Chapter 9

More about Latin Squares

9.1 Uses of Latin squares

Let S be an $n \times n$ Latin square. Figure 9.1 shows a possible square S when $n = 4$, using the symbols 1, 2, 3, 4 for the 'letters'. Such a Latin square can be used to construct a design for an experiment in surprisingly many ways.

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

Figure 9.1: A Latin square of order 4, used to construct four types of design

9.1.1 One treatment factor in a square

The square S is used to allocate n treatments to a row-column array with n rows and columns, as in Chapter 6. This can be summarized by the skeleton analysis of variance in Table 9.1, assuming the random-effects model.

9.1.2 More general row-column designs

The square S is used to allocate n treatments to a row-column array with nm rows and nl columns, where m and l are positive integers, as described in Chapter 6. The rectangular array is split into ml squares of size $n \times n$, and a copy of S is put into each one. The skeleton analysis of variance is shown in Table 9.2.

Stratum	source	df
mean	mean	1
rows	rows	$n-1$
columns	columns	$n-1$
plots	treatments	$n-1$
	residual	$(n-1)(n-2)$
	total	$(n-1)^2$
Total		n^2

Table 9.1: Skeleton analysis of variance in Section 9.1.1

Stratum	source	df
mean	mean	1
rows	rows	$nm-1$
columns	columns	$nl-1$
plots	treatments	$n-1$
	residual	$n^2ml-nm-nl-n+2$
	total	$(nm-1)(nl-1)$
Total		n^2ml

Table 9.2: Skeleton analysis of variance in Section 9.1.2

9.1.3 Two treatment factors in a block design

Suppose that there are n blocks of size n and that there are two n-level treatment factors F and G. If we can assume that the F-by-G interaction is zero then we need estimate only the main effects of F and G. We can use S to allocate levels of F and G to the plots in such a way that all combinations of F and G occur and that each of F and G is orthogonal to blocks. This design depends on the interaction being zero, so it is called a *main-effects-only* design. It is also a *single-replicate* design, because each treatment occurs exactly once.

Construction Rows are blocks. Columns are levels of F, say $f_1, f_2, \ldots, f_n$. ‘Letters’ are levels of G, say $g_1, g_2, \ldots, g_n$. If the letter in row i and column j is l then treatment combination $f_j g_l$ comes in block i.

The square in Figure 9.1 gives the following block design.

Block 1			
f_1g_1	f_2g_2	f_3g_3	f_4g_4

Block 2			
f_1g_2	f_2g_4	f_3g_1	f_4g_3

Block 3			
f_1g_3	f_2g_1	f_3g_4	f_4g_2

Block 4			
f_1g_4	f_2g_3	f_3g_2	f_4g_1

Randomization Now blocks do not all contain the same treatments, so we need to randomize whole blocks among themselves and well as randomizing plots within blocks.

(i) Randomize blocks (that is, randomly permute the order or the names of the four whole blocks);

(ii) randomize plots within blocks (that is, within each block independently, randomly permute plots).

Be careful not to think of this as randomizing levels of either F or G, because we do not want to destroy the sets of n treatment combinations that come together in a block.

Skeleton analysis of variance The null analysis of variance has strata for the mean, blocks and plots, just like any block design. By construction, blocks and F and G are all orthogonal to each other (use Theorems 4.1 and 5.1), so the skeleton analysis of variance is as shown in Table 9.3.

Stratum	source	df	
mean	mean		1
blocks	blocks		$n-1$
plots	F	$n-1$	
	G	$n-1$	
	residual	$(n-1)(n-2)$	
	total		$n(n-1)$
Total			n^2

Table 9.3: Skeleton analysis of variance in Section 9.1.3

9.1.4 Three treatment factors in an unblocked design

Suppose that F, G and H are three n-level treatment factors. If we can assume that all interactions are zero then we can construct a main-effects-only design in n^2 plots with no blocking. Not all the treatment combinations occur, so this is called a *fractional replicate*.

Construction Rows are levels of F. Columns are levels of G. Letters are levels of H. If the letter in row i and column j is l then the treatment $f_i g_j h_l$ occurs in the design.

Thus the Latin square in Figure 9.1 gives the following 16 treatments out of the possible 64. The design is called a *quarter-replicate*.

$$f_1g_1h_1,\ f_1g_2h_2,\ f_1g_3h_3,\ f_1g_4h_4,\ f_2g_1h_2,\ f_2g_2h_4,\ f_2g_3h_1,\ f_2g_4h_3,\ f_3g_1h_3,$$
$$f_3g_2h_1,\ f_3g_3h_4,\ f_2g_4h_2,\ f_4g_1h_4,\ f_4g_2h_3,\ f_4g_3h_2,\ f_4g_4h_1$$

Randomization Completely randomize all the plots. Once again, do not think of this as randomizing levels of any of the treatment factors, or the careful choice of fractional replicate may be ruined.

Skeleton analysis of variance There are no blocks, so Section 2.13 shows that the strata are V_0 and $V_0^\perp$. By construction, F, G and H are all orthogonal to each other, so we obtain the skeleton analysis of variance in Table 9.4.

Stratum	Source	df
mean	mean	1
plots	F	$n-1$
	G	$n-1$
	H	$n-1$
	residual	$(n-1)(n-2)$
	total	n^2-1
Total		n^2

Table 9.4: Skeleton analysis of variance in Section 9.1.4

9.2 Graeco-Latin squares

Definition Two $n \times n$ Latin squares are *orthogonal* to each other if each letter of the first square occurs in the same position as each letter of the second square exactly once.

Such a pair is often called a *Graeco-Latin* square, because traditionally Latin letters are used for the first square and Greek letters for the second square.

Example 9.1 (Mutually orthogonal Latin squares of order 3) The following two Latin squares are orthogonal to each other.

A	B	C
C	A	B
B	C	A

α	β	γ
β	γ	α
γ	α	β

When the two squares are superimposed, we obtain the following Graeco-Latin square.

$A\ \alpha$	$B\ \beta$	$C\ \gamma$
$C\ \beta$	$A\ \gamma$	$B\ \alpha$
$B\ \gamma$	$C\ \alpha$	$A\ \beta$

Note that not every Latin square has another orthogonal to it. For example, there is no Latin square orthogonal to the one in Figure 6.2 (if you try to construct one by trial and error you will soon find that it is impossible).

However, there are some simple methods for constructing pairs of mutually orthogonal Latin squares.

Prime numbers If $n = p$, a prime number, label the rows and columns of the square by the integers modulo p. Choose a to be an integer modulo p which is not equal to 0 or 1. In the first square, the letter in row x and column y should be $x+y$; in the second it should be $ax+y$.

in the first square

	$\dots$	y	$\dots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$
x	$\dots$	$x+y$	$\dots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$

in the second square

	$\dots$	y	$\dots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$
x	$\dots$	$ax+y$	$\dots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$

Once the first square has been constructed it is easy to construct the second: for each value of x, take the row labelled ax in the first square and put it in the row labelled x in the second square.

For example, taking $n = p = 5$ and $a = 2$ gives the following pair of mutually orthogonal Latin squares of order 5.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

	0	1	2	3	4
0	0	1	2	3	4
1	2	3	4	0	1
2	4	0	1	2	3
3	1	2	3	4	0
4	3	4	0	1	2

In general, different values of a give squares orthogonal to each other.

Finite fields If n is a power of a prime but not itself prime, use a similar construction using the *finite field* $\mathrm{GF}(n)$ with n elements. Do not worry if you know nothing about finite fields. For practical purposes it is enough to know about $\mathrm{GF}(4)$, $\mathrm{GF}(8)$ and $\mathrm{GF}(9)$.

The elements of the field $\mathrm{GF}(4)$ can be written as 0, 1, x and x^2, where all operations are modulo 2 and $x^2 = x+1$. For example, $x^3 = x(x^2) = x(x+1) = x^2 + x = (x+1) + x = 1$.

The elements of the field $\mathrm{GF}(8)$ can be written in the form $m_1 + m_2 y + m_3 y^2$, where m_1, m_2 and m_3 are integers modulo 2. All operations are modulo 2, and $y^3 = y+1$.

The elements of the field $\mathrm{GF}(9)$ can be written in the form $m_1 + m_2 i$, where m_1 and m_2 are integers modulo 3. All operations are modulo 3, and $i^2 = -1 = 2$. For example, if we take $a = i$ then we obtain the pair of mutually orthogonal 9×9 Latin squares in Figure 9.2. It may be less confusing to rewrite these squares as in Figure 9.3, where the elements of $\mathrm{GF}(9)$ have been replaced by letters.

Product method If S_1 and T_1 are mutually orthogonal Latin squares of order n_1 and S_2 and T_2 are mutually orthogonal Latin squares of order n_2 then the product squares $S_1 \otimes S_2$ and $T_1 \otimes T_2$ are orthogonal to each other and have order $n_1 n_2$.

There is no Graeco-Latin square of order 1, 2 or 6. However, Graeco-Latin squares exist for all other orders. The above methods can be combined to give a pair of orthogonal Latin squares of order n whenever n is odd or divisible by 4. If n is even but not divisible by 4 then the construction is more complicated. A pair of mutually orthogonal Latin squares of order 10 is shown in Figure 9.4.

9.3 Uses of Graeco-Latin squares

There are many ways of using a Graeco-Latin square of order n to construct an experiment for n^2 plots. All the designs described here have $(n-1)(n-3)$ residual degrees of freedom.

9.3.1 Superimposed design in a square

If last year's experiment was a Latin square, and there are the same number of treatments this year, on the same plots, then this year's experiment should be a Latin square orthogonal to last year's. If you know before you do the first experiment that there will be a second one, then choose any Graeco-Latin square, randomize its rows and columns, use the Latin letters for Year 1 and the Greek letters for Year 2.

If the experimental units are expensive and long-lived, such as trees, then an experiment next year is quite likely, so it is safer to always use a Latin square that has another Latin square orthogonal to it.

	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
0	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
1	1	2	0	$1+i$	$2+i$	i	$1+2i$	$2+2i$	$2i$
2	2	0	1	$2+i$	i	$1+i$	$2+2i$	$2i$	$1+2i$
i	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$	0	1	2
$1+i$	$1+i$	$2+i$	i	$1+2i$	$2+2i$	$2i$	1	2	0
$2+i$	$2+i$	i	$1+i$	$2+2i$	$2i$	$1+2i$	2	0	1
$2i$	$2i$	$1+2i$	$2+2i$	0	1	2	i	$1+i$	$2+i$
$1+2i$	$1+2i$	$2+2i$	$2i$	1	2	0	$1+i$	$2+i$	i
$2+2i$	$2+2i$	$2i$	$1+2i$	2	0	1	$2+i$	i	$1+i$

	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
0	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
1	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$	0	1	2
2	$2i$	$1+2i$	$2+2i$	0	1	2	i	$1+i$	$2+i$
i	2	0	1	$2+i$	i	$1+i$	$2+2i$	$2i$	$1+2i$
$1+i$	$2+i$	i	$1+i$	$2+2i$	$2i$	$1+2i$	2	0	1
$2+i$	$2+2i$	$2i$	$1+2i$	2	0	1	$2+i$	i	$1+i$
$2i$	1	2	0	$1+i$	$2+i$	i	$1+2i$	$2+2i$	$2i$
$1+2i$	$1+i$	$2+i$	i	$1+2i$	$2+2i$	$2i$	1	2	0
$2+2i$	$1+2i$	$2+2i$	$2i$	1	2	0	$1+i$	$2+i$	i

Figure 9.2: Pair of mutually orthogonal Latin squares of order 9

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
B	*C*	*A*	*E*	*F*	*D*	*H*	*I*	*G*
C	*A*	*B*	*F*	*D*	*E*	*I*	*G*	*H*
D	*E*	*F*	*G*	*H*	*I*	*A*	*B*	*C*
E	*F*	*D*	*H*	*I*	*G*	*B*	*C*	*A*
F	*D*	*E*	*I*	*G*	*H*	*C*	*A*	*B*
G	*H*	*I*	*A*	*B*	*C*	*D*	*E*	*F*
H	*I*	*G*	*B*	*C*	*A*	*E*	*F*	*D*
I	*G*	*H*	*C*	*A*	*B*	*F*	*D*	*E*

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
D	*E*	*F*	*G*	*H*	*I*	*A*	*B*	*C*
G	*H*	*I*	*A*	*B*	*C*	*D*	*E*	*F*
C	*A*	*B*	*F*	*D*	*E*	*I*	*G*	*H*
F	*D*	*E*	*I*	*G*	*H*	*C*	*A*	*B*
I	*G*	*H*	*C*	*A*	*B*	*F*	*D*	*E*
B	*C*	*A*	*E*	*F*	*D*	*H*	*I*	*G*
E	*F*	*D*	*H*	*I*	*G*	*B*	*C*	*A*
H	*I*	*G*	*B*	*C*	*A*	*E*	*F*	*D*

Figure 9.3: Pair of mutually orthogonal Latin squares of order 9, rewritten using letters

A	*H*	*I*	*D*	*J*	*F*	*G*	*B*	*C*	*E*
H	*I*	*A*	*J*	*C*	*D*	*E*	*F*	*G*	*B*
I	*E*	*J*	*G*	*A*	*B*	*H*	*C*	*D*	*F*
B	*J*	*D*	*E*	*F*	*H*	*I*	*G*	*A*	*C*
J	*A*	*B*	*C*	*H*	*I*	*F*	*D*	*E*	*G*
E	*F*	*G*	*H*	*I*	*C*	*J*	*A*	*B*	*D*
C	*D*	*H*	*I*	*G*	*J*	*B*	*E*	*F*	*A*
F	*B*	*E*	*A*	*D*	*G*	*C*	*H*	*I*	*J*
D	*G*	*C*	*F*	*B*	*E*	*A*	*J*	*H*	*I*
G	*C*	*F*	*B*	*E*	*A*	*D*	*I*	*J*	*H*

α	β	γ	λ	ε	μ	ν	η	ζ	δ
γ	δ	λ	ζ	μ	ν	β	α	η	ε
ε	λ	η	μ	ν	γ	δ	β	α	ζ
λ	α	μ	ν	δ	ε	ζ	γ	β	η
β	μ	ν	ε	ζ	η	λ	δ	γ	α
μ	ν	ζ	η	α	λ	γ	ε	δ	β
ν	η	α	β	λ	δ	μ	ζ	ε	γ
η	ζ	ε	δ	γ	β	α	λ	μ	ν
ζ	ε	δ	γ	β	α	η	μ	ν	λ
δ	γ	β	α	η	ζ	ε	ν	λ	μ

Figure 9.4: Pair of mutually orthogonal Latin squares of order 10

9.3.2 Two treatment factors in a square

If the experimental units form an $n \times n$ square and there are two n-level treatment factors, F and G, whose interaction is assumed to be zero, then a Graeco-Latin square can be used to construct a main-effects only single-replicate design. The Latin letters give the levels of F, the Greek letters the levels of G. Randomize rows and columns independently.

9.3.3 Three treatment factors in a block design

Suppose that there are n blocks of size n and three n-level treatment factors F, G and H. If all interactions among the treatment factors can be assumed to be zero then we can use a Graeco-Latin square to construct a main-effects-only fractional replicate. Rows are blocks; columns are levels of F; Latin and Greek letters are levels of G and H respectively. If the Latin and Greek letters in row i and column j are l and m respectively then the treatment combination $F_j G_l H_m$ occurs in block i.

Randomization is as in Section 9.1.3.

9.3.4 Four treatment factors in an unblocked design

If there are no interactions among the four n-level treatment factors F, G, H and I then a Graeco-Latin square can be used to construct a main-effects-only fractional replicate in n^2 plots. Rows are levels of F; columns are levels of G; Latin and Greek letters are levels of H and I respectively. If the Latin and Greek letters in row i and column j are l and m respectively then the treatment combination $F_i G_j H_l I_m$ occurs in the design.

Randomization is as in Section 9.1.4.

Questions for Discussion

9.1 Construct a 4×4 Graeco-Latin square and a 7×7 Graeco-Latin square.

9.2 A horticultural research station intends to investigate the effects of two treatment factors on the total weight of apples produced from apple trees. One treatment factor is the method of thinning; that is, removing fruitlets at an early stage of development so that those remaining will be able to grow larger. There are five methods of thinning, coded as A, B, C, D, E. The second treatment factor is the type of grass to grow around the base of the tree to prevent the growth of weeds. There are five types of grass, coded as a, b, c, d, e. It is assumed that there is no interaction between method of thinning and type of grass.

There are 25 trees available for the experiment. They are arranged in a 5×5 rectangle. Construct a suitable design.

9.3 Construct a 12×12 Graeco-Latin square.

9.4 A road safety organization wishes to compare four makes of car-tyre. The organization has four test cars and four test drivers. One working day is needed to fit new tyres to a car, take it for an exhaustive test-drive, take relevant measurements on tyre treads, record all details of the test-drive, and prepare the car for the next session. The organization has only one week in which to perform its tests. To keep each car properly balanced, the organization has decided that all four tyres on a car at any one time should be of the same make.

Construct a suitable design for this trial.