Chapter 6

Row-Column Designs

6.1 Double blocking

As we noted in Section 4.1, sometimes more than one system of blocks is necessary.

Example 6.1 (Example 4.5 continued: Wine-tasting) In this experiment, each of eight judges tastes each of four wines. The plan given in Table 4.2 treats the judges as blocks and specifies the order of tasting for each judge. Perhaps the positions in tasting order should also be considered as blocks.

The plan is rewritten in Table 6.1. Wine 4 is almost always tasted first or second. By the fourth tasting the judges may be feeling happy (and so give good marks to the other wines) or bored (and so give bad marks to the other wines). In either case, the comparison of wine 4 with the other wines could be biased.

It would be better to regard 'judges' and 'positions in tasting order' as two systems of blocks.

For the rest of this chapter we shall call one system of blocks 'rows' and the other system 'columns'. Each design will therefore be written in a rectangle, like the one in Table 6.1. For simplicity, we assume that:

	Judge							
Tasting	1	2	3	4	5	6	7	8
1	2	4	4	2	1	2	4	4
2	1	3	1	4	4	4	2	3
3	3	2	2	3	3	1	1	1
4	4	1	3	1	2	3	3	2

Table 6.1: Another way of writing the plan in Table 4.2

(i) each row meets each column in a single plot;

(ii) all treatments occur equally often in each row;

(iii) all treatments occur equally often in each column;

(iv) there are m rows:

(v) there are n columns.

From these assumptions it follows that there are n plots per row and m plots per column; that $N = mn$; that t divides n and t divides m; and that every treatment has replication r, where $r = nm/t$.

6.2 Latin squares

The simplest way in which the above conditions can be satisfied is when $n = m = t$. Then the design is called a Latin square.

Definition A *Latin square of order t* is an arrangement of t symbols in a $t \times t$ square array in such a way that each symbol occurs once in each row and once in each column.

Example 6.2 (Latin square of order 6) Figure 6.1 shows a Latin square of order 6.

A	B	C	D	E	F
C	A	B	F	D	E
B	C	A	E	F	D
D	F	E	A	C	B
E	D	F	B	A	C
F	E	D	C	B	A

Figure 6.1: Latin square of order 6

A	B	C	D
D	A	B	C
C	D	A	B
B	C	D	A

Figure 6.2: Cyclic Latin square of order 4

Here are some simple methods of constructing Latin squares.

Cyclic method Write the symbols in the top row in any order. In the second row, shift all the symbols to the right one place, moving the last symbol to the front. Continue like this, shifting each row one place to the right of the previous row.

A cyclic Latin square of order 4 is shown in Figure 6.2.

More generally, each row can be shifted s places to the right of the previous one, so long as s is co-prime to t.

Group method (for readers who know a little group theory) Take any group G of order t. Label its elements $g_1, g_2, \ldots, g_t$. Label the rows and columns of the square by $g_1, g_2, \ldots, g_t$. In the cell in the row labelled g_i and the column labelled g_j put the element $g_i g_j$.

	g_1	g_2	$\cdots$	g_j	$\cdots$	g_n
g_1						
$\vdots$						
g_i				$g_i g_j$		
$\vdots$						
g_n						

For example, when $t = 4$ we can take G to be the cyclic group $\{1, g, g^2, g^3 : g^4 = 1\}$ to obtain

	1	g	g^2	g^3
1	1	g	g^2	g^3
g	g	g^2	g^3	1
g^2	g^2	g^3	1	g
g^3	g^3	1	g	g^2

.

Stripping off the border of labels at the top and the left, and relabelling the symbols, gives the Latin square

0	1	2	3
1	2	3	4
2	3	0	1
3	0	1	2

,

which is a cyclic square in which each row is shifted one place to the left (equivalently, three places to the right) of the row before.

When $t = 6$ we can take G to be the symmetric group S_3. This gives the Latin square in Figure 6.1.

Product method Given two Latin squares

$$S_1 \text{ of size } t_1 \text{ with letters } A_1, \ldots, A_{t_1}$$
$$S_2 \text{ of size } t_2 \text{ with letters } B_1, \ldots, B_{t_2},$$

we make a new Latin square $S_1 \otimes S_2$ with letters C_{ij} for $i = 1, \ldots, t_1$ and $j = 1, \ldots, t_2$.

Enlarge square S_2. Wherever letter B_j occurs, put the whole square S_1 but replacing letters $A_1, \ldots, A_{t_1}$ by $C_{1j}, \ldots, C_{t_1 j}$.

For example, when $t_1 = t_2 = 2$ we can take

$$S_1 = \begin{array}{|c|c|} \hline A_1 & A_2 \\ \hline A_2 & A_1 \\ \hline \end{array} \quad \text{and} \quad S_2 = \begin{array}{|c|c|} \hline B_1 & B_2 \\ \hline B_2 & B_1 \\ \hline \end{array}.$$

Then

$$S_1 \otimes S_2 = \begin{array}{|c|c||c|c|} \hline C_{11} & C_{21} & C_{12} & C_{22} \\ \hline C_{21} & C_{11} & C_{22} & C_{12} \\ \hline \hline C_{12} & C_{22} & C_{11} & C_{21} \\ \hline C_{22} & C_{12} & C_{21} & C_{11} \\ \hline \end{array},$$

which can be rewritten as

C	*D*	*E*	*F*
D	*C*	*F*	*E*
E	*F*	*C*	*D*
F	*E*	*D*	*C*

Note that this Latin square is not cyclic.

6.3 Construction and randomization

We return to the more general case that both m and n are multiples of t. To construct a row-column design and randomize it, proceed as follows.

(i) Divide the $m \times n$ rectangle into $t \times t$ squares.

(ii) In each $t \times t$ square, put any Latin square of order t, using the same symbols in each square.

(iii) Randomly permute the m rows (*not* the treatments within them).

(iv) Randomly permute the n columns (*not* the treatments within them).

Note that the division into $t \times t$ squares is artificial, just to help in the construction of the design. It must be ignored at the randomization stage.

Example 6.1 revisited (Wine-tasting) Here we have 4 wines, 4 rows and 8 columns, so we start by dividing the 4×8 rectangle into 4×4 squares.

	1	2	3	4	⋮	5	6	7	8
1					⋮				
2					⋮				
3					⋮				
4					⋮				

Then we put any two 4×4 Latin squares into the spaces. Here we use the symbols A, B, C and D for the wines, to avoid confusion with the numbering of the rows and columns.

	1	2	3	4	⋮	5	6	7	8
1	*A*	*B*	*C*	*D*	⋮	*C*	*D*	*A*	*B*
2	*D*	*A*	*B*	*C*	⋮	*D*	*C*	*B*	*A*
3	*C*	*D*	*A*	*B*	⋮	*A*	*B*	*C*	*D*
4	*B*	*C*	*D*	*A*	⋮	*B*	*A*	*D*	*C*

We rub out the construction lines (the dotted vertical line above).

To randomize the rows, we choose a random permutation of four objects by one of the methods described in Section 2.2. If the permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

then we randomize the rows of the previous rectangle to obtain

	1	2	3	4	5	6	7	8
1	*A*	*B*	*C*	*D*	*C*	*D*	*A*	*B*
3	*C*	*D*	*A*	*B*	*A*	*B*	*C*	*D*
4	*B*	*C*	*D*	*A*	*B*	*A*	*D*	*C*
2	*D*	*A*	*B*	*C*	*D*	*C*	*B*	*A*

.

Similarly, to randomize the columns we choose a random permutation of eight objects. The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 5 & 7 & 8 & 2 & 1 & 4 \end{pmatrix}$$

gives the following rectangle.

	3	6	5	7	8	2	1	4
1	*C*	*D*	*C*	*A*	*B*	*B*	*A*	*D*
3	*A*	*B*	*A*	*C*	*D*	*D*	*C*	*B*
4	*D*	*A*	*B*	*D*	*C*	*C*	*B*	*A*
2	*B*	*C*	*D*	*B*	*A*	*A*	*D*	*C*

Before giving the plan to the experimenter, replace the row and column numbers by the natural orders (or the explicit names of the rows and columns, if they have any). The final plan is shown in Table 6.2.

Tasting	Judge 1	2	3	4	5	6	7	8
1	*C*	*D*	*C*	*A*	*B*	*B*	*A*	*D*
2	*A*	*B*	*A*	*C*	*D*	*D*	*C*	*B*
3	*D*	*A*	*B*	*D*	*C*	*C*	*B*	*A*
4	*B*	*C*	*D*	*B*	*A*	*A*	*D*	*C*

Table 6.2: Randomized plan in Example 6.1

6.4 Orthogonal subspaces

As before, we define the treatment subspace V_T of dimension t, the one-dimensional subspace $V_0 = W_0$ consisting of constant vectors, and put $W_T = V_T \cap V_0^{\perp}$, which has dimension $t-1$. Spaces V_R and V_C are defined just like the blocks subspace V_B; that is, V_R consists of vectors which are constant on each row, and V_C consists of vectors which are constant on each column, so that $\dim V_R = m$ and $\dim V_C = n$. Then put

$$W_R = V_R \cap V_0^{\perp}$$

and

$$W_C = V_C \cap V_0^{\perp},$$

so that $\dim W_R = m-1$ and $\dim W_C = n-1$. Assumptions (i)–(iii) in Section 6.1 imply that the spaces W_T, W_R and W_C are orthogonal to each other: the proof is like the proof of Theorem 4.1. Therefore

$$V_R + V_C + V_T = W_0 \oplus W_R \oplus W_C \oplus W_T$$

orthogonally, and so

$$\dim(V_R + V_C + V_T) = 1 + (m-1) + (n-1) + (t-1).$$

Put

$$W_E = (V_R + V_C + V_T)^{\perp}$$

(the notation E for the subscript will be explained in Chapter 10). Then

$$\dim W_E = N - (1 + (m-1) + (n-1) + (t-1)) = (n-1)(m-1) - (t-1),$$

and

$$V = W_0 \oplus W_R \oplus W_C \oplus W_T \oplus W_E$$

orthogonally. This decomposition of V gives the analysis of variance, for both the fixed-effects model and the random-effects model.

6.5 Fixed effects: model and analysis

For each plot ω, write

$$\begin{aligned} R(\omega) &= \text{the row containing } \omega \\ C(\omega) &= \text{the column containing } \omega. \end{aligned}$$

We call R and C the *row* factor and the *column* factor respectively.

In the fixed-effects model, rows and columns both contribute to the expection but not to the covariance. That is,

$$\mathbb{E}(Y_\omega) = \tau_{T(\omega)} + \zeta_{R(\omega)} + \eta_{C(\omega)} \tag{6.1}$$

and

$$\mathrm{Cov}(\mathbf{Y}) = \sigma^2 \mathbf{I}.$$

Here $\zeta_1, \ldots, \zeta_m$ are unknown parameters associated with the rows and $\eta_1, \ldots, \eta_n$ are unknown parameters associated with the columns. We can rewrite Equation (6.1) in vector terms as

$$\mathbb{E}(\mathbf{Y}) = \tau + \zeta + \eta, \tag{6.2}$$

where $\tau \in V_T$, $\zeta \in V_R$, and $\eta \in V_C$. As in Section 4.5, put $\tau_0 = \bar{\tau}\mathbf{u}_0$, $\tau_T = \tau - \tau_0$, $\zeta_0 = \bar{\zeta}\mathbf{u}_0$, $\zeta_R = \zeta - \zeta_0$, $\eta_0 = \bar{\eta}\mathbf{u}_0$, and $\eta_C = \eta - \eta_0$. Then τ_0, ζ_0 and η_0 are all in W_0 while $\tau_T \in W_T$, $\zeta_R \in W_R$ and $\eta_C \in W_C$. Then Equation (6.2) gives

$$\begin{aligned} \mathbb{E}(\mathbf{Y}) &= (\tau_0 + \tau_T) + (\zeta_0 + \zeta_R) + (\eta_0 + \eta_C) \\ &= (\tau_0 + \zeta_0 + \eta_0) + \tau_T + \zeta_R + \eta_C, \end{aligned}$$

with $\tau_0 + \zeta_0 + \eta_0$ in W_0, τ_T in W_T, ζ_R in W_R and η_C in W_C. We cannot distinguish between τ_0, ζ_0 and η_0; that is, between the overall level of treatment means, row means and columns means. However, because W_T, W_R and W_C are orthogonal to each other, we can estimate treatment contrasts, row contrasts and column contrasts. In fact, Theorem 2.4 shows that

$$\mathbb{E}(P_{W_T}\mathbf{Y}) = P_{W_T}(\mathbb{E}\,\mathbf{Y}) = P_{W_T}(\tau_0 + \zeta_0 + \eta_0 + \tau_T + \zeta_R + \eta_C) = \tau_T$$

and

$$\mathbb{E}(\|P_{W_T}\mathbf{Y}\|^2) = \|\tau_T\|^2 + \dim(W_T)\sigma^2.$$

Similarly, $\mathbb{E}(P_{W_R}\mathbf{Y}) = \zeta_R$, $\mathbb{E}(P_{W_C}\mathbf{Y}) = \eta_C$,

$$\mathbb{E}(\|P_{W_R}\mathbf{Y}\|^2) = \|\zeta_R\|^2 + \dim(W_R)\sigma^2,$$

and

$$\mathbb{E}(\|P_{W_C}\mathbf{Y}\|^2) = \|\eta_C\|^2 + \dim(W_C)\sigma^2.$$

This gives us the anova table in Table 6.3.

source	sum of squares	degrees of freedom	EMS	variance ratio
mean	$\frac{\mathrm{sum}^2}{N}$	1	$\Vert\tau_0+\zeta_0+\eta_0\Vert^2+\sigma^2$	$\frac{\mathrm{MS(mean)}}{\mathrm{MS(residual)}}$
rows	$\sum_{i=1}^{m}\frac{\mathrm{sum}^2_{R=i}}{n}-\frac{\mathrm{sum}^2}{N}$	$m-1$	$\frac{\Vert\zeta_R\Vert^2}{m-1}+\sigma^2$	$\frac{\mathrm{MS(rows)}}{\mathrm{MS(residual)}}$
columns	$\sum_{i=1}^{n}\frac{\mathrm{sum}^2_{C=i}}{m}-\frac{\mathrm{sum}^2}{N}$	$n-1$	$\frac{\Vert\eta_C\Vert^2}{n-1}+\sigma^2$	$\frac{\mathrm{MS(columns)}}{\mathrm{MS(residual)}}$
treatments	$\sum_{i=1}^{t}\frac{\mathrm{sum}^2_{T=i}}{r}-\frac{\mathrm{sum}^2}{N}$	$t-1$	$\frac{\Vert\tau_T\Vert^2}{t-1}+\sigma^2$	$\frac{\mathrm{MS(treatments)}}{\mathrm{MS(residual)}}$
residual	$\leftarrow$ ..... by subtraction ..... $\rightarrow$		σ^2	–
Total	$\sum_{\omega} y_\omega^2$	N		

Table 6.3: Anova table for rows, columns and unstructured treatments under the fixed-effects model

6.6 Random effects: model and analysis

In the random-effects model we assume that rows and columns make no contribution to the expectation but that they do affect the pattern of covariance. Thus $\mathbb{E}(Z_\omega) = 0$ and so $\mathbb{E}(Y_\omega) = \tau_\omega$, for all plots ω, while, for a pair of plots α and β,

$$\mathrm{cov}(Z_\alpha, Z_\beta) = \begin{cases} \sigma^2 & \text{if } \alpha = \beta \\ \rho_1\sigma^2 & \text{if } \alpha \neq \beta \text{ but } R(\alpha) = R(\beta) \\ \rho_2\sigma^2 & \text{if } \alpha \neq \beta \text{ but } C(\alpha) = C(\beta) \\ \rho_3\sigma^2 & \text{otherwise.} \end{cases}$$

Typically $\rho_1 > \rho_3$ and $\rho_2 > \rho_3$. In matrix form

$$\begin{aligned} \mathrm{Cov}(\mathbf{Y}) &= \sigma^2\mathbf{I} + \rho_1\sigma^2(\mathbf{J}_R - \mathbf{I}) + \rho_2\sigma^2(\mathbf{J}_C - \mathbf{I}) + \rho_3\sigma^2(\mathbf{J} - \mathbf{J}_R - \mathbf{J}_C + \mathbf{I}) \\ &= \sigma^2[(1-\rho_1-\rho_2+\rho_3)\mathbf{I} + (\rho_1-\rho_3)\mathbf{J}_R + (\rho_2-\rho_3)\mathbf{J}_C + \rho_3\mathbf{J}], \end{aligned}$$

where $\mathbf{J}_R$ is the $N \times N$ matrix whose (α, β)-entry is equal to

$$\begin{cases} 1 & \text{if } R(\alpha) = R(\beta) \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbf{J}_C$ is defined similarly with respect to the column factor.

Theorem 6.1 *If*

$$\mathrm{Cov}(\mathbf{Y}) = \sigma^2[(1-\rho_1-\rho_2+\rho_3)\mathbf{I} + (\rho_1-\rho_3)\mathbf{J}_R + (\rho_2-\rho_3)\mathbf{J}_C + \rho_3\mathbf{J}],$$

then the eigenspaces of $\mathrm{Cov}(\mathbf{Y})$ *are* W_0, W_R, W_C *and* $(V_R + V_C)^\perp$, *with eigenvalues* ξ_0, ξ_R, ξ_C *and* ξ *respectively, where*

$$\begin{aligned} \xi_0 &= \sigma^2[1 + (n-1)\rho_1 + (m-1)\rho_2 + (m-1)(n-1)\rho_3] \\ \xi_R &= \sigma^2[1 + (n-1)\rho_1 - \rho_2 - (n-1)\rho_3] \\ \xi_C &= \sigma^2[1 - \rho_1 + (m-1)\rho_2 - (m-1)\rho_3] \\ \xi &= \sigma^2[1 - \rho_1 - \rho_2 + \rho_3]. \end{aligned}$$

Proof Put $\mathbf{C} = \mathrm{Cov}(\mathbf{Y})$. Let $\mathbf{x}$ be a vector in V. An argument similar to the one in Section 4.6 shows that if $\mathbf{x} \in V_R$ then $\mathbf{J}_R\mathbf{x} = n\mathbf{x}$ while if $\mathbf{x} \in V_R^\perp$ then $\mathbf{J}_R\mathbf{x} = \mathbf{0}$. Likewise, if $\mathbf{x} \in V_C$ then $\mathbf{J}_C\mathbf{x} = m\mathbf{x}$ while if $\mathbf{x} \in V_C^\perp$ then $\mathbf{J}_C\mathbf{x} = \mathbf{0}$. As always, if $\mathbf{x} \in V_0$ then $\mathbf{J}\mathbf{x} = N\mathbf{x} = nm\mathbf{x}$ while if $\mathbf{x} \in V_0^\perp$ then $\mathbf{J}\mathbf{x} = \mathbf{0}$.

Now, $\mathbf{u}_0 \in V_R \cap V_C \cap V_0$ so

$$\mathbf{C}\mathbf{u}_0 = \sigma^2[(1-\rho_1-\rho_2+\rho_3) + n(\rho_1-\rho_3) + m(\rho_2-\rho_3) + nm\rho_3]\mathbf{u}_0 = \xi_0\mathbf{u}_0.$$

If $\mathbf{x} \in W_R$ then $\mathbf{x} \in V_R \cap V_C^\perp \cap V_0^\perp$ so

$$\mathbf{C}\mathbf{x} = \sigma^2[(1-\rho_1-\rho_2+\rho_3) + n(\rho_1-\rho_3)]\mathbf{x} = \xi_R\mathbf{x}.$$

Similarly, if $\mathbf{x} \in W_C$ then

$$\mathbf{Cx} = \sigma^2[(1-\rho_1-\rho_2+\rho_3)+m(\rho_2-\rho_3)]\mathbf{x} = \xi_C\mathbf{x}.$$

Finally, if $\mathbf{x} \in (V_R+V_C)^{\perp}$ then $\mathbf{x} \in V_R^{\perp} \cap V_C^{\perp} \cap V_0^{\perp}$ and so

$$\mathbf{Cx} = \sigma^2(1-\rho_1-\rho_2+\rho_3)\mathbf{x} = \xi\mathbf{x}. \quad \blacksquare$$

Note that $\xi_C = \xi + \sigma^2 m(\rho_2 - \rho_3)$, which is bigger than ξ if $\rho_2 > \rho_3$. Similarly, $\xi_R > \xi$ if $\rho_1 > \rho_3$.

The treatment subspace W_T is contained in the eigenspace $(V_R+V_C)^{\perp}$, so we obtain the anova table shown in Table 6.4.

Estimation and testing of treatment effects is the same in the fixed-effects and the random-effects models. In the fixed-effects model we can also estimate row differences and column differences and test for these differences. These tests are one-sided. In the random-effects model we could do formal two-sided tests of the null hypotheses that $\xi_R = \xi$ and $\xi_C = \xi$. However, all that we normally do is compare MS(rows) with MS(residual) and MS(columns) with MS(residual) to draw broad conclusions.

If MS(rows) $>>$ MS(residual) then rows are good for blocking. Likewise, if MS(columns) $>>$ MS(residual) then columns are good for blocking. If either MS(rows) $<<$ MS(residual) or MS(columns) $<<$ MS(residual) then either there are some patterns of variability or management patterns which the experimenter has not told you about or he is fiddling the data.

Questions for Discussion

6.1 Look at the eelworm experiment from Assigment 3. Suppose that, in the year following the eelworm experiment, a similar experiment is to be conducted on the same plots, using four types of fertilizer at a single dose and no "control". How should the plots be divided into blocks?

6.2 The report-writing department of a technical research station needs to replace its old word-processing system. It plans to evaluate 5 new word-processors. The vendors of each word-processor have agreed to make several copies of it available to the department for a one-week trial period: the first full week in March. The head of the department has decided to abandon all normal work for that week and devote it to testing and evaluating the new word-processors. There are 15 typists in the department's typing pool. The minimum amount of time that a typist can spend with a new word-processor and sensibly be asked to evaluate it for ease of use is one day.

Design the experiment and produce a plan for the head of the department.

stratum	source		df	EMS	VR
V_0	mean	mean	1	$\|\tau_0\|^2 + \xi_0$	–
W_R	rows	rows	$m-1$	ξ_R	–
W_C	columns	columns	$n-1$	ξ_C	–
$(V_R + V_C)^\perp$	plots	treatments	$t-1$	$\frac{\|\tau_T\|^2}{t-1} + \xi$	$\frac{\text{MS(treatments)}}{\text{MS(residual)}}$
		residual	by subtraction	ξ	–
Total			N		

Table 6.4: Anova table for rows, columns and unstructured treatments under the random-effects model