{C}^{n+1})$, the complex K_f has a rational parameterization $\phi : \mathbb{P}^n \dashrightarrow C(f)$, given by

$$p_{ij} = t_i F_j(t_0, \ldots, t_n), \quad 1 \le i < j \le n.$$

The map ϕ is equal to the composition of the inverse of the projection $\Gamma_f \to \mathbb{P}^n$ and the map ι. If the closure of fixed points in dom(f) is a hypersurface in $\mathbb{P}^n$ of degree k, then this hypersurface is a fixed component of each isologue. After we subtract this hypersurface from the linear system, we obtain a linear system of hypersurfaces of degree $d+1-k$ without fixed components. It is called the *linear system of isologues*. Its members are *isologues*.

If f is non-degenerate, then the linear system of isologues is parametrized by the dual of the Plücker space. If $\mathbb{P}^n = |E|$ for some vector space E of dimension $n+1$, then isologues are elements of $|\Lambda^2 E^\vee|$. An isologue is called *special* if it belongs to the Grassmannian $G_1(|E^\vee|)$ of lines in $|E^\vee|$. Identifying a line in the dual $\mathbb{P}^n$ with a codimension 2 subspace Π in $\mathbb{P}^n$, we see that the corresponding isologue hypersurface V_Π is equal to the closure of the set

$$\{x \in \mathbb{P}^n : x \neq f(x), \langle x, f(x)\rangle \cap \Pi \neq \emptyset\}.$$

In other words, a special isologue is the pre-image of a hyperplane in $|\Lambda^2 E|$ which cuts out the Grassmannian along the special Schubert variety $\langle \Pi \rangle$ of lines intersecting Π. For example, when $n=2$, we can identify $G_1(|E^\vee|)$ with $|E|$, hence all isologues are special. They are parameterized by points $p \in |E|$ and equal to the closures of the sets

$$\{x \in \mathbb{P}^2 : f(x) \neq x, p \in \langle x, f(x)\rangle\}.$$

If f is degenerate, then some of these loci may coincide with the whole plane.

A Cremona transformation f is called *Arguesian* if $\dim C(f) < n$. This implies that the rational map $\phi : \mathbb{P}^n \dashrightarrow G_1(\mathbb{P}^2)$ has fibres of positive dimension. Generically, the fibres are lines $\langle x, f(x)\rangle$, and the transformation leaves these lines invariant. In the case $n=2$, this is equivalent to that f is degenerate. It is clear that any Arguesian transformation is of De Jonquières type, i.e. belongs to some De Jonquières group. The De Jonquières involutions defined by a submonoidal hypersurface are Arguesian.

If $n=3$, then f is Arguesian if $C(f)$ is an irreducible *congruence*[1] of lines in $G_1(\mathbb{P}^3)$. Recall that $G_1(\mathbb{P}^3)$ is isomorphic to a nonsingular quadric in the Plücker space $\mathbb{P}^5$. It contains two families of planes:

$$\begin{aligned} P(x) &= \{\text{lines containing a point } x \in \mathbb{P}^3\}, \\ P(\pi) &= \{\text{lines contained in a plane in } \mathbb{P}^3\}. \end{aligned}$$

The cohomology classes of these planes freely generate

$$H^4(G_1(\mathbb{P}^3), \mathbb{Z}) \cong A_2(G_1(\mathbb{P}^3)) \cong \mathbb{Z}^2.$$

The class of any congruence S in $G_1(\mathbb{P}^3)$ is determined by two numbers (m,n), called the *order* and *class*. The order is equal to the number of lines in S passing

[1] codimension 2 subvariety in the Grassmannian

through a general point in $\mathbb{P}^3$. The class is equal to the number of lines in S lying in a general plane in $\mathbb{P}^3$. Since through a general point in $\mathbb{P}^3$ passes only one fibre of the map ϕ, we see that the image of ϕ is a congruence of order 1.

Example 4.4.1. Consider the standard Cremona transformation T_{st} in $\mathbb{P}^n$. Its graph is the toric variety $X(A_n)$. The transformation has 2^n fixed points with affine coordinates $z_i = \pm 1$. The linear system of isologues is a linear system of hypersurfaces of degree $n+1$ passing through the fixed points and the base locus of T_{st}. If $n = 2$, then $X(A_2)$ blown up at 4 points is a weak Del Pezzo surface of degree 7 (weak means that the anticanonical class is nef and big but not ample). The map ι is a rational map of degree 2 onto the plane. The branch divisor is the union of 4 lines in a general position.

If $n = 3$, then isologues are quartic surfaces which contain the edges of the coordinate tetrahedron and pass through 8 fixed points of T_{st}. The linear system of isologues is 4-dimensional. It maps $X(A_3)$ onto a hyperplane section of $G_1(\mathbb{P}^3)$, a linear complex of lines. It is isomorphic to a 3-dimensional quadric. This shows that f is degenerate but not Arguesian . The degree of the map is equal to 6. Special isologues are quartic which are singular outside the vertices of the tetrahedron.

Remark 4.4.1. The linear system of isologues is not an invariant of the homaloidal linear system. It depends on the map itself, i.e. a choice of a basis in the homaloidal linear system. In the previous example, say $n = 3$, if switch the first two coordinates, we find that the fixed locus consists of 4 lines: $V(t_0, t_2 \pm t_3)$ and $V(t_1, t_2 \pm t_3)$.

4.5 Arguesian involutions

We know that there exist Arguesian involutions in $\mathbb{P}^n$ of any degree d. For example, De Jonquières involutions occur in every degree. They leave invariant a $(n-1)$-dimensional variety of lines passing through a fixed point. Here we will construct Arguesian involutions of $\mathbb{P}^3$ of degree $d = m+5$. which leaves invariant a congruence of lines of bidegree $(1, m+1)$.

The following proposition is a result of E. Kummer (see a modern proof in [Z.Ran, Crelle Journa, 1984]).

Proposition 4.5.1. *Let γ_1 be a line and γ_2 be a smooth rational curve of degree $m+1$ intersecting γ_1 transversally at m points. The closure of the set of lines intersecting γ_1 and γ_2 at one point consists of the union of m congruences of bidegree $(1, 0)$ formed by lines passing through one of the intersection points and an irreducible congruence of bidegree $(1, m+1)$. Any irreducible congruence of lines of bidegree $(1, m)$ is either obtained in this way, or is equal to the congruence of secant lines of a skew cubic (in this case $m = 3$).*

Let $S \subset \mathbb{G} = G_1(\mathbb{P}^3)$ be an irreducible congruence of lines obtained from a reducible curve $\gamma_1 \cup \gamma_2$ as above. Note that all lines in this congruence intersect γ_1, so the congruence is contained in the tangent hyperplane of $\mathbb{G}$ at the point corresponding to γ_1. In the case $m = 0$, it is a 2-dimensional quadric contained in another hyperplane. Take any point x on the line γ_1. The set of lines from S which pass through x is a ruling of a cone of degree $m+1$ with the vertex at x. In S, this defines a linear pencil $C(x), x \in \ell_1$, of curves of degree $m+1$ parameterized by ℓ_1. If $m > 0$, the line γ_1 itself belongs to all such cones, it is a base point s_0 of this pencil.

Now, we take any point on γ_2 and do the same. This time we obtain a pencil of lines $L(x), x \in \gamma_2$, in $\mathbb{G}$, parameterized by γ_2. A curve $C(x), x \in \gamma_1$, intersects a line $L(y), y \in \gamma_2$, at one point represented by the line $\langle x, y\rangle$.

Consider a rational map $\psi : \gamma_1 \times \gamma_2 \dashrightarrow S$ that sends a point (x, y) to the intersection point $C(x) \cap L(y)$. It is easy to see that the map is defined everywhere except at the intersection points $(x_i, y_i), x_i = y_i$. Thus it extends to a regular map from $\gamma_1 \times \gamma_2$ to S. The intersection points go to the same point s_0. This shows that ψ is a normalization map, and $\#f^{-1}(s_0) = m$. When $m > 0$, the surface S is a non-normal scroll with an isolated singular point. The map f is given by a linear system of dimension 4 contained in the linear system $|l_1 + (m+1)l_2|$, where $|l_1|$ and $|l_2|$ are the rulings. So, S is a projection of $\mathbf{F}_0$, embedded as a scroll $\Sigma_{0,2m+2}$ of degree $2m+2$ in $\mathbb{P}^{2m+3}$, to $\mathbb{P}^4$.

Suppose Γ is contained in a smooth quartic surface X. It follows from the theory of periods of K3 surfaces that quartic surfaces containing some Γ as above form a subvariety of codimension 2 in the projective space of quartics. Consider the following Cremona involution T in $\mathbb{P}^3$. Take a general point x, pass a secant ℓ_x of Γ through x which intersects Γ at two points a and b. Define the involution on ℓ_x by designating a and b to be its fixed points. Then $T(x)$ is defined to be conjugate to x under this involution.

Let us locate the base points of T. Obviously, the map is not defined at the curve $\gamma_1 \cup \gamma_2$. It is also not defined at the points where the secant line ℓ is tangent to X. Everywhere else the map is well defined. Let us determine the locus R of the tangency points.

Consider the birational involution τ of X that assigns to a point $x \in X$ the intersection point of the secant ℓ_x of Γ which is not contained in Γ. A birational involution of a K3 surface extends to a biregular involution. The conjugate points are on a line from the congruence S, so the fixed locus of the involution is the curve R we are looking for. The linear system of planes through the line γ_1 cuts out on X a pencil of elliptic cubic curves. It is easy to see that τ induces th involution on each cubic curve contained in a plane Π obtained by projection from the unique point in Π which lies on γ_2 but does not lie on γ_1. In particular, the divisor class

$F = H - \gamma_1$ is τ-invariant.

Consider the linear system in $\mathbb{P}^3$

$$\mathcal{H} = |(m+2)H - (m+1)\gamma_1 - \gamma_2|.$$

It is clear that any line from the congruence S is blown down to a point. The dimension of $\mathcal{H}$ is easy to compute. A homogeneous form of degree $m+2$ vanishing on a line $V(t_0, t_1)$ with multiplicity $\geq m+1$ is equal to a linear combination of monomials $t_0^i t_1^{m+2-i}$ and monomials $t_0^i t_1^{m+1-i} t_2, t_0^i t_1^{m+1-i} t_3$. Thus the dimension of $|(m+2)H - (m+1)\gamma_1| = 3m+6$. To contain γ_2 requires $(m+1)(m+2)+1-m(m+1) = 2m+3$ additional conditions. This gives $\dim \mathcal{H} = m+3$.

Let $|D| = |(m+2)h - (m+1)\gamma_1 - \gamma_2|$ be the restriction of $\mathcal{H}$ to X. It is a complete linear system and an easy computation gives $D^2 = 2m+4$. It is well-known that a complete linear system on a K3 surface without fixed components has no base points. Thus the linear system defines a regular map $f : X \to \mathbb{P}^{m+3}$ onto a surface of degree $2(m+4)/\deg f$. The linear pencil $|E| = |h - \gamma_1|$ of cubic curves defines an elliptic fibration on X with $E \cdot D = 2$. This shows that the linear system $|D|$ is a hyperelliptic linear system on X and the map f is of degree 2 on a scroll of degree $m+2$ in $\mathbb{P}^{m+3}$. Our curve R is the ramification curve of the map f.

Assume that X is general in the sense of periods of K3 surfaces. This means that the divisor classes h, γ_1 and γ_2 generate $\text{Pic}(X)$. The involution τ defined by the double cover f acts on the transendental lattice $\text{Pic}(X)^\perp \subset H^2(X, \mathbb{Z})$ as the minus identity. It has two invariant divisor classes D and $H - E$. Since $D = (m+1)(H - \gamma_1) + (H - \gamma_2)$ we may assume that the invariant part of $\text{Pic}(X)$ is generated by $H - \gamma_1$ and $H - \gamma_2$. We have

$$(H - \gamma_1)^2 = 0, (H - \gamma_2)^2 = -2m, \ (H - \gamma_1) \cdot (H - \gamma_2) = 2.$$

Since R is invariant with respect to τ, we can write

$$R \sim a(H - \gamma_1) + b(H - \gamma_2). \tag{4.8}$$

Since any nonsingular fibre of the elliptic fibration $|H - \gamma_1|$ has 4 fixed points, we get $R \cdot (H - \gamma_1) = 4$. Intersecting both sides of the previous equality with $H - \gamma_1$ we obtain $b = 2$. Since $R^2 = 16$, we obtain $16 = 4ab - 2mb^2 = 8a - 8m$. This gives $a = m+2$. Now

$$\deg R = R \cdot H = 3(m+2) + 2(4 - m - 1) = m + 12.$$

To summarize, we obtained that the base locus of T consists of a γ_1, γ_2 and a smooth curve R of degree $m+12$ and genus 9. Also observe that, using (4.8) with $a = m+2, b = 2$, we obtain that

$$R \cdot \gamma_1 = m+8,\ R \cdot \gamma_2 = 3m+8.$$

Let us determine the multiplicities of the base curves. Take a general point $x \in \gamma_2$ and consider the plane $\Pi = \langle x, \gamma_1 \rangle$. The transformation T leaves this plane invariant and fixes pointwisely the cubic curve $E = \Pi \cap X - \gamma_1$. We know also that the involution τ of X induces on E the involution defined by the projection from x. We immediately recognize that $T|_\Pi$ is a De Jonquières involution of degree 3 whose homaloidal linear system consists of cubic curves singular at x and having simple base points at the $4 = (m+12) - (m+8)$ intersection points of R with E outside of γ_1. This implies that the curve γ_1 is a curve of multiplicity $d-3$ for a general divisor from the homaloidal linear system defining T and the curve γ_2 is a double curve. The curve R enters with multiplicity 1.

Let Q be the image of a general plane H under T. It is a rational surface of degree d equal to the algebraic degree of T. The homaloidal linear system defining T restricts to H to define a linear system of curves of degree d passing through the set $\Sigma = (R \cup \gamma_1 \cup \gamma_2) \cap H$. From above we know that any intersection point with γ_1 is a point of multiplicity $d-3$ and the intersection point with γ_1 is a point of multiplicity 2. It has also $m+1$ simple points. Since it maps H to a surface of degree d we must have

$$d^2 - d = \sum_{p_i \in \Sigma} m_i^2 = (d-3)^2 + 4(m+1) + 12 + m.$$

This gives

$$d = m+5.$$

One can also describe the P-locus of the involution T. It consists of the ruled surface of degree $8+2m = R \cdot \mathcal{H}$ whose rulings are tangents to R, a quartic surface which is blown down to γ_2, and a surface of degree $2m+4$ which is blown down to γ_1.

Let us consider the case of the congruence S of bidegree $(1,3)$ of secants of a twisted cubic γ. The surface S, embedded in the Plücker space $\mathbb{P}^5$ is a Veronese surface. Again we choose a general quartic surface X containing γ. The map $f : X \to S$ is a double cover branched along a smooth plane sextic curve. It is defined by the linear system $|2h - \gamma|$. The ramification curve R belongs to the linear system $|6h - 3\gamma|$. Its genus is 10 and its degree is 15. The base curve is the union of the curves γ and R intersecting at 24 points. We skip the details and leave to the reader to check that the homaloidal linear system defining T consists of surfaces of degree 7 vanishing along γ with multiplicity 3. The P-locus consists of the tangential ruled surface of γ of degree 4 and a ruled surface of degree 12 whose rulings are tangent to R.

Example 4.5.1. We will discuss a special classical example of an Arguesian involutions of degree 5. It is obtained from the previous construction when the quartic surface is special and the curve R becomes equal to the union of 8 skew lines and a quartic elliptic curve intersecting each line at two points.

Fix a general pencil $\mathcal{Q}$ of quadrics in $\mathbb{P}^3$. The restriction of $\mathcal{Q}$ to a general $\ell \in S$ defines a pencil of degree 2, hence an involution on ℓ. This defines an Cremona involution on $\mathbb{P}^3$. Take a general point $x = [v]$ in $\mathbb{P}^3$, pass a unique line through x which intersects ℓ_1 and ℓ_2, and send x to $T(x)$ such that the pair $(x, T(x))$ is in the involution defined by $\mathcal{Q}$. This is an example of a De Jonquières type involution of level 1. Note that, for any filed K, an automorphism of order 2 of $\mathbb{P}^1_K$ is conjugate to an automorphism given by $(t_0, t_1) \to (t_1, at_0 + t_1)$ for some $a \in K$.

In formulas, let $\ell_1 = V(t_0, t_1), \ell_2 = V(t_2, t_3)$. For any $x = [a_0, a_1, a_2, a_3]$, the plane $\langle x, \ell_1\rangle$ is equal to $V(-a_1t_0 + a_0t_1)$. The plane $\langle x, \ell_2\rangle$ is equal to $V(-a_3t_2 + a_2t_3)$. The planes intersect along the line ℓ given in parametric form by $[sv_1 + tv_2]$, where $v_1 = (a_0, a_1, 0, 0), v_2 = (0, 0, a_2, a_3)$. The pencil of quadrics $V(u_0q_1 + u_1q_2)$ restricted to the line ℓ is equal to the pencil

$$A(u, a)s^2 + 2B(u, a)st + C(u, a)t^2 = 0, \tag{4.9}$$

where

$$\begin{aligned} A &= u_0q_1(v_1) + u_1q_2(v_1), \\ B &= u_0b_1(v_1, v_2) + u_1b_2(v_1, v_2), \\ C &= u_0q_1(v_2) + u_1q_2(v_2). \end{aligned}$$

Here b_i are the symmetric bilinear forms associated to the quadratic forms q_i. The coefficients A, B, C are bi-homogeneous functions in the coordinates $u = (u_0, u_1)$ and $a = (a_0, a_1, a_2, a_3)$ of degree 1 in u and degree 2 in a. The point x corresponds to the parameters $[s, t] = [1, 1]$. The quadratic form has 0 at $[1, 1]$. This allows one to express the coefficients of equation (4.9) as polynomials of degree 4 in the coordinates a_i:

$$[u_0 : u_1] = [q_2(v_1 + v_2), -q_1(v_1 + v_2)] = [q_2(v), -q_1(v)]$$

This gives the second solution $[s, t]$ of (4.9) as polynomials of degree 4

$$[s, t] = [C, A] = [q_2(v)q_1(v_2) - q_1(v)q_2(v_2), q_2(v)q_1(v_1) - q_1(v)q_2(v_1)]$$

Finally, the coordinates of the transform $T(x)$ of the point x become expressed as polynomials of degree 5 in the a_i's

$$T(x) = [a_0 s, a_1 s, a_2 t, a_3 t].$$

This shows that the involution T is of algebraic degree 5.

Let us see where our transformation is not defined. Obviously, it is not defined on the lines ℓ_1 and ℓ_2, as well on the base locus E of the pencil of quadrics. Also, it is not defined on any secant of C which intersects ℓ_1 and ℓ_2. In fact, the restriction of the pencil to this secant consists of two points which do not move. The set of secants of E is a congruence S_1 of lines in the Grassmannian $G_1(\mathbb{P}^3)$ of bi-degree $(2,6)$. The set of lines intersecting ℓ_1 and ℓ_2 is a congruence S_2 of bi-degree $(1,1)$. They intersect at 8 points. Thus the base locus contains 10 lines and a quartic curve. This can be confirmed by the explicit formulas for the transformation. They also show that the lines ℓ_1 and ℓ_2 enter with multiplicity 2.

Let us find the locus of fixed points of T. A general point x satisfies $T(x) = x$ if and only if $[s,t] = [1,1]$ is the unique solution of the quadratic equation (4.9). So, we get the equation

$$A - C = q_2(v)(q_1(v_2) - q_1(v_1)) + q_1(v)(q_2(v_1) - q_2(v_2)) = 0.$$

It is a quartic surface F with equation

$$q_2(t_0, t_1, t_2, t_3)(q_1(0,0,t_2,t_3) - q_1(t_0,t_1,0,0))$$

$$+q_1(t_0,t_1,t_2,t_3)(q_2(t_0,t_1,0,0) - q_2(0,0,t_2,t_3)) = 0.$$

It follows from the equations that the eight skew lines from the congruence S_2 lie on the quartic surface. The curve E itself also lies on the surface. Also, if we plug in $v_1 = 0$ (resp. $v_2 = 0$), we get zero. So, the quartic S also contains the lines ℓ_1 and ℓ_2.

Consider the elliptic pencils $|H - \ell_1|$ and $|H - \ell_2|$ cut out by planes through the lines ℓ_1 and ℓ_2. The linear system $|2H - \ell_1 - \ell_2|$ maps F two-to-one onto a nonsingular quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. Of course, this extends the rational map which assigns to $x \in F$ the points $(\ell_x \cap \ell_1, \ell_x \cap \ell_2)$, where ℓ_x is the line in the congruence S_1 passing though x.

Each pencil is the pre-image of a ruling. Since $\ell_i \cdot (H - \ell_1) = \ell_i \cdot (H - \ell_2) = 0$ for $i = 3, \ldots, 10$, we see that the eight secants $\ell_3, \ldots, \ell_{10}$ are blown down to singular points $q_1, \ldots, q_8$ of the branch curve W. The ramification curve of the double cover is our quartic curve E. It belongs to the divisor class $4H - 2\ell_1 - 2\ell_2 - \ell_1 - \ldots - \ell_{10}$, where H is a plane section of X.

We have $(H-\ell_1)\cdot\ell_1 = 3, (H-\ell_1)\cdot\ell_2 = 1$ and $(H-\ell_2)\cdot\ell_1 = 1, (H-\ell_2)\cdot\ell_1 = 1$. This implies that ℓ_1 (resp. ℓ_2) is mapped isomorphically to a curve R_1 (resp. R_2) of bi-degree $(3,1)$ (resp. $(1,3)$). They intersect at $q_1,\ldots,q_8$ and also at two more points a,b. The pre-image of R_i under the cover $S\to Q$ splits into the sum of two lines $\ell_i+\ell_i'$ with $\ell_i'\cdot\ell_j = 1, i\neq j$. The images of the two intersection points $\ell_1'\cap\ell_2$ and $\ell_1\cap\ell_2'$ are the intersection points a,b of R_1 and R_2.

Conversely, start with a set $\mathcal{P} = \{q_1,\ldots,q_8\}$ of 8 points on $Q = \mathbb{P}^1\times\mathbb{P}^1$ such that the dimension of the linear system of curves of bidegree $(4,4)$ with double points at each $q_i\in\mathcal{P}$ is a pencil containing a reducible curve R_1+R_2, where R_1,R_2 are smooth rational curves (necessary of bidegrees $(3,1)$ and $(1,3)$). One can show that the sets of such points depends on 14 parameters (by projecting from q_8, they correspond to base points of an Halphen pencils of elliptic curves of degree 6 with one reducible member equal to the union of two rational cubics).

Choose a smooth member W of the pencil and consider the double cover of Q branched over W. Let Q' be the blow-up of the set $\mathcal{P}$, and $\pi : X\to Q'$ be the double cover of Q' ramified along the proper transform W' of W.

The proper transform R_i' of the curves R_i splits into the disjoint union of two (-2)-curves $\ell_i+\ell_i'$ such that $\ell_1+\ell_2'\sim\ell_2+\ell_1'$ and $\ell_i\cdot\ell_j' = 2, i\neq j$. Let $|f_i|$ the pencils on Q' defined by the two rulings $Q\to\mathbb{P}^1$. We have

$$\pi^*(R_1-R_2) = \ell_1+\ell_1'-\ell_2-\ell_2'\sim 2(\ell_1-\ell_2)$$

$$\sim\pi^*(3f_1+f_2)-\pi^*(f_1+3f_2) = 2\pi^*(f_1)-2\pi^*(f_2).$$

This gives

$$\pi^*(f_1)+\ell_2\sim\pi^*(f_2)+\ell_1.$$

Set

$$H = \pi^*(f_2)+\ell_1.$$

We have $H^2 = 2(f_2+R_1)^2+R_1^2 = 6-2 = 4$. Also $H\cdot\ell_i = 1$. The linear system $|H|$ maps X onto a quartic surface in $\mathbb{P}^3$ and the images of ℓ_1 and ℓ_2 are lines.

Let $\ell_1,\ldots,\ell_8$ be the pre-images of the exceptional curves L_i of $Q'\to Q$ under π. These are (-2)-curves on X. We have

$$H\cdot\ell_i = (\pi^*(f_2)+\ell_1)\cdot\ell_i = \ell_1\cdot\ell_i = R_1\cdot L_i = 1, i = 1,\ldots,8.$$

So, the curves $\ell_i, i = 3,\ldots,10$, are mapped to lines on the quartic model of X. Finally,

$$E\sim 2(\pi^*(f_1)+\pi^*(f_2))-\ell_3-\ldots-\ell_{10},$$

and we easily check that $H\cdot E = 4$, so that E is mapped to an elliptic quartic curve. The lines $\ell_3,\ldots,\ell_{10}$ are common secants of E and $\ell_1+\ell_2$. So, we can reconstruct an involution T of $\mathbb{P}^3$ with fixed locus isomorphic to X.

Finally, observe that the set of involutions T depend on 24 parameters, 8 for the pairs of skew lines ℓ_1 and ℓ_2, and 16 for pencils of quadrics. Modulo projective transformations of $\mathbb{P}^3$, we have 9 parameters. The isomorphism classes of our quartic surfaces depend on the same number of parameters, 8 for sets of points $\mathcal{P}$ and one for a choice of its member.

Another class of Arguesian involutions which exist in any projective space of odd dimension are defined by varieties with an apparent double point. First let us recall the following definition.

Definition 4.5.1. *A closed subvariety X of $\mathbb{P}^n$ is called a* subvariety with one apparent double point *if a general point in $\mathbb{P}^n$ lies on a unique secant of X.*

A subvariety X of $\mathbb{P}^n$ with an apparent double point (an *OADP- variety*) defines a Cremona involution of $\mathbb{P}^n$ in a way similar to the definition of a De Jonquières involution. For a general point $x \in \mathbb{P}^n$ we find a unique secant of X intersecting X at two points (a, b), and then define the unique $T(x)$ such that the pair $\{x, T(x)\}$ is harmonically conjugate to $\{a, b\}$.

A most general example of a variety with one apparent double points is an *Edge variety*. The Edge varieties are of two kinds. The first kind is a general divisor $E_{n,2n+1}$ of bidegree $(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$ embedded by Segre in $\mathbb{P}^{2n+1}$. Its degree is equal to $2n + 1$. For example, when $n = 1$, we obtain a twisted cubic in $\mathbb{P}^3$. If $n = 2$, we obtain a Del Pezzo surface in $\mathbb{P}^5$. The second type is a general divisor of bidegree $(0, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$. For example, when $n = 1$, we get the union of two skew lines. When $n = 2$, we get a quartic ruled surface in $\mathbb{P}^5$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by the linear system of divisors of bidegree $(1, 2)$.

Let us recall what is known about smooth varieties X with one apparent double point. First, it is clear that an n-dimensional X spans $\mathbb{P}^{2n+1}$.

Theorem 4.5.2. *Suppose $n = 2$, then X is an Edge variety $E_{2,4}$ or $E_{2,5}$. If $n = 3$, then X is either an Edge variety $E_{3,6}$ or $E_{3,7}$, or a scroll of degree 8 in $\mathbb{P}^7$ equal to the intersection of a quadric hypersurface with the cone over the Segre variety $s(\mathbb{P}^1 \times \mathbb{P}^2) \subset \mathbb{P}^5$ with a line as a vertex.*

The case $n = 2$ is classical and goes back to F. Severi, and the case $n = 3$ was treated in a paper by Ciliberto, Mella, Russso in JAG.

The Cremona involution defined by $E_{1,2}$ (a pair of skew lines) is a projective transformation given (in notation of Example 4.5.1) by $[a_0, a_1, a_2, a_3] \mapsto [a_0, a_1, -a_2, -a_3]$. The Cremona involution defined by a skew cubic $E(1, 3)$ is the polarity with respect to the net of quadrics containing the cubic. It is a bilinear transformation which we considered in Example 3.4.4. The Cremona involution defined by a Del Pezzo surface X of degree 5 is also the polarity involution with respect to the

4-dimensional linear system of quadrics in $\mathbb{P}^5$. It is a bilinear transformation of degree 5. Its exceptional divisor is the tangential variety of X of degree 24. Its base locus is a double structure on X.

4.6 Geiser and Bertini involutions in $\mathbb{P}^3$

The following is a classical generalization of the Geiser involution in the plane. Assume $n = 3$, and consider a rational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by the linear system $\mathcal{H} = |2H - p_1 - \ldots - p_6|$, where $p_1, \ldots, p_6$ are points in general linear position. Take a general point $x \in \mathbb{P}^3$, then the quadrics from $\mathcal{H}$ passing through x form a net. The base locus of the net consists of 8 points, we have already seven base points $p_1, \ldots, p_6, x$. By definition, $T(x)$ is the eighth base point.

A general plane is mapped to a quartic surface, a projection of a Veronese surface. The pre-image of this quartic surface consists of the plane and a surface of degree 7. This shows that the degree of the involution is equal to 7. Let $\ell_{ij} = \langle p_i, p_j \rangle$ and R be a a twisted cubic through the six points (it is uniquely defined). The base locus of the involution consists of the union of 15 simple lines ℓ_{ij} and the curve R taken with multiplicity 3. The P-locus consists of the six quadrics from the linear system which have a singular point at one of the points p_j.

The set of fixed points of the involution is a famous *Weddle quartic surface* with nodes at $p_1, \ldots, p_6$. It is the locus of singular points of singular quadrics from the linear system $|2H - p_1 - \ldots - p_6|$. The rational map ϕ maps this surface birationally onto a Kummer quartic surface K. The 15 lines and the curve R are blown down to the 16 nodes of the Kummer surface.

One can show that the complex $C(T)$ defined by the linear system of isologues is the quadratic line complex associated to the Kummer surface K (see Griffiths-Harris book, or Topics).

Remark 4.6.1. The Geiser involution can be also defined using the geometry of the *Segre cubic primal* $\mathcal{S}_3$, a cubic hypersurface in $\mathbb{P}^4$ with 10 nodes. It is known that the linear system $|2H - p_1 - \ldots - p_6|$ defines birational isomorphism $\psi : \mathbb{P}^3 \dashrightarrow \mathcal{S}_3$. It blows down the 10 chords $\langle p_i, p_j \rangle$ to the singular points of $\mathcal{S}_3$. Consider the projection $\mathrm{pr} : \mathcal{S}_3 \dashrightarrow \mathbb{P}^3$ from a nonsingular point on $\mathcal{S}_3$. The composition $p \circ \psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is a degree 2 rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. The deck transformation is the Geiser involution.

The planar Bertini involution can be generalized to an involution in $\mathbb{P}^3$ as follows. Consider a net of quadrics in $\mathbb{P}^3$ passing through 7 general points $p_1, \ldots, p_7$. It has additional base point $\mathfrak{o}$. Take a general point $x \in \mathbb{P}^3$. The quadrics from the net passing through this point define a pencil $\mathcal{P}(x)$. Its base locus is an elliptic curve $E(x)$ of degree 4 with a fixed point $\mathfrak{o}$. We use it to define a group law on

$E(x)$. We set $T(x)$ to be equal to $-x$ in the group law. This is the definition of a *space Bertini involution* due to A. Coble. One can rephrase it in modern terms as follows. Let X be the blow-up of the set $\{p_1, \ldots, p_7, \mathfrak{o}\}$. The linear system $|2H - p_1 - \ldots - p_7 - \mathfrak{o}|$ defines an elliptic fibration $f : X \to \mathbb{P}^2$ with a distinguished section $E_{\mathfrak{o}} = \sigma^{-1}(\mathfrak{o})$. The open subset of nonsingular points of the fibres is a group scheme over $\mathbb{P}^2$. The Bertini involution restricted to this open set is the negation morphism $x \mapsto -x$. The fixed locus of the Bertini involution is the union of the section $E_{\mathfrak{o}}$ and a surface S of degree 6 with triple points at p_i's. The restriction of f to the proper transform $\bar{S}$ of S in X is a degree 3 cover of $\mathbb{P}^2$. Its branch locus is the dual of a plane quartic curve. All of this is well documented in Coble's book "Algebraic geometry and theta functions", and also in my book with D. Ortland (Asterisque, vol. 168).

Consider the linear system $\mathcal{H}$ of quartic surfaces with double points $p_1, \ldots, p_7$. Its dimension is equal to $34 - 7 \cdot 4 = 6$. Besides the seven double points, the base locus contains a curve C of degree 6 and genus 3, the locus of singular points of the elliptic fibration f. Let $Q_i = V(q_i), i = 1, 2, 3$, be a basis of the net of quadrics through these points. For any quadratic form Q in three variables, the quartics $V(Q(q_1, q_2, q_3))$ form a 5-dimensional subsystem $\mathcal{H}'$ of $\mathcal{H}$. The whole linear system is generated by this subsystem and some quartic $V(F_4)$. The linear system defines a regular degree 2 map $\pi : \mathrm{Bl}_{p_1,\ldots,p_7} \to \mathbb{P}^6$. Its image Y is the cone over a Veronese surface in $\mathbb{P}^5$ with the vertex equal to the image of the point $\mathfrak{o}$. The branch divisor is cut out by a cubic threefold. This is in a complete analog with the planar Bertini involution. In this way, the quotient of $\mathbb{P}^3$ becomes birationally isomorphic to a rational variety Y.

One can compute the degree of the Bertini involution as follows. The restriction of the linear system $\mathcal{H}$ to a general plane is a 6-dimensional linear system of quartic curves. The linear system $\mathcal{H}'$ maps it $4 : 1$ to a Veronese surface in $\mathbb{P}^5$. This shows that the projection of the image of the plane from the vertex of the cone is the intersection of the cone with a surface of degree 4. As in the Geiser involution, this implies that the degree of the Bertini involution is equal to 15.

The locus of fixed points of a general Bertini involution is surface of degree 6 with triple points at $p_1, \ldots, p_7$. The base locus consists of the union of 21 chords $\langle p_i, p_7 \rangle$ and the union of 7 skew cubics through 6 points $p_i' s$.

One can generalize a Geiser involutions as follows. A starting point is an observation that the base ideal $\mathcal{J}_Z$ of the linear system $|3H - p_1 - \ldots - p_7|$ is given by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(-3)^3 \to \mathcal{J}_Z \to 0.$$

This follows from the fact that the scheme of 7 points is a ACM subscheme of $\mathbb{P}^2$ of codimension 2. Arguing as in the case of biregular transformations, we deduce from this that the graph of the transformation is a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ of divisors of type $(1,1)$ and $(2,1)$. It defines a map of degree 2 from $\mathbb{P}^2$ to $\mathbb{P}^2$. The Geiser involuitiuon is the deck transformation of this map. We use this observation to define a degree 2 map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ whose graph is a complete intersection of $n-1$ divisors of type $(1,1)$ and one divisor of type $(2,1)$. The linear system defining this map has the base ideal given by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n-1} \oplus \mathcal{O}_{\mathbb{P}^n}(-2) \to \mathcal{O}_{\mathbb{P}^n}(-n-1)^{n+1} \to \mathcal{J}_Z \to 0.$$

Example 4.6.1. Take $n = 3$. Then, computing the Hilbert functions, we find that the base scheme Z is a curve of degree 11 and genus 14. For any smooth quartic S containing Z, a residual curve C in $|4h - Z|$ is a curve of genus 2 and degree 5. The linear system $|4h - Z|$ defines a degree 2 map from S onto $\mathbb{P}^2$. It is branched along a curve of degree 6. The linear system $|2h - C|$ consists of a skew cubic R. Conversely, a quartic surface containing a skew cubic R defines the linear system $|2h - R|$ of curves of degree 5 and genus 2, and the linear system $|2h + R|$ consists of curves of degree 11 and genus 14.

Let $X = \mathrm{Bl}_Z\mathbb{P}^3 \to \mathbb{P}^3$ be the degree 2 regular map defined by the linear system $|4H - Z|$. The branch divisor is a surface B of degree 6 in $\mathbb{P}^3$. The ramification divisor R is the proper transform of a surface of degree 12 which passes through Z with multiplicity 3. The exceptional divisor E is a ruled surface isomorphic to $\mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(4h - C)) \cong \mathbf{F}_{18}$. It is mapped by ϕ onto a ruled surface in $\mathbb{P}^3$ whose rulings are tritangent lines of the sextic surface B. Also, it follows from the Cayley formula for the number of quadrisecant lines of a curve of genus g and degree d

$$t_4 = \frac{(d-2)(d-3)^2(d-4)}{12} - \frac{(d^2 - 7d + 13 - g)g}{2},$$

that Z has 35 quadrisecant lines (see [Griffiths-Harris]). They are blown down to ordinary double points of B.

Note that the double cover of $\mathbb{P}^3$ branched along a general surface of degree 6 is a non-rational variety. In our case, it is a rational variety. Its intermediate Jacobian variety is isomorphic to the Jacobian variety of the curve Z. Finally, observe that the Geiser involution of this type is not conjugate to the Geiser involution described in above since the surfaces of fixed points are not birationally isomorphic.

Lecture 5

Factorization Problem

5.1 Elementary links

In this lecture we assume that the reader is somewhat familiar with some basics of the Minimal Model Program. Let $\mathfrak{M}$ be the full category of the category of projective algebraic varieties whose objects have terminal $\mathbb{Q}$-factorial singularities. We denote by $N^1(X)$ the vector space $\mathrm{Num}(X) \otimes \mathbb{R}$, where $\mathrm{Num}(X)$ is the Picard group modulo numerical equivalence. For any projective morphism $U \to V$, we denote the relative *Mori cone* by $\overline{\mathrm{NE}}_1(U/V)) \subset N_1(U/V)$. It is the real closure in $N_1(U) = A_1(U) \otimes \mathbb{R}$ of the cone of effective curves contained in fibres. The *nef cone* is the dual cone $\overline{\mathrm{NE}}^1(U/V)) \subset N^1(U/V) = \mathrm{Pic}(U) \otimes \mathbb{R}$. The real closure of the cone of effective divisors is denoted by $\overline{\mathrm{NM}}^1(U)$. Its elements are called *quasi-effective* divisor classes.

Definition 5.1.1. *A* Mori fibre space *is a projective morphism $\phi : X \to S$, where $X \in \mathfrak{M}$ and S is a normal variety with $\dim S < \dim X$. Its fibres are connected, $-K_X$ is relatively ample (i.e. for any curve C in fibres $K_X \cdot C < 0$) and $\dim\ N^1(X) = \dim\ N^1(S) + 1$. A morphism of fibre spaces is a diagram*

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X' \\ \phi \downarrow & & \downarrow \phi' \\ S & & S' \end{array}$$

where f is a birational map. There is no arrow at the bottom, so we keep the vertical arrows to remember that we have structures of Mori spaces on X and X'.

A morphism of Mori fibre spaces is called a square *(resp.* Sarkisov isomorphism*) if, additionally, there is a rational map $g : S \dashrightarrow S'$ such that the diagram*

$$\begin{array}{ccc} X & \stackrel{f}{\dashrightarrow} & X' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ S & \stackrel{g}{\dashrightarrow} & S' \end{array}$$

is commutative (resp. it is square and f is an isomorphism)

It is known that a Mori fibre space is obtained by contraction of an extremal ray in $\overline{NE}(X)$.

In dimension 2, a Mori fibre space is either a constant morphism of $\mathbb{P}^2$, or a projective bundle over a nonsingular curve.

In dimension 3, a Mori fibre space is one of the following three kinds:

(i) $X \to S =$ point, where X is a Fano variety and $\dim N^1(X) = 1$;

(ii) a conic bundle $X \to S$ over a normal surface S with rational singularities, the map contracts an extremal ray represented by an irreducible component of a singular fibre or a fibre if the map is smooth. Each singular fibre is either irreducible, non-reduced, or a bouquet of two $\mathbb{P}^1$'s.

(iii) a Del Pezzo fibration $X \to S$, where S is a nonsingular curve and a general fibre is a Del Pezzo surface of degree $d \neq 8$, or a quadric, or a plane. In the second case a singular fibre is a quadric with an isolated singular point. In the last case the fibration is a $\mathbb{P}^2$-bundle. The map contracts an extremal ray represented by the class of a line on a general fibre.

We will need a log-version of a Mori fibre space. Let (X, Δ) be a *log pair* that consists of a normal variety X and a $\mathbb{Q}$-divisor on Δ. We assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. For every divisorial valuation $v : K \to \mathbb{Z}$ of the field of rational functions K on X there exists a birational morphism $\nu : Z \to X$ of normal varieties and an irreducible divisor E on Z such that the valuation is defined by the discrete valuation of K with the ring of valuation equal to $\mathcal{O}_{Z,\eta}$, where η is a generic point of E. The difference $K_X + \nu^{-1}(\Delta) - \nu^*(K_X + \Delta)$ is a linear combination of exceptional divisors of ν. The coefficient at E is called the discrepancy of Δ and is denoted by $a_E(X, \Delta)$. It depends only on the valuation. We say that (X, Δ) has terminal (canonical, log-canonical) singularities if, for all $\nu : Z \to X$, the discrepancies $a_E(X, \Delta)$ are positive (non-negative, ≥ -1). In fact, to check the definitions, it suffices to look at the valuations defined by irreducible exceptional divisors in a log-resolution of the pair (X, Δ). If one takes $\Delta = 0$, we obtain the definition of terminal, canonical, log-canonical singularities.

One defines a *Mori log fibre space* $f : (X, \Delta_X) \to (S, \Delta_S)$ by requiring that $f(\Delta_X) = \Delta_S$, the pair (X, Δ) has terminal singularities, X is $\mathbb{Q}$-factorial, $\rho_{X/S} = 1$, and $-(K_X + \Delta_X)$ is relatively ample.

The importance of Mori fibre spaces is explained by the following main result of the Minimal Model program in dimension 3.

Theorem 5.1.1. *Let (X, Δ) be a $\mathbb{Q}$-factorial terminal log-pair. Then a sequence of elementary birational transformations $X \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_N = X'$, called divisorial contractions and flips, creates a log pair (X', Δ) with terminal $\mathbb{Q}$-factorial singularities such that one of the the following two alternatives occurs:*

(i) (X', Δ) *is a minimal model, i.e.* $K_{X'} + \Delta$ *is nef.*

(ii) (X', Δ) *admits a structure of a Mori log space* $f : (X', \Delta_{X'}) \to (S, \Delta_S)$.

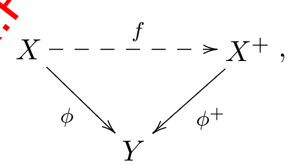
The goal of *Sarkisov's Program* is to factor any birational morphism between two Mori fibre spaces into a composition of *elementary links*, i.e. morphisms which elementary in the sense we will make precise.

Before we describe elementary links let us introduce one more definition.

Definition 5.1.2. *A birational map $f : X \dashrightarrow X'$ is called* small *if there exists an open Zariski subset U whose complement is of codimension ≥ 2 such that the restriction of f to U is an isomorphism.*

Note that there are no small Cremona transformations of degree > 1 since the P-locus is always a hypersurface. Also any small birational map of normal surfaces must be an isomorphism. However in dimension ≥ 3 small birational maps appear frequently.

Definition 5.1.3. *A* flip *(resp.* flop*) is a diagram of birational maps*



where ϕ and ϕ^+ are birational morphism onto a normal varieties with exceptional loci of codimension ≥ 2. Moreover, K_X is relatively ϕ-ample (resp. numerically trivial) and $-K_{X^+}$ is relatively ϕ^+-ample (resp. numerically trivial).

It is clear from this definition that a flip is a small birational map. Following Miles Reid, a birational transformation which is a flip or a flop, or its inverse, will be called a *flap*.

We will be often using the following well-known fact.

Lemma 5.1.2. *Let $X \subset \mathbb{P}^n \times \mathbb{P}^k$ be the graph of the projection map from a linear subspace L of dimension $n-k-1$. Then the first projection map $p_1 : X \to \mathbb{P}^n$ is isomorphic to the blow-up $\mathrm{Bl}_L\mathbb{P}^n$ with exceptional divisor $L \times \mathbb{P}^k$. The second projection $p_2 : X \to \mathbb{P}^k$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^k}^{\oplus n-k} \oplus \mathcal{O}_{\mathbb{P}^k}(1)$.*

Proof. The first assertion follows from the fact that the normal bundle of L is isomorphic to $\mathcal{O}_L(1)^{\oplus k}$. The variety X is a complete intersection of k of divisors of type $(1,1)$. Let $\mathcal{O}(1) \cong p_1^*\mathcal{O}_{\mathbb{P}^n}(1)$ be the tautological line bundle corresponding to the projection $\mathrm{pr}_2 : \mathbb{P}^n \times \mathbb{P}^k \to \mathbb{P}^k$. Then $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = (p_2)_*(\mathcal{O}_X(1))$. Assume first that $k = 1$. Then we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(-1,-1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1} \to \mathcal{O}_X \to 0.$$

Twisting by $p_1^*\mathcal{O}_{\mathbb{P}^n}(1)$ and taking the direct image, we obtain an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{n+1} \to \mathcal{E} \to 0.$$

Then $\mathcal{E}$ splits into the direct sum of n line bundles $\mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 0$. Taking the determinants, we get $\sum a_i = 1$. This gives $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}^{n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Assume $k > 1$. It is easy to see that $\mathcal{E}$ is a uniform vector bundle, i.e. the isomorphism class of the restriction of $\mathcal{E}$ to any line does not depend on the line.[1] This implies that $\mathcal{E}$ splits into the direct sum of vector bundles $\mathcal{O}_{\mathbb{P}^k}(a_i)$. The restriction of the projection $\mathrm{pr}_2 : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$ over ℓ is isomorphic to the projection of $\mathbb{P}^n \dashrightarrow \mathbb{P}^1$ from the space $\langle L, \ell\rangle \cong \mathbb{P}^{n-k+1}$ with center at L. Since the restriction of $\mathcal{E} = \oplus\mathcal{O}_{\mathbb{P}^k}(a_i)$ to ℓ is isomorphic to $\mathcal{O}_\ell^{\oplus n-k} \oplus \mathcal{O}_\ell$, we obtain that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^k}^{n-k} \oplus \mathcal{O}_{\mathbb{P}^k}(1)$. ☐

Example 5.1.1. Let Q be a quadric in $\mathbb{P}^4$ with isolated singular point o. We may consider Q as a cone with vertex at o over a nonsingular quadric Q_0 in a hyperplane H. The quadric Q has 2 rulings by planes arising from the two rulings $\mathcal{P}_1$ and $\mathcal{P}_2$ of Q_0 by lines. Consider the incidence variety

$$X_i = \{(x, P) \in Q \times \mathcal{P}_i : x \in P\}.$$

Let $\phi_i : X_i \to Q$ be the projections to Q. Then ϕ_i is an isomorphism over the complement of the point o, and its fibre R_i over o is equal to $\mathcal{P}_i \cong \mathbb{P}^1$. We claim that the birational transformation $f = \phi_2 \circ \phi_1^{-1} : X_1 \dashrightarrow X_2$ is a flop. First, observe that the second projection $X_i \to \mathcal{P}_i$ is a $\mathbb{P}^2$-bundle. It corresponds to the composition $X_i \setminus R_i \to Q \setminus \{0\} \to Q_0 \to \mathbb{P}^1$, where the last map corresponds to

[1] [Okonek, Vector bundles on complex projective spaces], Theorem 3.2.3

the rulings $\mathcal{P}_i$ on Q. The variety X_i embeds in $\mathbb{P}^4 \times \mathbb{P}^1$ as a complete intersection of two divisors of type $(1,1)$. We have an exact sequence

$$0 \to \mathcal{O}_\Gamma(-1,-1) \to \mathcal{O}_\Gamma \to \mathcal{O}_{X_i} \to 0,$$

where Γ is the graph of the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^1$ from the point o. Twisting by $p_1^*(\mathcal{O}_{\mathbb{P}^4}(1)$ and taking the direct image under $p_2 : \Gamma \to \mathbb{P}^1$, we get, by applying Lemma **??**, an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{E} \to 0.$$

This easily implies that $X_i \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}$. The curve R sits in X_i as the image of a section defined by the projection $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}$. Its normal sheaf $\mathcal{N}_{R_i/X_i}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. By adjunction, the anticanonical line bundle of X_i restricted to R_i is isomorphic to

$$\mathcal{O}_{R_i}(2) \otimes \Lambda^2(\mathcal{N}_{R_i/X_i}) \cong \mathcal{O}_{R_i}(2) \otimes \mathcal{O}_{R_i}(-2) \cong \mathcal{O}_{R_i}.$$

Thus $-K_{X_i}$ is relatively numerically trivial. This proves that the birational map f is a flop.

Let $\pi : Q' \to Q$ be the resolution of singularities of Q equal to the proper transform of Q under the blow up of the point o in $\mathbb{P}^4$. Its exceptional divisor E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. There are two projections $Q' \to X_i, i = 1, 2$, which restricts to E as the two projections $E \to \mathbb{P}^1$.

Example 5.1.2. Let $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the standard Cremona involution. Let X be the blow-up of the vertices of the coordinate simplex. Consider the birational map $T' : X \dashrightarrow X$ obtained by lifting T to X. It is a small birational map that restricts to a birational map of the proper transform of each facet of the simplex to the exceptional divisor over the opposite vertex of the simplex. More generally, for any $k > 1$, it sends the proper transform of any face of codimension k to the proper transform of the opposite face of codimension $n - k - 1$.

Let us see that T' is an example of a flop. We will do it only in the case $n = 3$ leaving the general case to the reader. Consider the linear system of quadrics in $\mathbb{P}^3$ that passes through the vertices of the coordinate simplex. Its dimension is equal to 5. Its proper transform on X has no base points and contracts the proper transforms R_{ij} of the six edges of the coordinate simplex. It is equal to the linear system $|2e_0 - e_1 - e_2 - e_3 - e_4|$, where e_0 is the pre-image of a plane in $\mathbb{P}^3$ and e_i are the divisor classes of the exceptional divisors over the base points. Since $-K_X = 4e_0 - 2e_1 - 2e_2 - 2e_3 - 2e_4$, we see that $-K_X$ is relatively numerically trivial. The image of X is a closed 3-dimensional subvariety Z of $\mathbb{P}^6$ with 6 singular points. Choose a basis in the linear system of quadrics formed by the quadrics

$V(t_0t_1), V(t_0t_2), \dots, V(t_2t_3)$. It gives the birational morphism $\phi : X \to \mathbb{P}^5$ which contracts the curves R_i to the points $[1, 0, \dots, 0], \dots [0, \dots, 0, 1]$. Then the composition $\phi' : T' \circ \phi$ differs from ϕ by the projective transformation g of $\mathbb{P}^5$ given by $[z_0, \dots, z_5] \mapsto [z_5, z_4, \dots, z_0]$. The composition $\phi^{-1} \circ g \circ \phi$ is equal to the composition of 6 flops.

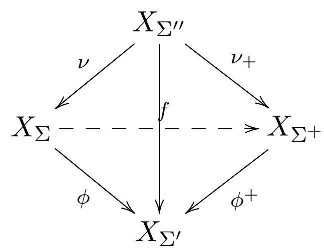
One can give a toric interpretation of each flop. Consider the toric variety $\mathbb{P}^3$ with the standard fan Σ_{st} defined by the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_0 = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ in $\mathbb{R}^3$. For any subset J of $\{0, 1, 2, 3|\}$ denote by F_J the convex cone spanned by the vectors $\mathbf{e}_j, j \in J$. The interior of F_J corresponds to the orbit O_J of dimension $3 - \#J$ and $O_J \subset \overline{O_K}$ if and only if $K \subset J$. More precisely, we can choose coordinates in $\mathbb{P}^3$ such that $\overline{O_J}$ is the coordinate subspace $t_j = 0,\ j \in J$. Blowing up a vertex of the coordinate simplex $O_J, \#J = 3$ we obtain the toric variety whose fan is generated by its 1-skeleton obtained by adding to the 1-skeleton of Σ_{st} the ray $R_J = \mathbb{R}_{\geq 0}(\sum_{j \in J} \mathbf{e}_j)$. Let us blow-up two vertices say $p_3 = O_{012}$ and $p_4 = O_{013}$. The new fan Σ acquires two new rays $R_{012} = \mathbb{R}_{\geq 0}(-\mathbf{e}_3)$, and $R_{013} = \mathbb{R}_{\geq 0}(-\mathbf{e}_2)$. Note that the cone spanned by these two rays does not belong to Σ because its intersection with the cone $F_{01} \in \Sigma$ is equal to the ray $\mathbb{R}(\mathbf{e}_0 + \mathbf{e}_1)$ which does not belong to Σ.

The proper transform ℓ_{34} of the edge $\langle p_3, p_4 \rangle$ corresponds to the cone $F_{01} \in \Sigma$. Let Σ' be the new fan obtained from Σ by deleting the cone F_{01}. Let $\phi : X_\Sigma \to X_{\Sigma'}$ be the toric morphism corresponding to the map of fans $\Sigma \to \Sigma_1$ under which F_{01} is mapped to the cone spanned by $R_{012}, R_{013}, F_0, F_1$. The morphism ϕ is a small contraction of the curve ℓ_{34} to the 0-dimensional orbit defined by the non-simplical cone $\langle R_{012}, R_{013}, F_0, F_1 \rangle \in \Sigma'$. Now let us consider the fan Σ^+ obtained from Σ_1 by adding the 2-face spanned by R_{012} and R_{013} (now there is no problem because we have deleted the face spanned by F_0, F_1). We have a toric morphism $\phi^+ : X_{\Sigma^+} \to X_{\Sigma'}$ which blows down the orbit corresponding to the face $\langle R_{012}, R_{013} \rangle$. This a commutative diagram of rational maps

$$\begin{array}{ccc} X_\Sigma & \overset{f}{\dashrightarrow} & X_{\Sigma^+} \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi^+} & \\ & X_{\Sigma'} & \end{array}$$

By adjunction formula, $\deg K_{\mathbb{P}^3} \cdot \langle p_3, p_4 \rangle = -2$. This easily implies that $K_{X_\Sigma} \cdot \ell_{34} = 0$. Thus f is a flop. Note that adding to Σ' the ray equal to the intersection of the 2-faces $\langle R_{012}, R_{013} \rangle$ and $\langle F_0, F_1 \rangle$, we obtain a fan Σ'' such that the natural morphism $\sigma : X_{\Sigma''} \to X_{\Sigma'}$ is a resolution of singularities with exceptional divisor isomorphic to a nonsingular 2-dimensional quadric. There are two projection $\Sigma'' \to \Sigma$ and $\Sigma'' \to \Sigma^+$ which define two new morphisms $\nu : X_{\Sigma''} \to X_\Sigma$ and

$\nu_+ : X_{\Sigma''} \to X_\Sigma$ which make the following diagram commutative:



For future use, observe that the fans Σ and Σ^+ are not isomorphic. The fan Σ_+ contains the two opposite faces $\langle \mathbf{e}_2, \mathbf{e}_3\rangle$ and $\langle -\mathbf{e}_2, -\mathbf{e}_3\rangle$. The fan Σ does not have such faces. Consider the projection to $N \to N' = \mathbb{Z}^3/\mathbb{Z}(-\mathbf{e}_2)$. The image of $\Sigma_+ \subset N_\mathbb{R} = \mathbb{R}^3$ is a N'-fan with 1-skeleton $\mathbb{R}_{\geq 0}$ equal to the images of the vectors $\mathbf{e}_1, \mathbf{e}_3, -\mathbf{e}_3, -(\mathbf{e}_1 + \mathbf{e}_3)$. It defines the toric surface $\mathbf{F}_1$. The 2-face $\langle \mathbf{e}_2, \mathbf{e}_3\rangle$ of Σ^+ is mapped to the 1-face $\mathbb{R}_{\geq 0}\mathbf{e}_3$. However, it is easy to see, looking at the extremal rays in the Mori cone, that X_Σ does not admit any morphisms onto $\mathbb{F}_1$.

Another type of birational transformation we will need is an elementary transformation of projective bundles.

Let $\mathbb{P}(\mathcal{E}) \to S$ be the projective bundle associated to a rank r locally free sheaf $\mathcal{E}$ over a smooth variety variety S. Let Z be an effective divisor on S considered as a closed subscheme $i : Z \hookrightarrow S$ and $\mathcal{F}$ be a locally free sheaf on Z of rank $r' < r$. Suppose we have a surjection $u : \mathcal{E} \to i_*\mathcal{F}$. It defines a closed embedding $s_u : \mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$. When $\mathcal{F}$ is of rank 1, this is a section over Z.

Since the projective dimension of $\mathcal{O}_Z$ is equal to 1, the sheaf $\mathcal{E}' = \mathrm{Ker}(u)$. is a locally free sheaf on S of rank r. The projective bundle $\mathbb{P}(\mathcal{E}^\vee)$[2]. is denoted by elm$(\mathbb{P}(\mathcal{E}), u, \mathcal{F})$ and is called the projective bundle obtained by an *elementary transformation* along the data $(\mathcal{F}, u)$. Note that

$$c(\mathcal{E}') = c(\mathcal{E})/c(i_*\mathcal{F}). \tag{5.1}$$

Since $\mathcal{E}'$ and $\mathcal{E}$ are isomorphic outside Z, the two projective bundles are isomorphic over the complement of Z. When Z is smooth, one can show that $\mathbb{P}(\mathcal{E}')$ is obtained from $\mathbb{P}(\mathcal{E})$ by blowing up $s_u(Z)$, followed by the blowing down the proper transform of the pre-image of Z in $\mathbb{P}(\mathcal{E})$.

Example 5.1.3. Let $S = \mathbb{P}^1$ and $X = \mathbf{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ Take a closed point $y \in \mathbb{P}^1$ and $\mathcal{F} = \mathcal{O}_y$ the structure sheaf of y.

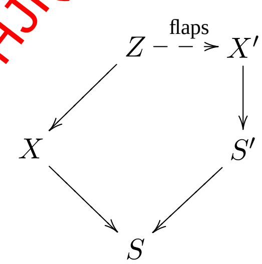
[2] We use Grothendieck's notation $\mathbb{P}(\mathcal{E}) := \mathrm{Proj} S^\bullet \mathcal{E}$

Assume $n > 0$. Let $s_n : \mathbb{P}^1 \to X$ be the section defined by the surjection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \to \mathcal{O}_{\mathbb{P}^1}(-n)$. The image of s_n is the unique curve on X with negative self-intersection equal to $-n$. It is called the *exceptional section*. Consider a surjective map of sheaves $u : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \to \mathcal{O}_y$ and let $\mathcal{E}'$ be its kernel. We know that every locally free sheaf over $\mathbb{P}^1$ is isomorphic to a sheaf $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$. Computing the degrees, we find that $a + b = n + 1$. If both $a, b < n$, we get $\mathcal{E}'$ is contained in the summand $\mathcal{O}_{\mathbb{P}^1}$, so the cokernel of u is not a sky-scrapper sheaf. So, up to switching a, b, we have only two possibilities $(a, b) = (0, n + 1)$ or $(a, b) = (1, n)$. In the former case, u induces an isomorphism on H^0, and hence u factors through the projection to $\mathcal{O}_{\mathbb{P}^1}(-n)$. This shows that u defines a point on the exceptional section $s_n(\mathbb{P}^1)$. The elementary transformation produces the surface $\mathbf{F}_{n+1} \cong \mathbb{P}$. In the latter case u factors through $\mathcal{O}_{\mathbb{P}^1}$. Then u defines a point outside of the exceptional section. The elementary transformation produces the surface $\mathbf{F}_{n-1}$.

Assume $n = 0$. Then, by similar argument, we obtain that $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, hence the elementary transformation produces the surface $\mathbb{F}_1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.

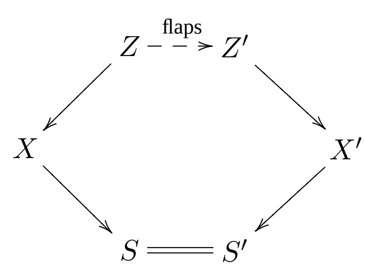
There are 4 types of elementary links between Mori fibre spaces $X \to S$ and $X' \to S'$.

Type I:



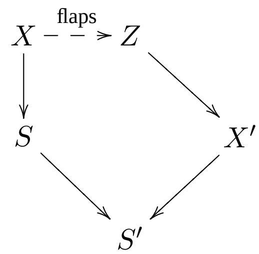
Here $Z \to X$ is a divisorial contraction of an extremal ray R with $K_Z \cdot R < 0$, and $S' \to S$ is a morphism with relative Picard number equal to 1.

Type II:



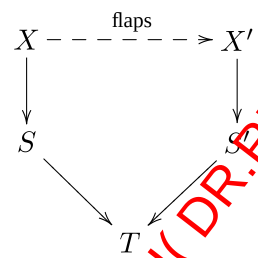
Here $Z \to X$ and $Z' \to X$ are extremal divisorial contractions as in the previous case.

Type III:



This link is the inverse of a link of type I.

Type IV:



Here the morphisms $S \to T$ and $S' \to T$ are morphisms to a normal variety T with relative Picard numbers equal to 1.

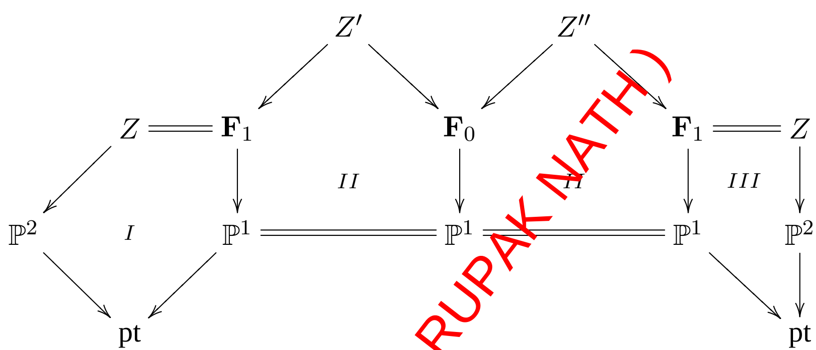
Example 5.1.4. Let us specialize to dimension 2. There are no flips here, and all varieties are nonsingular.

In type I, S could be a one-point variety. In this case X must a Del Pezzo surface with Picard number 1, i.e. $S = \mathbb{P}^2$. The morphism $Z \to X$ is the blow-up of one point. Then $S' = \mathbb{P}^1$ and $Z = X' \cong \mathbb{F}_1 \to S'$ is the $\mathbb{P}^1$-bundle structure on $\mathbb{F}_1$.

In type II, $S = \mathbb{P}^1$, $X \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle, $Z \to X$ blows up one point, and $Z \to X'$ blows down the proper transform of the fibre through this point. So, $X' \to \mathbb{P}^1$ is another projective bundle, and $X \dashrightarrow X'$ is an elementary transformation.

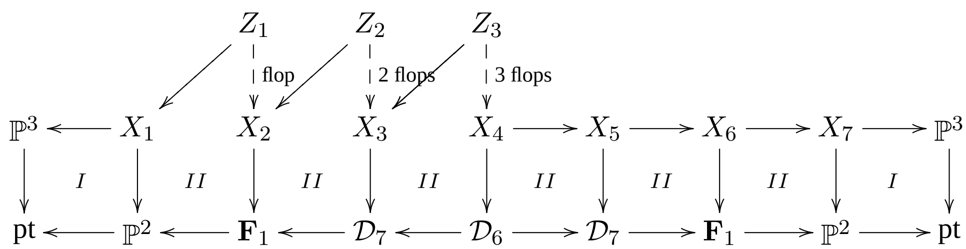
Type III is the inverse of type I. In Type IV, $X = X' = \mathbb{P}^1 \times \mathbb{P}^1$, T is the one-point varietry, and the two Mori fibrations are the two projections to $\mathbb{P}^1$.

Example 5.1.5. Consider the standard Cremona transformation $T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. We start with the Mori fibre space $X = \mathbb{P}^2 \to$ point. Then we blow up the point $p_1 = [1, 0, 0]$ to get a morphism $Z \to X$. The projection from the point p_1 defines a $\mathbb{P}^1$-bundle structure $Z \to \mathbb{P}^1$. This gives us a first link between $X \to$ point and $X' = Z \to \mathbb{P}^1$. It is of type I. Next we make an elementary transformation at the point $p_2 = [0,1,0]$. This is a link of type II leading to one of the projections $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Next we make an elementary transformation at the pre-image of the third base point $[0,0,1]$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This is a link of type II. We obtain the minimal ruled surface $\mathbf{F}_1 \to \mathbb{P}^1$. Finally, we blow down the exceptional section $\mathbf{F}_1 \to \mathbb{P}^2$. This is a link of type III.

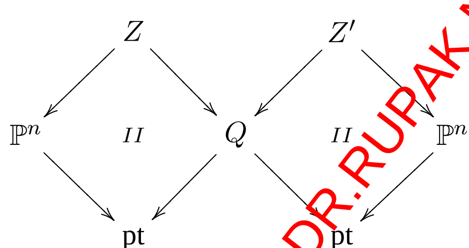


We leave to the reader the job of doing the same for degenerate standard Cremona transformations with two or one base points.

Example 5.1.6. Consider the standard Cremona transformation $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. We start with the Mori fibre space $X = \mathbb{P}^3 \to$ point. Next we blow up the first point $p_1 = [1,0,0,0]$. The projection from this point defines a $\mathbb{P}^1$-bundle $X_1 \to \mathbb{P}^2$. This is our first link. Next we blow-up the pre-image of the second base point $p_2 = [0,1,0,0]$ and make a flop along the proper transform of the edge of the coordinate simplex joining the points p_1, p_2. It follows from Example 5.1.2 that the new surface admits a morphism to $\mathbf{F}_1$. This is our new Mori fibre space. Then we blow up the the third vertex p_3, make two flops along the edges $\langle p_1, p_3\rangle, \langle p_2, p_3\rangle$ and get a morphism to $\mathcal{D}_7$, the blow-up of 2 points (a Del Pezzo surface of degree 7). Next we blow up the vertex p_4, make three flops along the edges $\langle p_i, p_4\rangle, i = 1,2,3$ and get a morphism to $X_3 \to \mathcal{D}_6$, a Del Pezzo surface of degree 6. The 3-fold X_3 is a toric variety, the corresponding fan contains 8 rays spanned by the vectors $\pm\mathbf{e}_i, i = 0,1,2,3$. It has 18 two-dimensional cones $\langle -\mathbf{e}_i, -\mathbf{e}_j\rangle, \langle -\mathbf{e}_i, \mathbf{e}_j\rangle, i \neq j$. It is isomorphic to the blow-up of 4 points in $\mathbb{P}^3$. We can start blowing down the four exceptional divisors making square morphisms of Mori fibre spaces.



Example 5.1.7. Consider a non-degenerate quadratic transformation in $\mathbb{P}^n, n > 2$. Recall that it is decomposed as the blow-up of a nonsingular quadric Q_0 in a hyperplane, followed by the projection from a point on the image quadric $Q \subset \mathbb{P}^{n+1}$ that does not lie in the proper transform of the hyperplane containing the quadric. Its decomposition into elementary links is as follows.

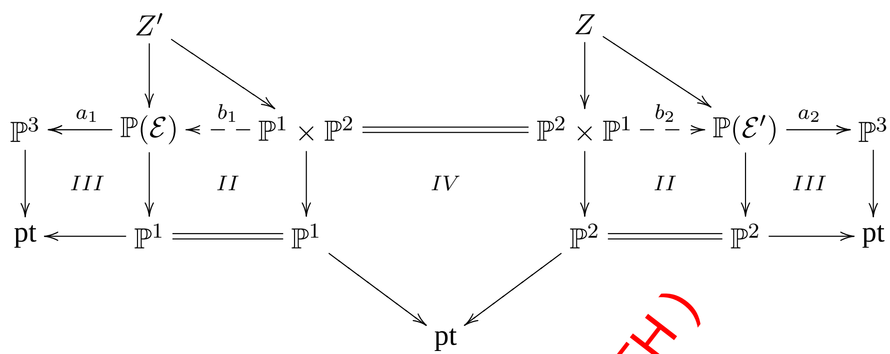


Note that a nonsingular quadric of dimension > 2 is an example of a Fano variety with Picard number equal to 1.

Example 5.1.8. Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) \cong \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the trivial $\mathbb{P}^2$-bundle. Take a point $y \in \mathbb{P}^1$ and a surjection $\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \to \mathcal{O}_y$. The kernel is isomorphic to $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The corresponding projective bundle $\mathbb{P}(\mathcal{E}'^{\vee}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is isomorphic to the blow-up of a line in $\mathbb{P}^3$ with the morphism to $\mathbb{P}^1$ defined by the projection from the line. Let $a_1 : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^3$ be the birational morphism which is the inverse of the blow-up of the line. Let $b_1 : \mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}(\mathcal{E})$ be the birational map defined by the elementary transformation.

Consider the other projection $X = \mathbb{P}^1 \times \mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2})^{\oplus 2} \to \mathbb{P}^2$ and make the elementary transformation with respect to the surjection $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to \mathcal{O}_\ell$, where ℓ is a line. The elementary transformation produces the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. It is isomorphic to the blow-up a point in $\mathbb{P}^3$ with the morphism to $\mathbb{P}^2$ defined by the projection from the point. Let a_2, b_2 be the similarly defined rational maps.

Thus we get a chain of elementary links decomposing the birational map $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ equal to the composition $(a_2 \circ b_2) \circ (b_1^{-1} \circ a_1^{-1})$.



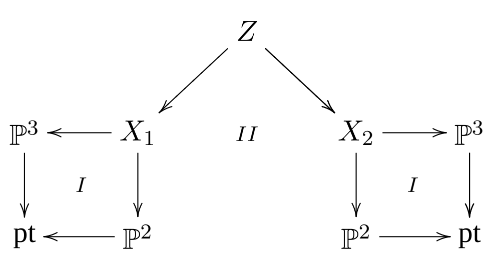
The map T can be described as follows. Fix a point x_i and a line ℓ_i not containing x_i and identify $\mathbb{P}^2$ with the family of lines through x_i and $\mathbb{P}^1$ with the family of planes through ℓ_i. The line ℓ corresponds to the point y over which we do the elementary transformation. The point x_1 is the base point of a_1^{-1}.

Fix a general plane Π_1 and a general line ℓ in $\mathbb{P}^3$. Identify Π with the family of lines through x_i by taking the intersection of a line $\langle x, x_i\rangle$ with the plane Π. Identify $\mathbb{P}^1$ with the family of planes through ℓ_i by taking the intersection of a plane $\langle x, \ell_i\rangle$ with ℓ. Now consider the following map. Take a general point $x \in \mathbb{P}^3$. The line $\ell_x = \langle x, x_1\rangle$ and the plane $\Pi_x = \langle x, \Pi_1\rangle$ defines a line through x_2 and the plane through ℓ_2. We take $T(x)$ to be the intersection point of this line and this plane. Let us see the indeterminacy points of T. Let $A_i = \ell_i \cap \Pi$ and $B_i = \langle x_i, \ell_i\rangle \cap \ell$. Obviously, the image $T(x)$ is not defined if $x = x_1$ or $x \in \ell_1$. Also, T is not defined at any point x in the intersection ℓ_1' of two planes $\langle \ell_1, B_2\rangle$ and $\langle x_1, A_2, B_2\rangle$. Thus we see that the base locus contains x_1 and the union of two intersecting lines $\ell_1 + \ell_1'$. It is easy to check that the pre-image of a general plane in the target $\mathbb{P}^3$ under the map $a_2 \circ b_2$ is a divisor of type $(1,1)$. The map $b_1^{-1} \circ a_1^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ is given by the 5-dimensional linear system of quadrics through x_1 and ℓ_1. It maps $\mathbb{P}^3$ to $\mathbb{P}^5$ with the image equal to $\mathbb{P}^2 \times \mathbb{P}^1$ embedded by the Segre map. This shows that $T^{-1}(\text{plane})$ is a quadric, thus T is a degenerate quadratic transformation.

Example 5.1.9. Consider the Cremona transformation defined by the homaloidal linear system of cubics through the double structure of a rational normal curve C in $\mathbb{P}^3$. Let $|L| = |2H - C|$ be the 2-dimensional linear system of quadrics through C. It defines a structure of a $\mathbb{P}^1$-bundle on the blow-up $q : X_1 = \mathrm{Bl}_C\mathbb{P}^3 \to \mathbb{P}^2$. For any point $x \in \mathbb{P}^2 = |L|^\vee$, the fibre $q^{-1}(x)$ is the secant line ℓ_x contained in the base locus of the pencil of quadrics in $|L|$ defined by this point. The proper transform F of the tangent surface $\mathrm{Tan}(C)$ is mapped to the conic K in the plane parameterizing the tangent lines of C. The exceptional divisor E_1 is a finite cover of $\mathbb{P}^2$ of degree 2. The two surfaces are tangent along the curve R which is a section of the bundle over the conic K (recall that $\mathrm{Tan}(C)$ contains the curve C as its cuspidal curve). The curve R is the ramification curve of the double cover $q|_{E_1} : E_1 \to \mathbb{P}^2$.

Let $\nu : Z = \mathrm{Bl}_R X_1 \to X_1$ be the blow-up of R. Computations from section 2.28 show that the normal bundle of R in X_1 is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(3)^{\oplus 2}$. This shows that the exceptional divisor E_2 of ν is isomorphic to $\mathbf{F}_0$. The proper transform F' of F in Z intersects E_2 along the curve R' of type $(1,1)$. The proper transform of E_1 intersects E_2 along the same curve. The union of E_2 and F' is the pre-image of the conic K under the projection $q \circ \nu : Z' \to X_1 \to \mathbb{P}^2$. Now we blow down F' to a curve R' on the new projective bundle $X_2 \to \mathbb{P}^2$. In other words we make an elementary transformation of $X_1 \to \mathbb{P}^2$ along the curve R given by an invertible bundle $\mathcal{L}$ on K of degree 1.

A fibre of $q : X_1 \to \mathbb{P}^2$ is naturally identified with a secant $\langle a, b\rangle$ of C and the plane $\mathbb{P}^2$ with the symmetric square $C^{(2)}$. The map $\{a,b\} \to \langle a,b\rangle$ defines an embedding of C into the Grassmannian of lines in $\mathbb{P}^3$. The image of the embedding in the Plücker space is a Veronese surface $V \subset G(2,4)$. The morphism q becomes isomorphic to the universal family of lines over the Veronese surface. Thus $X_1 \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is isomorphic to the pre-image of the universal subbundle over $G(2,4)$ restricted to V. Since a general point in $\mathbb{P}^3$ lies on a unique secant and any general plane contains three secants, we obtain that the cohomology class of V is of type $(1,3)$. his allows us to compute the Chern classes of $\mathcal{E}$. We have $c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) = 3$. The first equality implies that $\mathcal{E}'$ is self-dual. The Chern classes of $\mathcal{L}$ are $c_1(\mathcal{L}) = 2, c_2(\mathcal{L}) = 1$. Applying (5.1), we obtain that the new vector bundle $\mathcal{E}' \cong$ obtained by the elementary transformation has $c_1(\mathcal{E}') = 0, c_2(\mathcal{E}') = 2$. One can show that $\mathcal{E}' \cong \mathcal{E}(1)$, so $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}')$. After making the elementary transformation, we define the blowing down $X_2 \to \mathbb{P}^3$ with the exceptional divisor equal to the image of E_1.



5.2 Noether-Fano-Iskovskikh inequality

Let $\mathcal{H}_X$ be a linear system without fixed components on a smooth variety X which defines a rational map $X \dashrightarrow Y$ to a normal variety Y, so that $\mathcal{H}_X$ is equal to the proper transform of an ample linear system $\mathcal{H}_Y$ on Y. We will use a log resolution $\sigma : X' \to X$ of its base scheme. Recall that it is given by a sequence of birational morphisms

$$X' = X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_1 = X,$$

where each morphism $\sigma_i : X_{i+1} \to X_i$ is the blowing up of a smooth closed subscheme B_i of X_i, which we can always assume to be of codimension ≥ 2.

For any $N \geq a > b \geq 1$, we set

$$\sigma_{ab} = \sigma_a \circ \dots \circ \sigma_b.$$

We will often identify B_i with a closed subvariety of $X_j, j \leq i$, if the morphism $\sigma_{ij} : X_i \to X_j$ is an isomorphism at the generic point of B_i.

Let $\mathcal{H}_i = \sigma_{i1}^{-1}(\mathcal{H})$ be the proper transform of $\mathcal{H}$ in X_i. Then B_i is contained in its base scheme and we set

$$m_i = \min_{D \in \mathcal{H}_i} \text{mult}_{\eta_i} D,$$

where η_i is the generic point of B_i.

It follows from Hironaka's Theorem on resolution of singularities that we can also assume that the following *normal flatness* condition is satisfied

- $m_i > 0, m_i \geq m_j, 1 \leq i \leq j \leq N$.
- A general divisor of the linear system $\sigma_{i1}^{-1}(\mathcal{H})$, the scheme-theoretical intersection of two general divisors in $\sigma_{i1}^{-1}(\mathcal{H})$, and the base scheme of $\sigma_{i1}^{-1}(\mathcal{H})$ are normally flat along B_i (in particular, it is equimultiple along B_i).

Let $\bar{E}_i$ be the exceptional divisor of $\sigma_i : X_{i+1} \to X_i$. It is the projective bundle over B_i defined by the normal bundle $\mathbb{P}(\mathcal{N}_{B_i/X_i}^\vee)$ of B_i in X_i. It is a reduced effective divisor in X_i. We denote by E_i its proper transform in X and by $\mathcal{E}_i$ its scheme theoretical pre-image in X.

We define a partial order on the set of subvarieties B_i by writing $B_i > B_j$ if $\sigma_{ij}(\bar{E}_i) \subset \bar{E}_j$. The *Enriques diagram* of the resolution is an oriented graph whose vertices are the subvarieties B_i and an edge connects B_i with B_j if $B_i > B_j$ and B_i is contained in the proper transform of $\bar{E}_j$ in $\bar{E}_i$ under the map σ_{ij}. If additionally, $\sigma_{i,j} : \bar{E}_i \to X_j$ maps B_i isomorphically onto B_j, we say that B_i is infinitely near to B_j of order 1. By induction, we define B_i to be infinitely near of order k to B_j and write $B_i \succ_k B_j$ in this case.

For any $i \le j$ let γ_{ij} be the sum of the lengths of the paths connecting i and j in the Enriques diagram.

In the case of surfaces, all subschemes B_i's are closed points $x_i \in X_{i-1}$ and the exceptional divisors $\bar{E}_i$ are isomorphic to $\mathbb{P}^1$. In this case the notion of being infinitely near makes the partial order on the set of points B_i's. Two points x_i, x_j are connected by an edge (x_i, x_j) if and only if $x_i \succ_1 x_j$.

The standard formula for the behavior of the canonical class under the blow-up of a smooth closed subschemes shows that

$$K_X = \sigma^*(K_{X_0}) + \sum_{i=1}^N \delta_i \mathcal{E}_i = \sigma^*(K_{X_0}) + \sum_{j=1}^N (\sum_{j \le i} \gamma_{ij}\delta_j) E_i, \tag{5.2}$$

where $\delta_j = \text{codim}\,(B_i, X_{i-1}) - 1$.

Let $\mathcal{H}_i = \sigma_{i1}^{-1}(\mathcal{H})$ be the proper transform of the linear system $\mathcal{H}$ to X_i. It is equal to $\sigma_{i1}^*(\mathcal{H}_{i-1}) - m_i \bar{E}_i$. By induction, we obtain

$$\sigma^{-1}(\mathcal{H}) = \sigma^*(\mathcal{H}) - \sum_{i=1}^N m_i \mathcal{E}_i = \sigma^*(H) - \sum_{i=1}^N (\sum_{j \le i} \gamma_{ij} m_j) E_i, \tag{5.3}$$

For example, if $\mathcal{H} = |dH - \sum m_i Z_i|$ is a smooth homaloidal linear system in $\mathbb{P}^n$, we may take for a resolution the sequence of blow-ups of $B_i = (Z_i)_{\text{red}}$. In this case the Enriques diagram consists of disjoint vertices, and we obtain

$$\begin{aligned} K_X &= -(n+1)\sigma^*(H) + \sum a_i E_i, \\ \sigma^{-1}(\mathcal{H}) &= d\sigma^*(H) - \sum m_i E_i. \end{aligned}$$

Definition 5.2.1. *Let $\mathcal{H}$ be a linear system without fixed components on a variety X. We say that the pair $(X, b\mathcal{H})$, where $0 \le b \le 1$ is a rational number, has* canonical singularities *(resp.* terminal*) if there exists a log resolution of* Bs$(\mathcal{H})$ *with*

$$\delta_i - bm_i \ge 0, \ i = 1, \ldots, N, \quad \text{resp.} \ > 0.$$

The canonical threshold *(*terminal threshold *of $\mathcal{H}$ is maximal positive rational number $c(\mathcal{H})$ such that the pairs $(X, c\mathcal{H})$ has canonical (terminal) singularities. An irreducible component E_i is called* crepant *if $c(\mathcal{H}) = a_i/r_i$.*

One can show that these definitions do not depend on a choice of a log resolution and coincide with the definition of the canonical (terminal) pair (X, bD), where D is a general divisor in $\mathcal{H}$.

Thus $\mathcal{H}$ has canonical singularities if $\delta_i \geq bm_i \geq 0$ for all i. Also, we see that

$$c(\mathcal{H}) = \min\{\frac{\delta_i}{m_i}\} \tag{5.4}$$

The following theorem is called the *Noetehr-Fano-Iskovskikh inequality*.

Theorem 5.2.1. *Let $f : X \dashrightarrow Y$ be a birational map defined by a linear system $\mathcal{H}_X = f^{-1}(\mathcal{H}_Y)$ for some ample linear system $\mathcal{H}_Y$ on Y. Assume that, for some rational number t, we have $|t\mathcal{H}_Y + K_Y| = \emptyset$ but $|t\mathcal{H}_X + K_X| \neq \emptyset$. Then $t < c(\mathcal{H}_X)$.*

Proof. Let (X', π, σ) be a log resolution of f. We have

$$\pi^{-1}(\mathcal{H}_X) = \pi^*(\mathcal{H}_X) - \sum_{i=1}^{N} m_i\mathcal{E}_i = \sigma^*(\mathcal{H}_Y).$$

Write $K_{X'} = \pi^*(K_X) + \sum \delta_i\mathcal{E}_i$, multiply both sides by t, add $\pi^*(K_X) + \sum a_iE_i$, to the left-hand side and $K_{X'}$ to the right side to obtain

$$K_{X'} + \sigma^*(t\mathcal{H}_Y) = \pi^*(K_X + t\mathcal{H}_X) + \sum_{i=1}^{N}(\delta_i - tm_i)\mathcal{E}_i. \tag{5.5}$$

Assume $\delta_i \geq tm_i \geq 0$ for all i. Applying σ_* to both sides and using that $\sigma_*(K_{X'}) = K_Y$, we get

$$\emptyset = |K_Y + t\mathcal{H}_Y| = |\sigma_*(\pi^*(K_X + t\mathcal{H}_X) + \sum_{i=1}^{N}(\delta_i - tm_i)\sigma_*(\mathcal{E}_i)|.$$

Since the right-hand side is non-empty, we get a contradiction. Thus $\delta_i - tm_i < 0$ for some i, hence $t > c(\mathcal{H})$. □

Let $\alpha : X \to S$, and $\alpha'' : Y \to S'$ be Mori fibrations. Since the relative Picard numbers are equal to 1, we can write

$$\mathcal{H}_X \subset |-rK_X + \alpha^*(A)|,\ \mathcal{H}_X = -r'K_X + \alpha'^*(A')$$

for some rational positive numbers r, r' and some effective divisor classes on S and S'. Then $t\mathcal{H}_Y + K_Y| = \emptyset$ for $t > r'$. So if $r > r'$, then $|t\mathcal{H}_X + K_X| \neq \emptyset$ for $r \geq t > r'$. Thus

$$c(\mathcal{H}) < r' \tag{5.6}$$

Together with (5.4), this implies that

$$\max\{\frac{m_i}{\delta_i}\} > \frac{1}{r'}.$$

Let us apply this to the case when $X = Y = \mathbb{P}^n$, $\mathcal{H}_X \subset |dh|$ is a homaloidal linear system on $\mathbb{P}^n$ and $\mathcal{H}_Y = |h|$, where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Since $K_{\mathbb{P}^n} = -(n+1)h$, we obtain that $r' = \frac{n+1}{d}$, hence

$$c(\mathcal{H}_X) < \frac{n+1}{d}, \tag{5.7}$$

and

$$\max\{\frac{m_i}{\delta_i}\} > \frac{d}{n+1}.$$

For example, if $n = 2$, we obtain that there exist a base point of multiplicity $> d/3$. This of course follows from Noether's inequality

$$m_1 + m_2 + m_3 > d,$$

if $m_1 \geq m_2 \geq \ldots \geq m_N$.

Let us rewrite $\sum_{i=1}^N(\delta_i - tm_i)\mathcal{E}_i$ as the sum $\sum_{j\leq i}\gamma_{ij}(\delta_j - tm_j)E_i$. If all these sums are non-negative, then $t \leq c(\mathcal{H}_X)$. Thus

$$c(\mathcal{H}) = \min_{i,j}\Big\{\frac{\sum_{j\leq i}\gamma_{ij}\delta_j}{\sum_{j\leq i}\gamma_{ij}m_j}\Big\}.$$

Hence, in view of (5.6), we obtain

$$\max_{i,j}\Big\{\frac{\sum_{j\leq i}\gamma_{ij}m_j}{\sum_{j\leq i}\gamma_{ij}\delta_j}\Big\} > \frac{1}{r'}. \tag{5.8}$$

Definition 5.2.2. *The exceptional divisor E_i is called the* maximal singularity *of the linear system $\mathcal{H}$ if it satisfies* (5.8).

Assume that all $\delta_j \geq \delta_i$ for any $i \leq j$ (e.g. $\dim X = 2$). Let E_{j_0} corresponds to the minimal vertex in the Enriques diagram, i.e. $\sigma_{j_0 1} : X_j \to X$ is an isomorphism at the generic point of B_j. Then, by the choice of a resolution, we have $m_{j_0} \geq m_j$ for any $j \leq i$ with $\gamma_{ij} \neq 0$. This implies that

$$\frac{m_{j_0}}{\delta_{j_0}} = \frac{\sum_{j\leq i}\gamma_{ij}m_{j_0}}{\delta_{j_0}\sum_{j\leq i}\gamma_{ij}} \geq \frac{\sum_{j\leq i}\gamma_{ij}m_j}{\sum_{j\leq i}\gamma_{ij}\delta_j} > \frac{1}{r'}.$$

So, we can find the maximal singularity among the irreducible components of the base scheme and its multiplicity m satisfies

$$m \geq \frac{\delta}{r'},$$

where $\delta + 1$ is equal to the codimension of the component.

Example 5.2.1. Let us look at Example 2.5.3. The Enriques diagram consists of 4 minimal vertices corresponding to the for points in the base locus, and the fifth vertex corresponding to a line infinitely neat to the fourth vertex. All the multiplicities m_i are equal to 1. Any of the minimal vertices give $m_i/\delta_i = 1/2$. we have $r' = 2/4 = 1/2$. So, none of them is maximal. For the fifth vertex the ratio (5.8) is equal to $2/3$, and this is larger than $1/2$. So, the component E_5 of the exceptional divisor of the resolution is maximal.

For any morphism $\phi : X' \to X$ we denote by $\equiv_\phi$-equivalence of $\mathbb{Q}$-Weil divisors (where $D \equiv_\phi 0$ if $D \cdot C = 0$ for any curve C with $\sigma(C) =$ point).

The following is known as the *Negaivity Lemma*. We refer for the proof to [Kollar].

Lemma 5.2.2. *Let $\sigma : X' \to X$ be a birational morphism with exceptional irreducible divisors E_i. A $\mathbb{Q}$-Weil effective divisor D is called σ-effective if no E_i appears in D. Assume that*

$$\sum a_i E_i \equiv_\phi H + D$$

for some σ-effective divisor D and σ-nef $\mathbb{Q}$-Cartier divisor H. Then all a_i are non-positive. Moreover, if $\sigma(E_i) = x \in X$ and the restriction of D or H to $\sigma^{-1}(x)$ is not numerically trivial, then $a_i < 0$.

Corollary 5.2.3. *Let X, Y be $\mathbb{Q}$-factorial varieties, and $f : X \dashrightarrow Y'$ be a small birational map. Let H be an ample divisor on X such that $H' = f(H)$ is nef. Then f^{-1} is a morphism, hence f is an isomorphism if H' is ample.*

Proof. Choose a smooth resolution (X, π, σ). It suffices to show that, for any point $y \in Y$, $\dim \pi(\sigma^{-1}(y)) = 0$. Since f is small, the exceptional divisors E_i of π are the same as exceptional divisor of σ. Then $\sigma_*(\pi^*(H)) = H'$ and $\sigma_*(\sigma^*(H')) = H'$ implies

$$\sum a_i E_i = \sigma^*(H') - \pi^*(H) = (\pi - \text{nef}) + (\pi - \text{effective}).$$

Also

$$-\sum a_i E_i = -\pi^*(H) + \sigma^*(H') = (\sigma - \text{nef}) + (\sigma - \text{effective}).$$

By the Negativity Lemma, we get that all $a_i = 0$. Thus $\sigma^*(H') = \pi^*(H)$. Suppose $\dim \pi(\sigma^{-1}(y)) > 0$ for some point y. Take a curve C in $\sigma^{-1}(y)$ such that $\pi(C)$ is not a point. Then

$$0 = H' \cdot \sigma_*(C) = \sigma^*(H') \cdot C = \pi^*(H) \cdot C = H \cdot \pi_*(C) > 0.$$

This is a contradiction. □

Note that in the case of surfaces, this is well-known and follows from negative definiteness if the intersection matrix of exceptional divisors.

Theorem 5.2.4. *Let $f : (\phi : X \to S) \to (\phi' : X \to S')$ be a birational isomorphism of Mori fibre space defined by a linear system without fixed components $\mathcal{H} \subset |D|$. Write D in the form $-\mu K_X + \phi^*(A)$ for some effective divisor class A on S and a rational positive number μ (this is possible, by definition of a Mori fibre space). Let $\mathcal{H} = f^{-1}(\mathcal{H}')$, where $\mathcal{H}' \subset |D'| = |-\mu' K_{X'} + \phi'(A')|$ for some ample D' on X' and ample A' on S. Then*

- *(i) $\mu \geq \mu'$ and $\mu = \mu'$ if and only if f is square;*
- *(ii) if $(X, \frac{1}{\mu}\mathcal{H}_X)$ is a canonical pair and $|K_X + \frac{1}{\mu}\mathcal{H}_X|$ is nef, then f is a Sarkisov isomorphism.*

Proof. Let us consider equality (5.5). Substitute $K_{X'} = \sigma^*(K_Y) + \sum b_i F_i$ for some exceptional divisors F_i of σ. Then we get

$$\pi^*(K_X + t\mathcal{H}_X) + \sum (a_i - t r_i) E_i = \sigma^*(K_Y + t\mathcal{H}_Y) + \sum b_i F_i. \qquad (5.9)$$

Let Γ be a general curve contained in fibres of $X \to S$ which is disjoint from $\phi^*(A)$ and all $\pi(E_i)$'s. The latter condition allows one to identify Γ with its pre-image in X. Intersecting both sides with Γ, we obtain

$$\pi^*(K_X + t\mathcal{H}_X) \cdot \Gamma = (\sigma^*(K_Y + t\mathcal{H}_Y) \cdot \Gamma + \sum b_i F_i \cdot \Gamma.$$

Take $t = 1/\mu'$, then the right-hand side is equal to

$$\frac{1}{\mu'}\sigma^*(\phi'^*(A')) \cdot \Gamma + \sum b_i F_i \cdot \Gamma.$$

Since X has terminal singularities all b_i's are positive. Also A' is ample. This implies that the right-hand side is non-negative. Since the left-hand side is equal to $(1 - \frac{\mu}{\mu'})K_X \cdot \Gamma$, we get $1 - \frac{\mu}{\mu'} \leq 0$, i.e. $\mu' \leq \mu$. Moreover, if $\mu = \mu'$, then $\phi'(A') \cdot \cdot \sigma_*(\Gamma) = 0$, thus $f(\Gamma) = \sigma_*(\pi^*(\Gamma))$ is contained in fibres of ϕ'. This means that f is square. This checks (i).

Assume that $(\mathcal{H}_X, \mu)$ is canonical. Then, if we take $t = 1/\mu$ in (5.9), we obtain that the coefficients $a' = a_i - \frac{r_i}{\mu}$ in the sum $\sum(a_i - \frac{r_i}{\mu})E_i$ are all non-negative. Replacing now Γ with a general curve Γ' contained in fibres of ϕ', and repeating the previous argument, we obtain $\mu \leq \mu'$. By (i), f is square.

Let $E_i = F_j$ for some subset K of indices $i \in I$. Consider the equality

$$\pi^*(K_X + \frac{1}{\mu}\mathcal{H}_X) + \sum a_i' E_i = \sigma^*(K_Y + \frac{1}{\mu}\mathcal{H}_Y) + \sum b_i F_i$$

It gives

$$\sum a_i' E_i - \sum b_i F_i \equiv_\pi \sigma^*(K_Y + \frac{1}{\mu}\mathcal{H}_Y),$$

$$\sum b_i F_i - \sum a_i' E_i \equiv_\sigma \pi^*(K_X + \frac{1}{\mu}\mathcal{H}_X).$$

If $E_i = F_j$ for some i and j, then E_i enters in the equality with coefficients $a_i' - b_j$ and, in the second equality, with the coefficient $b_j - a_i$. The right-hand sides are the sums of a relative nef and relative effective divisors. Applying the Negativity Lemma, we get that all non-zero coefficients and both equalities are non-positive, in particular $a_i' = b_j$ if $E_i = F_j$. So, we obtain that each exceptional irreducible divisor of σ is equal to a non-crepant exceptional irreducible divisor of π and all other exceptional E_i' are crepant.

Now let $\nu : Z \to X$ be the extraction of the crepant exceptional divisors of π and let $f' = f \circ \nu : Z \dashrightarrow Y$ be the composition of the rational maps. Then the exceptional irreducible divisors of f and f^{-1} are the same. In particular, f' is a small birational map. Let us show that f' is an isomorphism. Choose a very ample divisor on Z and $D' = f'(D)$ be its image in Y. Since

$$K_Z + \frac{1}{\mu}\mathcal{H}_Z + \delta D = (\phi \circ \nu)^*(A) + \delta D$$

is ample (and terminal) for $0 < \delta << 1$ and similarly

$$K_Y + \frac{1}{\mu}\mathcal{H}_Y + \delta D = (\phi \circ \nu)^*(A) + \delta D$$

Then σ has the same set of irreducible components and we have

$$\pi^*(K_X + \frac{1}{\mu}\mathcal{H}_X) + \delta D' = (\phi \circ \nu)^*(A') + \delta D'$$

is ample (and terminal) for $0 < \delta << 1$. Applying Lemma **??**, we obtain that f' is an isomorphism. In particular, $g = f^{-1}$ is a *crepant morphism* (i.e. all of its exceptional divisors are crepant).

We have

$$g^*(K_X + \frac{1}{\mu}\mathcal{H}_X) = K_Y + \frac{1}{\mu}\mathcal{H}_Y = \phi'^*(\frac{1}{\mu}A').$$

This implies that g factors through $\phi' : Y \to S'$, hence g, and hence f, is an isomorphism. Since we know that f sends fibres of ϕ to fibres of ϕ, f induces an isomorphism $S \cong S'$. □

5.3 The untwisting algorithm

In this section we explain how the Sarkisov program works. The main idea (called a *2-ray game*) is as follows. We start with a birational map $f : (X \xrightarrow{\phi} S) \to (X' \xrightarrow{\phi'} S')$ between two Mori fibre spaces. It is given by a linear system $\mathcal{H}$ on X without fixed components such that $\mathcal{H}_X = f^{-1}(\mathcal{H}_{X'})$, where $\mathcal{H}_{X'}$ is a very ample linear system on X'. We write $\mathcal{H}_X = -\mu K_X + \phi^*(A), \mathcal{H}_{X'} = \mu' K_{X'} + \phi'^*(A')$.

We introduce the invariants, called the *Sarkisov degree*. They form a triple (μ, c, e) which consists of the number μ such that $\mathcal{H} = -\mu K_X + \phi^*(A)$, the canonical discrepancy $c(\mathcal{H})$ and the number e of crepant divisorial valuations for $\mathcal{H}$. We order lexicographically the Sarkisov degrees.

Suppose f is not an isomorphism of Mori fibre spaces. By Theorem 5.2.4, there are two cases to consider

(i) $(X, K_X + \frac{1}{\mu}\mathcal{H})$ does not have canonical singularities;

(ii) $(X, K_X + \frac{1}{\mu}\mathcal{H})$ has canonical singularities but $K_X + \frac{1}{\mu}\mathcal{H}$ is not nef.

If the first case occurs, then $c < \frac{1}{\mu}$. We apply Proposition 5.3.1 below to find a blow-up $\nu : Z \to X$ with $\rho_{Z/X} = 1$ such that $K_Z + c\mathcal{H}_Z = f^*(K_X + c\mathcal{H}_X)$, where $\mathcal{H}_Z = \sigma^{-1}(\mathcal{H}_X)$. It is called a *maximal extraction*. In the case of surfaces, this is just the blow-up a base point with maximal multiplicity. One shows that a maximal extraction strongly decreases the Sarkisov degree. Since $\rho(Z/S) = 2$, there will be two relative extremal rays in $\overline{NE}(Z)$. One of them P defines a relative contraction $(Z/S) \to (X/S)$. We look at another one Q and consider its contraction $(Z/S) \to (Y/S)$. If Y is in the Mori category, we obtain a new Mori fibre space Y/S. This is our first link of type I or II. If Y/S is not a Mori fibre space, we make a small contraction $(Z'/S) \to (Y/S)$ with flapping ray P'. Again the relative Picard number of (Z'/S) is equal to 2, so we can find another extremal ray Q'. We repeat the game hoping that a sequence of flaps terminates (it does in dimension 3).

In case (ii) we make the link by choosing some contraction $S \to T$ as follows. Let $P \in \overline{NE}(X)$ be the ray defining the Mori fibration $X \to S$. Since $K_X + \frac{1}{\mu}\mathcal{H}$ is not nef, we find an extremal ray Q with $(K_X + \frac{1}{\mu}\mathcal{H}_X) \cdot Q < 0$. Since $K_X + \frac{1}{\mu}\mathcal{H} = \phi^*(A)$, the rays P and Q are different. Then we show that the extremal face $\langle P, Q \rangle$ exists, so we can contract this face to get $X \to Y \to T$ equal to $X \to S \to T$. This step decreases the Sarkisov degree.

The process ends when the new μ becomes equal to μ' and new $K + \frac{1}{\mu}\mathcal{H}$ becomes canonical and nef. Then we obtain a Sarkisov isomorphism.

Proposition 5.3.1. *Let $\mathcal{H}$ be a linear system without fixed components defining a birational map $f : X \dashrightarrow Y$. There exists Z with at most $\mathbb{Q}$-factorial terminal singularities and a Mori extremal contraction $\nu : Z \to X$ such that $(Z, c(\nu^{-1}(\mathcal{H})))$ has canonical singularities and*

$$K_Z + c\nu^{-1}(\mathcal{H}) = \nu^*(K_X + c\mathcal{H}).$$

Here c is the canonical threshold of $\mathcal{H}$. The morphism c is called the maximal extraction.

Proof. If all dimensions of the base schemes B_i in a resolution satisfy $\dim B_i \leq \dim B_j$ if $B_i > B_i$, this is easy. We choose a maximal singularity represented by an irreducible component of the base scheme, and blow it up. The corresponding birational morphism $\nu : Z \to X$ is our maximal extraction.

Let us consider the general case. Consider a log resolution $\pi : Z \to X$. Let c be the canonical threshold of $(X, \mathcal{H})$. Let t_0 be minimal $t \geq 0$ such that

$$K_Z + t_0\mathcal{H}_Z = \pi^*(K_X + t_0\mathcal{H}_X) + \sum (a_i - t_0 r_i)E_i$$

is nef. Consider an extremal ray R in $\overline{NE}(Z/X)$ such that $K_Z \cdot R < 0$ and $(K_Z + t_0\mathcal{H}_Z) \cdot R = 0$. We find such R represented by a curve in some E_i such that $a_i - t_0 r_i = 0$

Let $Z \to Z_1$ be the contraction of R and $\mathcal{H}_{Z_1}$ be the image of $\mathcal{H}_Z$. We define $t_1 \geq t_0$ as above, and continue by induction until we define $(Z_i, \mathcal{H}_{Z_i}, t_i)$. The process terminates when we are left with only one contraction $Z_k \to X$ with $c = t_k$. This is our maximal extraction.

□

5.4 Noether Theorem

Let us see how the Sarkisov Program allows to prove Noether's Theorem on factorizations of a Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Proposition 5.4.1. *Let $f : Q \dashrightarrow Q$ be a birational automorphism of a nonsingular quadric $Q \cong \mathbf{F}_0$ in $\mathbb{P}^3$. Consider it as a Mori fibre space $X_0 \to S$ by choosing one of the rulings $\phi_1 : Q \to S = \mathbb{P}^1$. Then f is a composition of links of type II and IV between Mori fibre spaces $X_i \to \mathbb{P}^1$, where $X_i = \mathbf{F}_0$ for even i and $X_i = \mathbf{F}_1$ for odd i.*

Proof. Let $\mathcal{H}$ be a linear system on $X_0 = Q$ defining a birational map $f : Q \dashrightarrow Q$. We choose the very ample linear system $\mathcal{H}' = \mathcal{O}_Q(1)$ on the target quadric. The linear system $\mathcal{H}$ is contained in the complete linear system $|af_1+bf_2|$, where f_1, f_2 are the standard generators of Pic(Q). Obviously, we may assume that $a \geq b > 0$. We have $-K_Q = 2f_1 + 2f_2$. Let us consider the Mori fibration on Q defined by the projection $p_1 : Q \to \mathbb{P}^1$ with fibre of the divisor class f_1 so that f_2 generates the relative Picard group. We have $\mathcal{H}_0 = -\frac{b}{2}K_{X_0} + a - bf_1$, so that $\mu = \frac{b}{2}$. Obviously, $\mu' = 1/2$. Applying Theorem 5.2.4, we obtain $b > 1$, unless the map is an isomorphism, in which case the assertion is obvious. Since $|K_Q + \mathcal{H}'| = \emptyset$ but $|K_Q + \frac{1}{\mu}\mathcal{H}| \neq \emptyset$, we obtain that the canonical threshold c of $(Q, \mathcal{H})$ satisfies $c > \frac{1}{\mu} = \frac{2}{b}$. Thus we can find a base point x_1 of maximal multiplicity $r_1 > b/2$. Let $\nu_1 : Z \to Q$ be the blow-up of x_1 with exceptional divisor E_1. Let F_1 be the proper transform of the fibre of $p_1 : Q \to \mathbb{P}^1$ passing through x_1. We have $\nu_1^*(f_1) = E_1 + F_1$. Thus we can write

$$-K_Z = 2\nu_1^*(f_2) + 2\nu_1^*(f_1) - E_1 = 2\nu_1^*(f_2) + 2F_1 + E_1,$$

$$\mathcal{H}_1 = \nu_1^{-1}(\mathcal{H}) = a\nu_1^*(f_2) + b\nu_1^*(f_1) - r_1E_1 = a\nu_1^*(f_2) + bF_1 + (b - r_1)E_1.$$

Let $\sigma_1 : Z \to X_1 \cong \mathbf{F}_1$ be the blowing down of F_1. Then

$$-K_{X_1} = \sigma_1(E_1) + 2\sigma_1(\nu_1^*(f_2)) = g + 2s,$$

where $g = [\sigma_1(E_1)]$ is the class of a fibre of the projection $p : \mathbf{F}_1 \to \mathbb{P}^1$ and s is a section from the divisor class $g + s_0$, where s_0 is the exceptional section. We also have

$$\mathcal{H}_1 = (\sigma_1)_*(\mathcal{H}_1) = as + (b - r_1)g = -\frac{a}{2}K_{X_1} + (b - \frac{a}{2} - r_1)g.$$

Since $b \geq 2$, we have $a \geq 2$, hence $2/a \leq \mu'$ and the canonical threshold c' of $\mathcal{H}_1$ is greater than 1. This gives a maximal base point x_2 of multiplicity $> a/2$. Since $s_0 \cdot \mathcal{H}_1 = b - r_1 < b/2 < r_1$, the point x_2 cannot lie on the exceptional section s_0. Let $\nu_1 : Z_1 \to X_1$ be the blow-up of x_2 and then $\sigma_1 : Z_1 \to X_2$ be the blow-down of the proper transform of the fibre of $p : X_1 \to \mathbb{P}^1$ through the point x_2. The surface X_2 is isomorphic to $\mathbf{F}_0$ and the birational transformation

$$t_{x_1,x_2} := (\sigma_1 \circ \nu_1^{-1}) \circ (\sigma \circ \nu^{-1}) : Q \dashrightarrow X_2$$

is equal to the composition of the two elementary transformations $\mathrm{elm}_{x_2} \circ \mathrm{elm}_{x_1}$ on Q, where we identify x_1 and x_2 with their images on Q. Note that the point x_2 could be infinitely near to x_1.

□

Let us fix a point $x_0 \in Q$. The linear projection $p_{x_0} : Q \setminus \{x_0\} \to \mathbb{P}^2$ defines a birational map. Let l_1, l_2 be two lines on Q passing through x_0 and q_1, q_2 be their projections. The inverse map $p_{x_0}^{-1}$ blows up the points q_1, q_2 and blows down the proper transform of the line $\overline{q_1, q_2}$. For any birational automorphism T of $\mathbb{P}^2$ the composition $p_{x_0}^{-1} \circ T \circ p_{x_0}$ is a birational transformation of Q. This defines an isomorphism of groups

$$\Phi_{x_0} : \mathrm{Bir}(Q) \stackrel{\cong}{\to} \mathrm{Bir}(Q),\ \sigma \mapsto p_{x_0} \circ \sigma \circ p_{x_0}^{-1}.$$

Explicitly, $\Phi_{x_0}^{-1}$ is given as follows. Choose coordinates in $\mathbb{P}^3$ such that $Q = V(z_0z_3 - z_1z_2)$ and $x_0 = [0,0,0,1]$. The inverse map $p_{x_0}^{-1}$ can be given by the formulas

$$[t_0, t_1, t_2] \mapsto [t_0^2, t_0t_1, t_0t_2, t_1t_2].$$

If T is given by the polynomials f_0, f_1, f_2, then $\Phi_{x_0}(T)$ is given by the formula

$$[z_0, z_1, z_2, z_3] \mapsto [f_0(z')^2, f_0(z')f_1(z'), f_0(z')f_2(z'), f_1(z')f_2(z')], \tag{5.10}$$

where $f_i(z') = f_i(z_0, z_1, z_2)$.

Remark 5.4.1. Let $z_1, \ldots, z_n \in Q$ be base points of T different from x_0. Let $T^{-1}(x_0)$ be a point if T^{-1} is defined at x_0 or the principal curve of T corresponding to x_0 with x_0 deleted if it contains it. The Cremona transformation $\Phi_{x_0}(T)$ is defined outside the set $q_1, q_2, p_{x_0}(z_1), \ldots, p_{x_0}(z_n), p_{x_0}(T^{-1}(x_0))$. Here, we also include the case of infinitely near fundamental points of T. If some of z_i's lie on a line l_i or infinitely near to points on l_i, their image under p_{x_0} is considered to be an infinitely near point to q_i.

Let $\mathrm{Aut}(Q) \subset \mathrm{Bir}(Q)$ be the subgroup of biregular automorphisms of Q. It acts naturally on $\mathrm{Pic}(Q) = \mathbb{Z}f + \mathbb{Z}g$, where $f = [l_1], g = [l_2]$. The kernel $\mathrm{Aut}(Q)^o$ of this action is isomorphic to $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(2) \times \mathrm{PGL}(2)$. The quotient group is of order 2, and its nontrivial coset can be represented by the automorphism τ of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ defined by $(a,b) \mapsto (b,a)$.

Proposition 5.4.2. *Let $\sigma \in \mathrm{Aut}(Q)^o$. If $\sigma(x_0) \neq x_0$, then $\Phi_{x_0}(\sigma)$ is a quadratic transformation with fundamental points $q_1, q_2, p_{x_0}(\sigma^{-1}(x_0))$. If $\sigma(x_0) = x_0$, then $\Phi_{x_0}(\sigma)$ is a projective transformation.*

Proof. It follows from Remark 5.4.1 that $\Phi_{x_0}(\sigma)$ has at most 3 fundamental points if $\sigma(x_0) \neq x_0$ and at most 2 fundamental points if $\sigma(x_0) = x_0$. Since any birational map with less than 3 fundamental points (including infinitely near) is regular, we see that in the second case $\Phi_{x_0}(\sigma)$ is a projective automorphism. In the first case, the image of the line $\overline{q_1, q_2}$ is equal to the point $p_{x_0}(\sigma(x_0))$. Thus $\Phi_{x_0}(\sigma)$ is not projective. Since it has at most 3 fundamental points, it must be a quadratic transformation. □

Take two points x, y which do not lie on the same fibre of each projection $p_1 : \mathbf{F}_0 \to \mathbb{P}^1, p_2 : \mathbf{F}_0 \to \mathbb{P}^1$. Let $x = F_1 \cap F_2$, $y = F_1' \cap F_2'$, where F_1, F_1' are two fibres of p_1 and F_2, F_2' are two fibres of p_2. Then $t_{x,y}$ is a birational automorphism of $\mathbf{F}_0$.

Proposition 5.4.3. *$\Phi_{x_0}(t_{x,y})$ is a product of quadratic transformations. If $x_0 \in \{x, y\}$, then $\Phi_{x_0}(t_{x,y})$ is a quadratic transformation. Otherwise, $\Phi_{x_0}(t_{x,y})$ is the product of two quadratic transformation.*

Proof. Assume first that y is not infinitely near to x. Suppose x_0 coincides with one of the points x, y, say $x_0 = x$. It follows from Remark 5.4.1 that $\Phi_{x_0}(T)$ is defined outside $q_1, q_2, p_{x_0}(y)$. On the other hand, the image of the line $\overline{q_1, p_{x_0}(y)}$ is a point. Here we assume that the projection $\mathbf{F}_0 \to \mathbb{P}^1$ is chosen in such a way that its fibres are the proper transforms of lines through q_1 under $p_{x_0}^{-1}$. Thus $\Phi_{x_0}(T)$ is not regular with at most three F-points, hence is a quadratic transformation.

If $x_0 \neq x, y$, we compose $t_{x,y}$ with an automorphism σ of Q such that $\sigma(x_0) = x$. Then

$$\Phi_{x_0}(t_{x,y} \circ \sigma) = \Phi_{x_0}(t_{x_0,\sigma^{-1}(y)}) = \Phi_{x_0}(t_{x,y}) \circ \Phi_{x_0}(\sigma).$$

By the previous lemma, $\Phi_{x_0}(\sigma)$ is a quadratic transformation. By the previous argument, $\Phi_{x_0}(t_{x_0,\sigma^{-1}(y)})$ is a quadratic transformation. Also the inverse of a quadratic transformation is a quadratic transformation. Thus $\Phi_{x_0}(t_{x,y})$ is a product of two quadratic transformations.

Now assume that $y \succ x$. Take any point $z \neq x$. Then one can easily checks that $t_{x,y} = t_{z,y} \circ t_{x,z}$. Here we view y as an ordinary point on $t_{x,z}(\mathbf{F}_0)$. □

Theorem 5.4.4. *Any Cremona transformation of the plane is a composition of projective transformations and the standard quadratic transformation T_0.*

Proof. Combining the previous propositions, we obtain that any Cremona transformation of the plane is a composition of projective and quadratic transformations. A quadratic transformation with reduced base locus is a composition of the standard Cremona transformation T_{st} and a projective transformation.

It is enough to show that a quadratic transformations T with two or one base points is a composition of $T_{\rm st}$ and projective transformations. Suppose T has fundamental points at p_1, p_2 and an infinitely near point $p_3 \succ_1 p_1$. Choose a point q different from p_1, p_2, p_3 and not lying on the line $\overline{p_1, p_2}$. Let f be a quadratic transformation with base points at p_1, p_2, q. It is easy to check that $f \circ T$ is a quadratic transformation with three base points $(p_1, p_2, T(q))$. Composing it with projective automorphisms we get the standard quadratic transformation $T_{\rm st}$.

Finally, let us consider a quadratic transformation T with base points $p_3 \succ p_2 \succ p_1$. Take a point q which is not on the line passing through p_1 with tangent direction p_2. Consider a quadratic transformation f with base points p_1, p_2, q. It is easy to see that $f \circ T$ is a quadratic transformation with F-points $p_1, p_2, \tau_3(q)$. By the previous case we can write this transformation as a composition of $T_{\rm st}$ and projective transformation. □

5.5 Hilda Hudson's Theorem

Noether's Theorem has no analogs in higher dimension. In fact, the following negative result was proven by Hilda Hudson 1927.

Theorem 5.5.1. *Any set of generators of the Cremaon group* $\mathrm{Cr}(3)$ *must contain uncountable number of transformations of degree* $d > 1$.

By a different method, and for arbitrary $n \geq 3$, this result was proven recently by Ivan Pan. We will reproduce it here.

First we generalize the construction of a dilated Cremona transformation.

Obviously, the Cremona transformation defined by $(F_0, QG_1, \ldots, QG_n)$ contains the hypersurface $V(Q)$ in its P-locus. Conversely, for any hypersurface $V(Q')$ of degree k with a point q of multiplicity $\geq k-1$ one can construct, using the previous lemma, a Cremona transformation of degree $d > k$ that contains the hypersurface in its P-locus. For example, we can take $G_i = t_i, i = 1, \ldots, n$, $Q = H^{d-k-1}Q'$, where $V(H)$ is a general hyperplane containing q, and F_0 is a hypersurface of degree d with $d-1$-multiple point at q with $(F_0, HQ') = 1$.

Now we are ready to prove Hudson-Pan's Theorem. Consider a family of uncountably many degree $d > 1$ hypersurfaces $X_i, i \in I$, no two birationally isomorphic, which all vanish at some point q with multiplicity d (for example, cones over plane pairwise non-isomorphic cubic curves[3]). For each such $i \in I$ we construct a Cremona transformation T_i which contains X_i in its P-locus. Suppose we have a set of generators of $\mathrm{Cr}(n)$ which contains only countably many

[3]Here we use that $n > 2$

non-projective transformations. Each generator contains only finitely many hypersurfaces in its P-locus. If we have a product of generators $g = g_{i_1} \cdots g_{i_s}$, then each irreducible hypersurface in the P-locus of g is contained in the P-locus of some g_{i_k}. Thus the set of hypersurfaces contained in the P-locus of some Cremona transformation is countable. This contradictions proves the theorem.